John von Neumann and the Theory of Operator Algebras *

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After some earlier work on single operators, von Neumann turned to families of operators in [1]. He initiated the study of rings of operators which are commonly called von Neumann algebras today. The papers which constitute the series "Rings of operators" opened a new field in mathematics and influenced research for half a century (or even longer). In the standard theory of modern operator algebras, many concepts and ideas have their origin in von Neumann's work. Since its inception, operator algebra theory has been in intimate relation to physics. The mathematical formalism of quantum theory was one of the motivations leading naturally to algebras of Hilbert space operators. After decades of relative isolation, again physics fertilized the operator algebra theory by mathematical questions of quantum statistical mechanics and quantum field theory.

The objective of the present article is two-fold. On the one hand, to sketch the early development of von Neumann algebras, to show how the fundamental classification of algebras emerged from the lattice of projections. These old ideas of von Neumann and Murray revived much later in connection with Jordan operator algebras and the K-theory of C*-algebras. On the other hand, to review briefly some relatively new developments such as the classification of hyperfinite factors, the index theory of subfactors and elements of Jordan algebras. These developments are connected to the programs initiated by von Neumann himself. The last part of the paper is devoted to those topics of operator algebra theory which are closest to physical applications.

Our overview of the legacy of von Neumann in operator algebra theory is neither entirely historical nor is it complete. It reflects the scientific taste and knowledge of the authors. The theory of operator algebras is a technical subject and to present a readable account of the development of many years is a difficult task. To facilitate reading, each section begins with an informal review of the essential ideas discussed in the section.

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Von Neumann algebras and the lattice of projections

In this section we first give the mathematical definition of von Neumann algebras which consist of linear Hilbert space operators. The characteristic feature of the concept of von Neumann algebra is its very rich structure. A von Neumann algebra contains the spectral projections of all selfadjoint operators belonging to the algebra. In particular, there are many orthogonal projections in the algebra itself. Roughly speaking, the point in the concept of von Neumann algebra is that formation of product and spectral diagonalization of selfadjoint elements are posssible within the algebra. It turns out that the projections of a von Neumann algebra form a lattice in the sense that any two of them determine a least upper bound and a greatest lower bound with respect to an appropriate and natural ordering. The lattice of projections is the starting point in the classification of von Neumann algebras and a ground for quantum logics. Von Neumann algebras are classified in terms of the range of a dimension function defined on the lattice of projections. The dimension function is the extension of the simple concept of rank (for matrices) and the peculiarity of the subject begins with the observation that in nontrivial cases this “rank” could be a noninteger. In this section the classification of von Neumann algebras is described. Also, the influence of measure theory on the early operator algebra theory is demonstrated by a comparison of a measure theoretic construction of Alfréd Haar with the dimension function of Murray and von Neumann. This example shows that the connection to measure theory and ergodic theory has been very important for operator algebras since the very beginning.

In the sequel, we denote by $B(\mathcal{H})$ the set of all bounded operators acting on the Hilbert space $\mathcal{H}$. For a subset $\mathcal{S} \subseteq B(\mathcal{H})$, its commutant $\mathcal{S}'$ is defined to consist of all operators commuting with $\mathcal{S}$:

$$\mathcal{S}' = \{K \in B(\mathcal{H}) : KS = SK \quad \text{for all} \quad S \in \mathcal{S}\}\,.$$

Note that $\mathcal{S} \subseteq (S')'$ holds obviously for any $\mathcal{S} \subseteq B(\mathcal{H})$. A family of operators acting on a Hilbert space is called von Neumann algebra if it contains the adjoint, the linear combinations and the products of its elements and forms a closed subspace of the space of all bounded operators with respect to the topology of pointwise convergence. A von Neumann algebra is linearly spanned by its selfadjoint elements and the spectral resolution of the latter ones lies conveniently in the algebra. One of the first results of von Neumann, the von Neumann's double commutant theorem, was an equivalent algebraic definition of von Neumman algebras. Von Neumann's double commutant theorem asserts that a family of operators is a von Neumann algebra if and only if it contains the adjoint of its elements and coincides with its second commutant (that is, the commutant of its commutant). The remarkable point in the double commutant theorem is the lack of any topological requirement. In the concept of von Neumann algebra, topology and pure algebra are in great harmony.

The selfadjoint idempotents, called projections, of a von Neumann algebra form an orthomodular, complete lattice with respect to the lattice operations $\wedge, \vee, \perp$ and the partial ordering $\leq$. Below we describe how these operations are defined in terms of the algebaric operations. The projections are in natural correspondence with the closed subspaces of the underlying Hilbert space and the set theoretical inclusion of subspaces induces a partial ordering on the projections. This ordering is equivalently defined as

$$p \leq q \quad \text{if} \quad pq = p\,. \tag{1}$$

The smallest projection with respect to this ordering is 0 and the largest one is the identity. For projections p and q, their meet (that is, greatest lower bound) $p \wedge q$ is the orthogonal projection onto the intersection of the range spaces of p and q. The projection $p \wedge q$ may be obtained as the pointwise limit of the sequence $(pq)^n$ of operators. The orthocomplementation $\perp$ is defined as $p^\perp = I - p$. The orthomodularity of the lattice of projections means that the following so-called orthomodularity condition is fulfilled in the lattice:

$$q = p \vee (p^\perp \wedge q) \quad \text{for} \quad p \leq q \tag{2}$$

This relation is a weakening of the distributivity condition and is an essential property of the lattice of projections.

Let p and q be two projections in a von Neumann algebra $\mathcal{M}$. The projections p and q are called equivalent (with respect to $\mathcal{M}$), $p \sim q$ in notation, if there is an operator x in $\mathcal{M}$ such that $p = x^*x$ and $q = xx^*$. In terms of the underlying Hilbert space, the equivalence of p and q means that there exists a partial isometry x in the given von Neumann algebra which sends the range space of p isometrically onto the range of q. An extended positive-valued function $D : \mathcal{P}(\mathcal{M}) \to [0, \infty]$ on the set $\mathcal{P}(\mathcal{M})$ of all projections of $\mathcal{M}$ is called dimension function if it satisfies the following requirements:

(a) $D(p) > 0$ if $p \neq 0$ and $D(0) = 0$.

(b) $D(p) = D(q)$ if p and q are equivalent projections.

(c) $D(\sum_i p_i) = \sum_i D(p_i)$ if $p_i q_j = 0$ whenever $i \neq j$.

It is fundamental in the theory of von Neumann algebras that the dimension function is determined up to a positive multiple if the center of the algebra is trivial.

We sketch how the dimension function was obtained in [3]. A nonzero projection is called finite if it is not equivalent with a smaller projection. "Smaller" is understood here in the sense of the partial ordering (1). Murray and von Neumann proved in [3] that if f is a finite and e is an arbitrary projection then there exists a unique integer k such that

$$f = q_1 + q_2 + \ldots + q_k + p\,,$$

where q_1, $q_2, \ldots, q_k$ are pairwise orthogonal projections equivalent to e, p is a projection orthogonal to all q_i and equivalent to a subprojection of f. This integer k is denoted by

$$\left[\frac{f}{e}\right] \tag{3}$$

and this is the number of projections equivalent to e which may be packed into f in a pairwise orthogonal way. (3) is an integer and is only an approximate measure of the ratio of the subspaces corresponding to f and e. The limit

$$\lim_n \frac{\left[\frac{f}{e_n}\right]}{\left[\frac{e_0}{e_n}\right]} = \left(\frac{f}{e_0}\right) \tag{4}$$

forms a quantitative ratio of relative dimensionality, where the sequence e_n is not detailed here.

The relative dimension was defined in [3] as

$$D(e) = \begin{cases} 0 & \text{if } e = 0, \\ \left(\dfrac{e}{e_0}\right) & \text{if } e \text{ is finite,} \\ +\infty & \text{if } e \text{ is not finite.} \end{cases}$$

The use of the relative dimension in the classification of factors will be discussed in the next section. Now we make a detour and compare the construction of the dimension function with that of the Haar measure on a locally compact topological group. The existence of a measure on an abstract locally compact group which is invariant under the right translations was proven in 1932 by a Hungarian mathematician Alfréd Haar [2]. Von Neumann was in contact with Haar and knew his celebrated result before it was published. It is instructive to trace back the dimension function of a ring of operators to Haar's beautiful idea for the construction of the invariant measure.

Let G be a locally compact topological group and for a precompact $B \subset G$ and an open $U \subset G$ denote by $h(B;U)$ the number which gives at least how many right-translates of the set U are needed to cover the set B. This is an integer showing the size of B compared with U. $h(B;U)$ is translation invariant by construction. Of course, the smaller is U, the larger is $h(B;U)$. The latter one may increase to infinity when U runs on the neighbourhoods of a point. We need a normalization of $h(B;U)$. A compact set S of nonempty interior is chosen to normalize the measure. (S will be a set of unit measure.)

$$\lim_n \frac{h(B;R_n)}{h(S;R_n)} = \mu(B) \tag{5}$$

gives the measure of a compact set B if (R_n) is the filter of neighbourhoods of a point. The set function μ is translation invariant and additive on disjoint compact sets. After the measure μ of compacts are obtained, measure theoretic arguments are used to extend μ to a larger class of sets.

It is difficult to refrain from comparing Haar's idea with the contstruction of dimension function of projections in a von Neumann algebra: the similarity of the formulas (5) and (4) is striking. (5) yields the translation invariant size of subsets of a group G and (4) defines an invariant under partial isometries for projections in a von Neumann algebra. This example demonstrates how measure theoretic arguments can survive in the apparently different discipline of operator algebras. Von Neumann devoted two papers to the Haar measure. In [4], he gave another proof for the existence and uniqueness in the compact case and in [5] he obtained uniqueness in the general locally compact case. Von Neumann presented several courses on measure theory and invariant measures at the Institute of Advanced Studies. For him operator algebra theory was a noncommutative outgrowth of measure theory.

Now we continue the comparison of the relative dimension and the Haar measure. The objective of integration theory is to construct a linear functional, called integral, from a certain measure. Murray and von Neumann extended the relative dimension functional to arbitrary selfadjoint elements of the given von Neumann algebra. Let $A = A^* \in \mathcal{M}$ and let $\int \lambda \, dE(\lambda)$ be its spectral resolution with a projection-valued measure E on the real line. Then thanks to property (c) of the relative dimension, $D(E)$ is a common measure and

$$\mathrm{Tr}_{\mathcal{M}}(A) = \int \lambda \, dD(E)(\lambda) \tag{6}$$

determines a real number when the integral on the right-hand side exist. The inconveniency of definition (6) is in the fact that for noncommuting selfadjoint operators A and B one cannot say much about the spectral resolution of $A + B$ in terms of the spectral resolutions of A and B. Murray and von Neumann expected that

$$\mathrm{Tr}_{\mathcal{M}}(A + B) = \mathrm{Tr}_{\mathcal{M}}(A) + \mathrm{Tr}_{\mathcal{M}}(B)$$

but this was proven in [3] only for commuting A and B. The general case remained for the subsequent paper [6]. It was established here that the abstract trace functional $\mathrm{Tr}_{\mathcal{M}}$ is linear. $\mathrm{Tr}_{\mathcal{M}}$ yields an analogue of an integral. This analogy has developed into an operator algebraic integration theory, including L^p spaces, measurable operators and so on. Since a commutative von Neumann algebra admits representations by functions, Segal proposed the term "noncommutative integration" in [7]. The subject attracted interest decades later. There is no room here for the details. Noncommutative integration theory wears the label 46L50 in the Mathematics Subject Classification of the Mathematical Reviews. It has turned out that any function μ on the projections of a von Neumann algebra which possesses the additivity property

$$\mu\Big(\sum_i p_i\Big) = \sum_i \mu(p_i) \quad \text{if} \quad p_i p_j = 0 \quad \text{whenever} \quad i \neq j$$

extends to a linear functional of the von Neumann algebra. This was proved by Gleason in 1957 ([8]) when $\mathcal{M} = B(\mathcal{H})$ and in the early 80's by others in the general case. (See [9] for a survey.) Hence to any "noncommutative measure" μ, a "noncommutative integral" is associated in the setting of von Neumann algebras. Gleason theorem and its extension to arbitrary von Neumann algebras fit very well in von Neumann's view of quantum logic. We can say that a state of a von Neumann algebra is a probability measure of the corresponding quantum logic.

Classification of factors

Factors are the building blocks of von Neumann algebras hence the understanding of their structure has primary interest. According to the range of the dimension function of projections, a factor might be "trivial", "regular" or "singular". The trivial or type I is characterized by integer dimension, in the regular or type II case the dimension function has a continuous range and the singular or type III case is free of finite projections. To investigate the type I and type II cases Murray and von Neumann could utilize the dimension function; however, that tool was insufficient for the type III factors. To have a feeling about the "singularity" of type III factors, one can think of a measure space in which all nonempty sets have infinite measure. The full understanding of the type III case needed half a century and bore the Fields Medal for Alain Connes. Ergodic theory was the first source of factors. Classification of von Neumann algebras is strongly related to conjugacy classes of transformations of measure spaces. The Tomita-Takesaki theory provided the new tools and revolutionized operator algebras in the 1970's.

In [1] von Neumann established the structure of commutative von Neumann algebras: The selfadjoint part of a commutative von Neumann algebra consists of all bounded measurable functions of a certain selfadjoint operator. The classification of nonabelian algebras was carried out in [3]. Murray and von Neumann recognized that the center of the algebra plays an important role in the structure problem. The center of a von Neumann algebra $\mathcal{M}$ is a von Neumann algebra again and if it contains a projection z, then $\mathcal{M}$ becomes the direct sum of $z\mathcal{M}$ and $(I-z)\mathcal{M}$. Hence to decrease the complexity of an algebra, one may assume that its center does not contain a nontrivial projection. A von Neumann algebra is called factor if its center is trivial, that is, if it contains the multiples of the identity operator only. On a von Neumann factor, the dimension function is unique up to a scalar multiple. Murray and von Neumann proved that there are the following possibilities for the range of the dimension function of projections:

(I_n) $\{0, 1, \dots, n\}$, where n is a natural number.

(I_∞) $\{0, 1, \dots, n, \dots, \infty\}$.

(II_1) The interval $[0, 1]$.

(II_∞) The interval $[0, +\infty)$.

(III) The two-element set $\{0, +\infty\}$.

In this classification all von Neumann factors were found to belong to the classes type I, type II and type III. (However, it is worth mentioning that at the time of the discovery of the classification it was not known whether type III factors exist.)

Factors of type I are characterized by the existence of minimal projections. If a maximal pairwise orthogonal family of minimal projections has cardinality n, then the factor is isomorphic to $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space of dimension n. In particular, for every $s \in \mathbb{N} \cup \{+\infty\}$, there exists only one factor of type I_s. The existence of factors of type II and type III is not at all apparent, however. Murray and von Neumann constructed factors of type II_1 and type II_∞ by means of ergodic theory in [3]. Below we describe a method called “group measure space construction”. This construction yields factors of different type.

Let $(X, \mathcal{B}, \mu)$ be a measure space and G be a countable group of measure-preserving transformations. The group measure space construction yields a von Neumann algebra acting on the Hilbert space $L^2(\mu) \otimes l^2(G)$ which is regarded as a set of functions defined on G and with values in $L^2(\mu)$. For every $f \in L^\infty(\mu)$ define

$$\big((M_f\xi)(g)\big)(x) = f(g^{-1}x)\big(\xi(g)(x)\big) \qquad (\xi \in L^2(\mu) \otimes l^2(G)\,) \tag{7}$$

and for every $g \in G$ set

$$V_g(\xi)(g')(x)) = \xi(g^{-1}g')(x) \qquad (\xi \in L^2(\mu) \otimes l^2(G)\,). \tag{8}$$

Let $\mathcal{M}(\mu, G)$ be the von Neumann algebra generated by the operators

$$\{M_f : f \in L^\infty(\mu)\} \cup \{V_g : g \in G\}\,.$$

Then the choice of the unit cirle with the Lebesgue measure and (the powers of) an irrational rotation yields a factor of type II_1. The real line with the Lebesgue measure and the rational translations give a factor type II_∞. A factor of type III was constructed only in the third paper of the “Rings of Operators” series ([10]). Von Neumann modified the above measure theoretic procedure by allowing measurable transformations preserving measure 0, nowadays they are called nonsingular transformations. In this way he produced a factor of type III from the Lebesgue measure of the real line and the group of all rational linear transformations.

The group measure space construction was the root of the concept of crossed product of a von Neumann algebra and a group action. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let G be a countable group of automorphisms of $\mathcal{M}$. Similarly to (7) and (8) one can set two kinds of operators acting on the tensor product $\mathcal{H} \otimes l^2(G)$, which is realized as square integrable $\mathcal{H}$-valued function on G. For $A \in \mathcal{M}$,

$$\pi(A)\xi(g) = g^{-1}(A)\xi(g) \qquad (\xi \in \mathcal{H} \otimes l^2(G),\, g \in G) \tag{9}$$

and for $g \in G$

$$(V_g\xi)(h) = \xi(g^{-1}h) \qquad (\xi \in \mathcal{H} \otimes l^2(G),\, h \in G). \tag{10}$$

The crossed product $\mathcal{M} \times G$ is the von Neumann algebra generated by all operators $\pi(A)$ and V_g. In the case of the group measure space construction, the von Neumann algebra $\mathcal{M}$ is the abelian algebra of L^∞-functions acting by multiplication on L^2 and the automorphisms are induced by nonsingular transformations of the measure space. Although Murray and von Neumann used the group measure space construction for the production of factors, called today as Krieger factors, the difficult question of isomorphism of factors that arised from different actions was clarified only 40 years later ([11]). Krieger proved that two ergodic nonsingular transformations of a

Lebesgue space give rise to isomorphic factors if and only if the two transformations are orbit equivalent.

Von Neumann believed that among all factors the case II_1 has the strongest interest and expected that not all factors of type II_1 are isomorphic to each other. Von Neumann preferred the type II_1 case for two main reasons. One of these is the nice behavior of the unbounded operators affiliated with a type II_1 factor. It is well-known that addition and multiplication of such operators are particularly troublesome. The crux of the difficulty lies in the unrelatedness of the domain and range of such an operator with the domain of another one. Much of the difficulties evaporates, however, if one considers selfadjoint operators with spectral resolution in a factor of type II_1. The other reason why von Neumann attributed great importance to the continuous finite factors is that he interpreted this lattice as the proper logic of a quantum system. The lattice of projections of such a factor is modular, that is, in addition to the orthomodularity property (2), the stronger condition

$$p \vee (p' \wedge q) = (p \vee p') \wedge q \quad \text{for} \quad p \leq q \tag{11}$$

holds for every p' (and not only $p' = p^{\perp}$). (Non-modularity of the projection lattice of an infinite dimensional factor of type I was considered by von Neumann as a pathology of the usual Hilbert space quantum mechanics as a non commutative probability theory.)

The paper “Rings of Operators IV” [12] has two important achievements concerning the type II_1 factors. He proved that there exist nonisomorphic type II_1 factors, and that there is only one hyperfinite type II_1 factor. A von Neumann factor is called hyperfinite if it is generated by an increasing sequence of finite dimensional subalgebras. (Nowadays such algebras are preferred to be called approximately finite dimensional, or AFD for short.) The hyperfinite type II_1 factor $\mathcal{R}$ may be produced many different ways; for example, the above group measure space construction yields $\mathcal{R}$. The uniqueness of $\mathcal{R}$ reminds us of the uniqueness of a finite, atomless separable measure space. Factors of type II_1 did not play much role in the theory of von Neumann algebras until the recent years. After Jones founded his index theory ([13]), the study of subfactors of type II_1 factors has received much interest. Even a concise review of the index theory would require a lot of space (cf. [14]) but its flavour is given below.

Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and having commutant $\mathcal{N}'$. Assume that both $\mathcal{N}$ and $\mathcal{N}'$ are type II_1 factors and let $\mathrm{Tr}_{\mathcal{N}}$ and $\mathrm{Tr}_{\mathcal{N}'}$ be the canonical normalized traces. For any vector $\xi \in \mathcal{H}$ the projection $[\mathcal{N}\xi]$ onto the closure of $\mathcal{N}\xi$ belongs to $\mathcal{N}'$ and similarly $[\mathcal{N}'\xi] \in \mathcal{N}$. The quotient

$$\dim_{\mathcal{N}}(\mathcal{H}) \equiv \frac{\mathrm{Tr}_{\mathcal{N}'}\big([\mathcal{N}\xi]\big)}{\mathrm{Tr}_{\mathcal{N}}\big([\mathcal{N}'\xi]\big)} \tag{12}$$

is known to be independent of the vector ξ and is called the coupling constant since the work of Murray and von Neumann. In a certain sense the coupling constant is the dimension of the Hilbert space $\mathcal{H}$ with respect to the von Neumann algebra $\mathcal{N}$. (When $\mathcal{N} \equiv \mathbb{C}I$ then the coupling constant is the usual dimension of $\mathcal{H}$, hence the notation $\dim_{\mathcal{N}}(\mathcal{H})$.) V. Jones used the coupling constant to define a size of a subfactor of a finite factor. He was inspired by the notion of the index of a subgroup of a group, he therefore called this the relative size index.

Let $\mathcal{N}$ be a subfactor of a type II_1 von Neumann factor $\mathcal{M}$ possessing a unique canonical normalized trace $\mathrm{Tr}_{\mathcal{M}}$. The index is obtained as the quotient

$$[\mathcal{M}:\mathcal{N}]=\frac{\dim_{\mathcal{N}}(\mathcal{H})}{\dim_{\mathcal{M}}(\mathcal{H})}\,. \tag{13}$$

The number $[\mathcal{M}:\mathcal{N}]$ is not always an integer, and the possible values of the index form the following set.

$$\{t\in\mathbb{R}:t\geq 4\}\cup\{4\cos^2(\pi/p):p\in\mathbb{N},\,p\geq 3\} \tag{14}$$

This is the fundamental result of Jones which influenced a huge subsequent research and renewed the almost forgotten coupling constant. Vaughan F.R. Jones was awarded the Fields Medal in 1992 for discovering a surprising relationship between von Neumann algebras and geometric topology (see [15] for a review). The index theorem was the first step towards his discovery.

Construction of factors was the main activity in the field of operator algebras after the papers ”Rings of operators” for many years. It is out of the scope of this overview to summarize the constructions that were used to get more and more factors. Instead, we turn to the very end of the story. By the time the paper “Rings of Operators IV” was published (year 1943) it was known that each of the classes of type I_n, II_1, II_∞ contained a unique (up to algebraic isomorphism) hyperfinite von Neumann factor. However, the type III case remained unclear for many years, until the discovery of new invariants. Operator algebras achieved a revolutionary development in the late 60’s after a relative isolation of 30 years. The Tomita-Takesaki theory introduced a completely new machinery into the von Neumann algebra theory and provided new tools for type III factors and other basic problems. Next we devote some space to the fundamentals of the Tomita- Takesaki theory ([16]).

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Assume that $\varOmega\in\mathcal{H}$ is a so-called cyclic and separating vector, which means that the sets

$$\{A\varOmega:A\in\mathcal{M}\}\quad\text{and}\quad\{A'\varOmega:A'\in\mathcal{M}'\}$$

are dense in $\mathcal{H}$. So the formula

$$S_0:A\varOmega\mapsto A^*\varOmega\qquad(A\in\mathcal{M})$$

determines a densely defined (conjugate) linear operator which has a closure S. The polar decomposition $J\varDelta^{1/2}$ of S defines the antiunitary J and the positive selfadjoint operator $\varDelta$, called modular operator. One of the fundamental results of the Tomita-Takesaki theory is the fact that for every $t\in\mathbb{R}$ and $A\in\mathcal{M}$ the operator $\varDelta^{\mathrm{i}t}A\varDelta^{-\mathrm{i}t}$ is in $\mathcal{M}$. Hence

$$\sigma_t(A)=\varDelta^{\mathrm{i}t}A\varDelta^{-\mathrm{i}t} \tag{15}$$

defines a group of automorphisms of $\mathcal{M}$, the modular automorphisms associated with Ω. The modular group can be used to distinguish the type *III* from the other types because the following relation holds

$$\sigma_t(A) = U_t A U_t^* \qquad (A \in \mathcal{M})$$

for certain unitaries in the factor $\mathcal{M}$ if and only if $\mathcal{M}$ is of type *I* or type *II*. The fixed point algebra

$$\mathcal{M}^\sigma = \{A \in \mathcal{M} : \sigma_t(A) = A \text{ for every } t \in \mathbb{R}\}$$

is called the centralizer of Ω (or that of the vector state induced by Ω). The centralizer is a von Neumann subalgebra. For each projection e in the center of $\mathcal{M}^\sigma$, let Δ_e be the modular operator on the closure of $e\mathcal{M}e\Omega$ associated to the vector Ω and the von Neumann algebra $e\mathcal{M}e$. When $\mathcal{M}$ is a type *III* factor then the intersection

$$\Gamma = \cap_e \text{Spectrum}(\Delta_e)$$

is a closed subgroup of the multiplicative group of positive reals and is independent of the cyclic and separating vector Ω. This is a new invariant for type *III* factors and it is known as the Connes spectrum ([17]). There are the following possibilities:

$$\Gamma = \begin{cases} \{1\}, & \\ \{\lambda^n : n \in \mathbb{Z}\} & \text{for certain } 0 < \lambda < 1, \\ \{t \in \mathbb{R} : t > 0\}. & \end{cases}$$

Accordingly, Connes introduced the III_0, III_λ, III_1 types among the type *III* factors ($0 < \lambda < 1$).

Type *III* factors may be produced by means of infinite tensor product. Let $M_2(\mathbb{C})$ be the algebra of 2-by-2 matrices. Fixing $0 < \lambda < 1$ we can define a state φ on this algebra as follows.

$$\varphi(A) = \text{Tr}(AD), \quad \text{where} \quad D = \begin{pmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{\lambda+1} \end{pmatrix}$$

(The matrix D is called density matrix, inducing φ.) A representation of the inductive limit of the n-fold tensor product of copies of $M_2(\mathbb{C})$ can be constructed by means of tensor product states of copies of φ. (The so-called Gelfand-Naimark-Segal construction is involved here, but we do not want to give more details.) The generated von Neumann algebra is a hyperfinite factor. For $\lambda = 1$, the type II_1 factor shows up, for $\lambda = 0$ one obtains a type I_∞ factor and for $0 < \lambda < 1$ a type III_λ factor $\mathcal{R}_\lambda$ appears. In fact, $\mathcal{R}_\lambda$ is the only hyperfinite type III_λ factor. Confined to hyperfinite type III_λ factors with $0 < \lambda < 1$ the Connes spectrum is a complete invariant due to the results of Connes. He received the Fields Medal in 1983 for his work on von Neumann algebras including the classification of type *III* factors, approximately finite dimensional factors and automorphisms of the hyperfinite type II_1 factor ([17]). After the work of Connes, the uniqueness of the hyperfinte type III_1 factor remained undecided. This question was answered in the positive somewhat later by Haagerup [19]. (In case of type III_0, there are infinitely many nonisomorphic hyperfinite factors.)

Operator algebras motivated by physics

Quantum mechanics was one of the motivations to create a theory of operator algebras. It seems that post-Hilbert space quantum physics has developed along a path somewhat different from the one imagined by von Neumann, who, as we have indicated considered the type II_1 factor as the most promising structure; nevertheless the relation of von Neumann algebras to physics has always been important and fruitful for both the operator algebra theory and physics. In fact, von Neumann algebras are only one type among the several kinds of operator algebras used in mathematical physics. Jordan algebras and C*-algebras are almost as old as von Neumann algebras and equally important. The formalism of operator algebras has been utilized most deeply in quantum statistical mechanics and in quantum field theory in the last twenty years. Although type III factors seem to be pathological from the point of view of dimension function, they occur naturally in the algebraic quantum field theory.

In the Hilbert space formalism of quantum mechanics the bounded observables are represented by the selfadjoint part of the set $B(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$. $B(\mathcal{H})$ has a rich algebraic and topological structure which is utilized in the theory of von Neumann algebras. However, the usual (composition) product of selfadjoint operators A and B is not selfadjoint, in general, unless A and B commute. It was realized by Jordan, Wigner and von Neumann that the symmetric (or Jordan as it is now called) product defined by

$$A \bullet B = (AB + BA)/2 \tag{16}$$

is selfadjoint even if A and B are non-commuting selfadjoint operators ([20]). The Jordan product is commutative, but non-associative in general. It has the following properties:

(a) $A \bullet (B \bullet A^2) = (A \bullet B) \bullet A^2$,

(b) $\|A \bullet B\| \leq \|A\| \, \|B\|$.

Replacement of the associative product by the non-associative one $\bullet$ leads to the concept of Jordan algebras. Thus the (bounded) observables described in the Hilbert space formalism of quantum mechanics form a Jordan algebra (more precisely a JB algebra, [21]). The main idea of the so called "algebraic approach" to quantum mechanics is that in modelling the quantum system it is this Jordan algebra structure of the observables that is essential, therefore, this should be taken as a primitive concept. Furthermore, if the observables are wanted to satisfy the functional calculus of spectral theory, then they are assumed to form a JB algebra. This connection is thoroughly discusssed in the book of Emch [22]. We have to admit that at present time C*-algebras are more commonly used than JB algebras as a setting for observables. A C*-algebra is a norm closed *-subalgebra of $B(\mathcal{H})$ and in particular, von Neumann algebras are C*-algebras. Gelfand and Naimark succeeded in finding a simple abstract characterization of normed *-algebras which are isometrically and algebraically isomorphic to a norm closed *-subalgebra of $B(\mathcal{H})$ ([23]).

The selfadjoint part of a C*-algebra is a JB algebra with the product defined in complete analogy with (16). Thus, all C*-algebras, and in particular all von Neumann algebras define a Jordan algebra, and, in view of the highly non-$B(\mathcal{H})$-type-character of some von Neumann algebras, their Jordan algebra structure, too, is far from the usual structure of $B(\mathcal{H})$. JB algebras arising in this way are, nevertheless, special in that the Jordan product in them is determined by a multiplication in an associative algebra. JB algebras in which the Jordan product is not coming from a product of an associative algebra, are called exceptional. The role of exceptional JB algebras in physical applications is not clear. (The only finite dimensional exceptional JB algebra is the algebra of hermitian 3-by-3 matrices over the Cayley numbers, see [21].)

Historically, the study of Jordan algebras began with Jordan's 1933 papers, and the first result on classification was obtained soon after in the finite dimensional case by Jordan, von Neumann and Wigner in [20]. It was emphasized already in [20] that the assumption of finite dimensionality is a very restrictive one, and it was von Neumann who undertook an investigation of non-finite dimensional Jordan algebras in [24]. In the infinite dimensional case it is necessary to introduce topology into the algebra. In choosing the topology in an abstract Jordan algebra, von Neumann was motivated by his research on von Neumann algebras and he mimicked the weak operator topology. By the time of working on the Jordan algebraic generalization of quantum mechanics, his work with Murray on rings of operators containing the classification of factors had already appeared, and so von Neumann was led to the analysis of the set of idempotents in a Jordan algebra. Von Neumann proved that the set of idempotents is a complete lattice. Having established the existence of the analogue in the Jordan algebra of the projection lattice of a von Neumann algebra, von Neumann opened the way to a classification of Jordan algebras along the line of classification of von Neumann algebras. This classification was supposed to be the subject of Part II of the paper, however, the second part never appeared, and, as far as one can see from his published works, von Neumann never returned to the topic of weakly closed Jordan algebras. The systematic study of Jordan algebras from the point of view of functional analysis was resumed after von Neumann only in the mid-1960s. The monograph [21] gives all details on the structure theory of JB and JBW algebras and their relation to von Neumann algebras.

In quantum mechanics, the position operator Q and the momentum operator P obey the canonical commutation relation

$$QP - PQ = \mathrm{i}I \tag{17}$$

on a dense subset of the underlying Hilbert space. ((17) is also called the Heisenberg commutation relation.) Relation (17) can be viewed as the infinitesimal form of another commutation relation and, accordingly, it can be reformulated in terms of the one parameter groups U, V of unitaries determined by Q, P as infinitesimal genarators:

$$U(a)V(b) = e^{\mathrm{i}ab}V(b)U(a) \qquad (a, b \in \mathbb{R}). \tag{18}$$

To study the commutation relation in its form (18), the so-called Weyl-form, von Neumann introduced the two-parameter family of unitary operators

$$W(a,b) \equiv U(a)V(b)\exp\big(-\tfrac{1}{2}\mathrm{i}ab\big)$$

in terms of which (18) becomes

$$W(a,b)W(c,d) = W(a+c,b+d)\exp\big(\tfrac{1}{2}\mathrm{i}(ad-bc)\big)\,. \qquad (19)$$

A map $(a,b) \mapsto W(a,b)$ from the two dimensional space $\mathbb{R}^2$ into the set of bounded operators $B(\mathcal{H})$ that has the property (19) is called the representation of the canonical commutation relation (CCR). Von Neumann proved in [25] what has become known as von Neumann's theorem on the uniqueness of the representation of the CCR: If the irreducible representation W of CCR is continuous in the weak operator topology, then it is unique, and is isomorphic to the "Schrödinger" representation. Two assumptions are essential in von Neumann's uniqueness theorem: the continuity property of the map $(a,b) \mapsto W(a,b)$ and that $\mathbb{R}^2$ is finite dimensional as a linear space. If one replaces $\mathbb{R}^2$ by a possibly infinite dimensional linear space H with a symplectic bilinear form σ taking the place of $(a,b) \mapsto \frac{1}{2}(ad-bc)$ in (18), then the uniqueness theorem is no longer valid. This fact was realized only as late as in the 1950s.

One can also replace $B(\mathcal{H})$ by an arbitrary abstract C*-algebra $\mathcal{A}$, and call a map $W : H \to \mathcal{A}$ the representation of CCR, if it has the following two properties:

(i) $W(-f) = W(f)^*$,

(ii) $W(f)W(g) = W(f+g)\exp(\mathrm{i}\sigma(f,g))$,

where σ is a simplectic form on H. The C*-algebra CCR(H,σ) generated by $\{W(f) : f \in H\}$ is called the C*-algebra of the canonical commutation relations determined by H and σ. The CCR(H,σ) was shown to be unique (up to a *-isomorphism preserving the labelling of the unitaries W) by Slawny [26]; the existence of CCR(H,σ) can be shown by constructing $W(f)$ explicitly on the Hilbert space of complex-valued functions on H with countable support (that is, $l^2(H)$, cf. [27]). Similarly to the representation and the algebra of the CCR relations one can define the representation and algebra of the canonical anticommutation relation (CAR). Both the CAR and the CCR algebras are simple in the sense that their closed ideals are trivial. For a systematic description of the CCR and CAR algebras see [28], [27] and [29].

In the algebraic modeling of infinitely extended quantum spin systems one considers the infinite lattice $\mathbb{Z}^n$ and it is assumed that a copy of the same N-dimensional Hilbert space $\mathcal{H}$ is assigned to each site x of the lattice. For a finite set Λ of lattice points one forms $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$. The (selfadjoint part of the) algebra $\mathcal{A}(\Lambda) = B(\mathcal{H}_\Lambda)$ represents then the local observables confined to the region Λ. The inductive limit of the finite dimensional algebras $\mathcal{A}_\Lambda$ is called the "quasilocal algebra of the lattice gas". $\mathcal{A}$ represents the set of all (i.e. not necessarily strictly local) observables of the infinite system. It is in this framework in which the thermodynamic limit can be carried out in a mathematically rigorous way.

A typical thermodynamic limit process is the construction of the dynamic of the infinite system: One prescribes the dynamic of the strictly local system pertaining to a finite region Λ in the "Heisenberg picture", i.e. by setting

$$\alpha_t^{\Lambda}(A) = e^{\mathrm{i}tH(\Lambda)} A e^{-\mathrm{i}tH(\Lambda)} \qquad (A \in \mathcal{A}(\Lambda)),$$

where the selfadjoint $H(\Lambda) \in \mathcal{A}(\Lambda)$, the generator of the local dynamic, is the energy operator of the local system, it is determined by the interaction between the spins at the different sites. One then wants to show that the limit

$$\lim_{\Lambda \to L} e^{\mathrm{i}tH(\Lambda)} A e^{-\mathrm{i}tH(\Lambda)} = \alpha_t(A)$$

exists under suitable specification of $\Lambda \to L$ and in appropriate topology. The infinite system is represented then by a C*-dynamical system $(\mathcal{A}, \{\alpha_t : t \in \mathbb{R}\})$.

Given a C*-dynamical system as a model of the quantum statistical system in thermodynamic limit, one wants to single out the equilibrium states of the system, which is a precondition of any investigation on coexistent phases, phase transitions etc. Consider first the Gibbs equilibrium state in the ususal Hilbert space formalism: If H is the energy operator and $\exp(-\beta H)$ is a trace class operator (with $\beta > 0$ as the inverse temperature) then

$$\varphi_G(A) = \frac{\mathrm{Tr}(e^{-\beta H} A)}{\mathrm{Tr}(e^{-\beta H})} \tag{20}$$

is the Gibbs state. φ_G is stationary with respect to the dynamic $A \mapsto A_t$ given by the Hamiltonian H; however, it also has the following two, much stronger properties:

(1) The function $t \mapsto \varphi_G(AB_t)$ can be analytically extended to the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \beta\}$ of the complex plane.

(2) $\varphi_G(AB_{\mathrm{i}\beta}) = \varphi_G(BA)$.

The conditions 1 and 2 above are called the Kubo-Martin-Schwinger (KMS) conditions, a state φ of a C*-algebra having these two properties with respect to a dynamic $A_t \equiv \alpha_t(A)$ is called an (α, β)-KMS state. This condition was proposed in [30] to be taken as a definition of the equilibrium state of the infinite system at the inverse temperature β. The definition can be justified by proving, if not in full generality then for lattice gases, a number of properties of φ that are characteristic of the equilibrium. For instance an (α, β)-KMS state φ is α-invariant, it maximizes appropriately defined entropy, it has stability properties with respect to perturbations of the dynamic α etc. see [28] for a detailed analysis of the KMS states. Let us mention one property of a KMS state φ that establishes a link to the Tomita-Takesaki modular theory: A vector state induced by a cyclic and separating vector Ω satisfies the KMS condition with respect to the corresponding modular group of automorphisms (15). This link is a strong contact point between the Tomita-Takesaki theory and equilibrium quantum statistical mechanics.

The idea of the algebraic approach to relativistic quantum field theory, proposed by Haag and Kastler in 1964 ([31]), is that only local (in the sense of localization in the Minkowski spacetime M) observables, state preparations, measurements etc. do make physical sense, and so every physical information about the quantum field should be contained in the net of strictly local C*-algebras $\mathcal{A}(V)$), where V is a bounded, open region in the spacetime M. The postulates of relativity theory are formulated in terms of the net as follows.

(i) isotony: $\mathcal{A}(V_1) \subset \mathcal{A}(V_2)$ if $V_1 \subset V_2$

(ii) microcausality: $\mathcal{A}(V_1)$ commutes with $\mathcal{A}(V_2)$ if V_1 and V_2 are specelike separated.

(iii) relativistic covariance: there is a representation R of the Poincare group $\mathcal{P}$ by automorphisms on $\mathcal{A}$ such that $R(g)\mathcal{A}(V) = \mathcal{A}(gV)$ for every $g \in \mathcal{P}$ and every V.

It also is part of the axioms of algebraic relativistic quantum field theory that there exists at least one physical representation of the algebra $\mathcal{A}$, which means mathematically that one postulates the existence of a Poincare invariant state φ (vacuum) such that the spectrum condition (below), which expresses that the energy is positive, is fulfilled in the corresponding cyclic (GNS) representation. In this representation of the algebra, R is implemented by unitaries, and there are generators P_i, $i = 0, 1, 2, 3$, of the translation subgroup of the Poincare group $\mathcal{P}$ such that

(iv) spectrum condition: $P_0^2 \geq 0$, $P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0$

Algebraic quantum field theories given by local nets $(\mathcal{A}, \mathcal{A}(V))$ satisfying the postulates (i)–(iv) only are very general, and the net typically has some further properties, which are either consequences of how the net is constructed, or are required extra, motivated by physical considerations. Such an additional property is weak additivity:

(v) Let $\mathcal{A}(V)$ be a net of von Neumann algebras on a Hilbert space $\mathcal{H}$. We say that weak additivity holds for $\mathcal{A}(V)$, if for any (possibly unbounded) region O in M $\mathcal{A}(V) = \{\mathcal{A}(V) : V \subset O\}''$

Typical unbounded regions are the so-called wedge regions of spacetime. Condition (v) means that the spacetime is homogeneous, there does not exist "minimal distance". Another reason why (v) is important is that it is an assumption needed in the Reeh-Schlieder theorem: If the net $\mathcal{A}(V)$ satisfies (i)–(v), then the vacuum vector Ω is both cyclic and separating for any local algebra $\mathcal{A}(V)$ such that the causal complement of V is non-empty. Thus, in this case the vacuum state is faithful on (non)trivial local algebras, and, again, the modular theory applies. It follows that the vacuum state is a KMS state with respect to the modular dynamic, and there exists a temperature associated with the vacuum.

We mention finally the very important fact that the local algebras in a net of von Neumann algebras are "typically" type III algebras; for instance the local algebras $\mathcal{A}(\mathcal{W})$ pertaining to the wedge regions $\mathcal{W}$ are type III. The appearance of type III

algebras is a very characteristic difference between the local relativistic and non-local, non-relativistic quantum mechanics, and has important consequences, which can not be detailed here. We refer to the review [32] and to the recent monograph [33] of some of the facts related to the type of local algebras.

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