Solutions

1. Linear algebra

1. $x = 1$, $y = -1$, $z = -1$, $t = 1$.

2. $x_1 = 2i$, $x_2 = -2 - 2i$, $x_3 = 2 + i$.

3. $x_1 = -\frac{53}{6}x_4 + \frac{5}{2}x_5$, $x_2 = -\frac{17}{6}x_4 + \frac{3}{2}x_5$, $x_3 = \frac{7}{2}x_4 - \frac{7}{2}x_5$.

4. $a = 4$.

5. (a) -5
   (b) 40
   (c) $1 + ab + ad + cd + abcd$

6. $x = \dfrac{\begin{vmatrix} 9 & 2 & 5 \\ 6 & -7 & 3 \\ -5 & 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 5 \\ 8 & -7 & 3 \\ 2 & 5 & 1 \end{vmatrix}} = \frac{-265}{244}$,

   $y = \dfrac{\begin{vmatrix} 1 & 9 & 5 \\ 8 & 6 & 3 \\ 2 & -5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 5 \\ 8 & -7 & 3 \\ 2 & 5 & 1 \end{vmatrix}} = \frac{-257}{244}$,

   $z = \dfrac{\begin{vmatrix} 1 & 2 & 9 \\ 8 & -7 & 6 \\ 2 & 5 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 5 \\ 8 & -7 & 3 \\ 2 & 5 & 1 \end{vmatrix}} = \frac{595}{244}$.

7. Apply Gauss Jordan elimination to the matrix

$$[\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}] = \begin{bmatrix} 2 & 1 & -2 & 3 \\ -2 & 9 & -4 & 7 \\ -4 & 3 & 1 & -1 \end{bmatrix}$$

   to obtain

$$\begin{bmatrix} 1 & 0 & -\frac{7}{10} & 1 \\ 0 & 1 & -\frac{3}{5} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   . This shows that $\{\mathbf{a_1}, \mathbf{a_2}\}$ is a maximal linearly independent system, while $\mathbf{a_3} = -\frac{7}{10}\mathbf{a_1} - \frac{3}{5}\mathbf{a_2}$ and $\mathbf{a_4} = \mathbf{a_1} + \mathbf{a_2}$.

8. (a) Apply Gauss-Jordan elimination to

$$[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] = \begin{bmatrix} 1 & 1+i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2-i \\ -i & i & 1 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

, which has full rank, hence the vectors are linearly independent.

(b) Apply Gauss-Jordan elmination to

$$[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] = \begin{bmatrix} 1 & 1+i & 3+2i \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ -i & i & i \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

, which does *not* have full rank, hence the vectors are linearly dependent. In particular $\mathbf{v_3} = \mathbf{v_1} + 2\mathbf{v_2}$.

9. Apply Gauss-Jordan elimination to

$$[\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{b}] = \begin{bmatrix} i & 1+i & 1-i & 1 \\ -1 & 2i & -2i & i \\ -i & -2+2i & -2-2i & 1+i \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & -\frac{1}{4}-\frac{3}{4}i \\ 0 & 0 & 1 & -\frac{1}{4}-\frac{1}{4}i \end{bmatrix}$$

, which shows that $\mathbf{b} = (1-i)\mathbf{a_1} + (-\frac{1}{4}-\frac{3}{4}i)\mathbf{a_2} + (-\frac{1}{4}-\frac{1}{4}i)\mathbf{a_3}$.

10. (a) $[\mathbf{w}]_B = [\mathbf{v_1}\ \mathbf{v_2}]^{-1}[\mathbf{w}]_T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 9 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{17}{3} \\ \frac{10}{3} \end{bmatrix}$.

(b) $[\mathbf{u}]_T = [\mathbf{v_1}\ \mathbf{v_2}][\mathbf{w}]_B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 15 \end{bmatrix}$.

11.

$$[\mathbf{a}]_{B'} = [\mathbf{v_1}\ \mathbf{v_2}]^{-1}[\mathbf{u_1}\ \mathbf{u_2}][\mathbf{a}]_B = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 15 & 8 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} -1 & -8 \\ 2 & 15 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 369 \\ -833 \end{bmatrix}.$$

12. $A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so the matrix of A is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

13. The matrix of the transformation in the natural base:

$$[\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}] = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ -1 & -3 & 4 \end{bmatrix}.$$

14. $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, hence the matrix of T:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

15. Express the vectors $\mathbf{y_i}$ as linear combinations of the vectors $\mathbf{x_j}$. TO do so apply Gauss-Jordan elmination to the matrix $[\mathbf{x_1}\ \ \mathbf{x_2}\ \ \mathbf{x_3}\ \ \mathbf{y_1}\ \ \mathbf{y_2}\ \ \mathbf{y_3}]$. This results in

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

.

Now the matrix of A in the base $B = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ is:

$$[[\mathbf{y_1}]_B\ \ [\mathbf{y_2}]_B\ \ [\mathbf{y_3}]_B] = \begin{bmatrix} 2 & 0 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

16. (a)

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -77 & 19 \\ -358 & 88 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix}^{-1} \begin{bmatrix} -77 & 19 \\ -358 & 88 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix} = \begin{bmatrix} -3304 & -8963 \\ 1222 & 3315 \end{bmatrix}.$$

17. (a) Choose two non parallel vectors in S, for instance fixing $x = 1,\ y = 0$ we get $z = -5$; while fixing $x = 0,\ y = 1$ we get $z = 6$. Consider the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 6 \end{bmatrix}$. Now the projection onto the plane S has matrix:

$$M(M^T M)^{-1} M^T = \frac{1}{62} \begin{bmatrix} 37 & 30 & -5 \\ 30 & 26 & 6 \\ -5 & 6 & 61 \end{bmatrix}.$$

The image of the point $P = (8, 1, 3)$ is:

$$\frac{1}{62} \begin{bmatrix} 37 & 30 & -5 \\ 30 & 26 & 6 \\ -5 & 6 & 61 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{311}{62} \\ \frac{142}{31} \\ \frac{149}{62} \end{bmatrix}.$$

(b) For instance $M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 5 \end{bmatrix}$. Now the matrix of the projection onto the plane T is

$$M(M^T M)^{-1} M^T = \frac{1}{30} \begin{bmatrix} 29 & -5 & 2 \\ -5 & 5 & 10 \\ 2 & 10 & 26 \end{bmatrix}.$$

The image of the point $P = (8,1,3)$ is:

$$\frac{1}{30}\begin{bmatrix} 29 & -5 & 2 \\ -5 & 5 & 10 \\ 2 & 10 & 26 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{233}{30} \\ -\frac{1}{6} \\ \frac{52}{15} \end{bmatrix}.$$

18. With c, d arbitrary:

(a) $\lambda_1 = 3,\ \mathbf{v_1} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = -2,\ \mathbf{v_2} = c\begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

(b) $\lambda_1 = 2,\ \mathbf{v_1} = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = -1,\ \mathbf{v_2} = c\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$; $\lambda_3 = 1,\ \mathbf{v_3} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(c) $\lambda_1 = -1,\ \mathbf{v_1} = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = \lambda_3 = 1,\ \mathbf{v_2} = c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

19. The eigenvalues of the matrix A and the corresponding eigenvectors are:

$$\lambda_1 = 5 + 2\sqrt{2},\ \mathbf{v_1} = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix},\ \lambda_2 = 5 - 2\sqrt{2},\ \mathbf{v_2} = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}.$$

Hence $A = \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5+2\sqrt{2} & 0 \\ 0 & 5-2\sqrt{2} \end{bmatrix}\begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix}^{-1}$.

If the normed eigenvectors $\frac{\mathbf{v_1}}{|\mathbf{v_1}|} = \begin{bmatrix} \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}$, $\frac{\mathbf{v_2}}{|\mathbf{v_2}|} = \begin{bmatrix} \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$ are considered, then, using the fact that A is symmetric:

$$A = \begin{bmatrix} \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}\begin{bmatrix} 5+2\sqrt{2} & 0 \\ 0 & 5-2\sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}^T.$$

20. The eigenvalues of the matrix A and the corresponding eigenvectors are:

$$\lambda_1 = 12,\ \mathbf{v_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix},\ \lambda_2 = -8,\ \mathbf{v_2} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Hence $A = \lambda_1 \mathbf{v_1}\mathbf{v_1}^T + \lambda_2 \mathbf{v_2}\mathbf{v_2}^T = 12\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} - 8\begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$.

21. $\mathbf{b_1} = \mathbf{a_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b_2} = \mathbf{a_2} - \frac{\mathbf{a_2 b_1}}{\mathbf{b_1 b_1}}\mathbf{b_1} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{b_3} = \mathbf{a_3} - \frac{\mathbf{a_3 b_1}}{\mathbf{b_1 b_1}}\mathbf{b_1} - \frac{\mathbf{a_3 b_2}}{\mathbf{b_2 b_2}}\mathbf{b_2} = \begin{bmatrix} \frac{14}{3} \\ -\frac{2}{3} \\ -3 \\ \frac{1}{3} \end{bmatrix}$ this gives an orthogonal base for the spanned subspace. Then $\{\frac{\mathbf{b_1}}{|\mathbf{b_1}|}, \frac{\mathbf{b_2}}{|\mathbf{b_2}|}, \frac{\mathbf{b_3}}{|\mathbf{b_3}|}\}$, that is, $\left\{\begin{bmatrix} \frac{2}{15} \\ \frac{1}{15} \\ \frac{1}{5} \\ \frac{1}{15} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{2}{9} \\ -\frac{1}{18} \\ -\frac{1}{18} \end{bmatrix}, \begin{bmatrix} \frac{14}{\sqrt{282}} \\ -\frac{2}{\sqrt{282}} \\ -\frac{3\sqrt{3}}{\sqrt{94}} \\ \frac{1}{\sqrt{282}} \end{bmatrix}\right\}$ is an orthonormal base.

22. The coefficients of the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ satisfy the following system of linear equations:

$$\begin{aligned} a_0 + (-3)a_1 + (-3)^2 a_2 + (-3)^3 a_3 &= 2 \\ a_0 + (-1)a_1 + (-1)^2 a_2 + (-1)^3 a_3 &= 4 \\ a_0 + (1)a_1 + (1)^2 a_2 + (1)^3 a_3 &= 3 \\ a_0 + (2)a_1 + (2)^2 a_2 + (2)^3 a_3 &= 0 \end{aligned}$$

Apply Gauss-Jordan elimination to obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{83}{20} \\ 0 & 1 & 0 & 0 & -\frac{49}{120} \\ 0 & 0 & 1 & 0 & -\frac{13}{20} \\ 0 & 0 & 0 & 1 & \frac{11}{120} \end{bmatrix}$$

Hence $p(x) = \frac{83}{20} - \frac{49}{120}x - \frac{13}{20}x^2 + \frac{11}{120}x^3$.

23. (a) The coefficients of the line $y = a + bx$ satisfy

$$\begin{bmatrix} 4 & -1 \\ -1 & 15 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$$

hence $a = \frac{128}{59}$, $b = -\frac{19}{59}$.

(b) The coefficients of the line $y = a + bx$ satisfy

$$\begin{bmatrix} 5 & 8 \\ 8 & 34 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 44 \\ 125 \end{bmatrix},$$

hence $a = \frac{248}{53}$, $b = \frac{273}{106}$.

24. The system to solve is $A^T A x = A^T b$, thus

(a) $x_1 = \frac{149}{46}$, $x_2 = -\frac{2}{115}$. The error is $||Ax - b|| = \frac{9}{\sqrt{230}}$.

(b) $x_1 = -\frac{46}{85}$, $x_2 = \frac{3}{17}$. The error is $||Ax - b|| = 12\sqrt{\frac{14}{85}}$.

25. (a) The matrix $A^T A = \begin{bmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$ has eigenvalues and (unit length) eigenvectors $\lambda_1 = 9$, $\mathbf{v_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$, $\lambda_2 = 1$, $\mathbf{v_2} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$. Így $V = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, hence the diagonal matrix of singular values is $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. $\mathbf{u_1} = \frac{1}{3}A\mathbf{v_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{u_2} = \frac{1}{1}A\mathbf{v_2} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$, tehát $U = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. Ezekkel $A = UDV^T$.

(b)

$$B = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(As before, $\mathbf{u_1} = \frac{1}{3}C\mathbf{v_1}$, $\mathbf{u_2} = \frac{1}{1}C\mathbf{v_2}$, while $\mathbf{u_3}$ is obtained by augmenting $\{\mathbf{u_1}, \mathbf{u_2}\}$ to an orthonormal base. TO do so, solve the system $\mathbf{u_1}^T\mathbf{u_3} = 0$, $\mathbf{u_2}^T\mathbf{u_3} = 0$ for $\mathbf{u_3}$, that is, find the solutions of

$$\begin{aligned} \frac{\sqrt{2}}{\sqrt{3}}u_{13} + \frac{1}{\sqrt{6}}u_{23} + \frac{1}{\sqrt{6}}u_{33} &= 0 \\ -\frac{1}{\sqrt{2}}u_{23} + \frac{1}{\sqrt{2}}u_{33} &= 0 \end{aligned}$$

($u_{13} = -c,\ u_{23} = c,\ u_{33} = c$), and pick one that has unit length.)

26.

$$A = \begin{bmatrix} \frac{5}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{11}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{14}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

27.

$$A = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

$$e^{2A} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} e^6 & 0 \\ 0 & e^{-8} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4e^{-8} + e^6 & -2e^{-8} + 2e^6 \\ -2e^{-8} + 2e^6 & e^{-8} + 4e^6 \end{bmatrix}.$$

$$B = \begin{bmatrix} 4 & -4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{4} & 1 \end{bmatrix},$$

$$e^{-3B} = \begin{bmatrix} 4 & -4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-6} & 0 \\ 0 & e^9 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{4} & 1 \end{bmatrix} = \begin{bmatrix} -2e^{-6} + 3e^9 & 4e^{-6} - 4e^9 \\ -\frac{3}{2}e^{-6} + \frac{3}{2}e^9 & 3e^{-6} - 2e^9 \end{bmatrix}.$$

2. Partial differential equations

28. (a) $f(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x)$

(b) $g(x) = \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2-1)} \cos(2kx)$

(c) $h(x) = \frac{2\pi}{3} + \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{k+1}}{k^2} \cos(kx)$

29. $f(x) = 1 + \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^k}{2k-1} \cos(\frac{(2k-1)\pi}{4}x)$

30. $f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(kx)$

31. $f(x) = \sum_{k=1}^{\infty} b_k \sin(\frac{k\pi}{2}x)$, where $b_k = \frac{8(-1)^{k+1}}{k\pi} - \frac{16}{k^3\pi^3}(1 - (-1)^k)$.

32. (a) $u(x,t) = \sum_{k=1}^{\infty}(A_k \cos(kt) + B_k \sin(kt)) \sin(kx)$, where $A_k = \frac{1}{k}$ (the Fourier sine series coefficients of $\frac{\pi - x}{2}$, see problem 30.), $B_k = 0$ for any positive integer k.

(b) $u(x,t) = \sum_{k=1}^{\infty}(A_k \cos(\frac{k\sqrt{2}\pi}{5}t) + B_k \sin(\frac{k\sqrt{2}\pi}{5}t)) \sin(\frac{k\pi}{5}x)$, where

$$\begin{aligned} A_k &= \frac{2}{5}\int_0^5 3x \sin\left(\frac{k\pi}{5}x\right) \mathrm{d}x = \frac{2}{5}\int_0^\pi \frac{15y}{\pi} \sin(ky)\frac{5}{\pi} \mathrm{d}y = \frac{15}{\pi}\frac{2}{\pi}\int_0^\pi y \sin(ky) \mathrm{d}y \\ &= \frac{15}{\pi}(-1)^{n+1}\frac{2}{k} = (-1)^{n+1}\frac{30}{k\pi} \end{aligned}$$

(integration by substitution reduces the problem to the Fourier coefficients of the function x, which was computed in class), while

$$\cos\left(\frac{3\pi}{5}x\right)\sin(\pi x) = \frac{1}{2}\sin\left(\frac{8\pi}{5}x\right) + \frac{1}{2}\sin\left(\frac{2\pi}{5}x\right)$$

so $B_2 = \frac{5}{2\sqrt{2}\pi}\frac{1}{2}$, $B_8 = \frac{5}{8\sqrt{2}\pi}\frac{1}{2}$, and for $k \notin \{2,8\}$ $B_k = 0$.

(c) $u(x,t) = \sum_{k=1}^{\infty}(A_k\cos(k\pi t) + B_k\sin(k\pi t))\sin(\frac{k\pi}{2}x)$, where $A_3 = 1$, $k \neq$ 1-re $A_k = 0$, and $B_k = \frac{2}{2k\pi}\left(\frac{8(-1)^{k+1}}{k\pi} - \frac{16}{k^3\pi^3}(1-(-1)^k)\right) = \frac{8(-1)^{k+1}}{k^2\pi^2} - \frac{16}{k^4\pi^4}(1-(-1)^k)$, see problem 31. above.

33. Initially the shape of the string is given by

$$f(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \le x \le 2, \\ 3, & \text{if } 2 \le x \le 3, \\ -x+6, & \text{if } 3 \le x \le 6. \end{cases}$$

Extend this as an odd function, and then periodically (by period 12), a function to be denoted by $f(x)$. By d'Alambert formula $u(x,t) = \frac{1}{2}(f(x+2t) + f(x-2t))$, so at $t = 1,5$:

$$u(x,1,5) = \begin{cases} -\frac{1}{2}x, & \text{if } 0 \le x \le 1, \\ \frac{1}{4}x - \frac{3}{4}, & \text{if } 1 \le x \le 5, \\ -\frac{1}{2}x + 3, & \text{if } 5 \le x \le 6. \end{cases}$$

34. Extend g as an odd function, and then periodically (with period 4). Then by D'Alembert formula: $u\left(x, \frac{1}{4}\right) = \frac{1}{2\cdot 4}\int_{x-1}^{x+1} g(u)\,\mathrm{d}u$. For the relevant segment:

$$g(x) = \begin{cases} 3x+2, & \text{if } -1 < x < 0, \\ 3x-2, & \text{if } 0 < x < 2, \\ 3x-10, & \text{if } 2 < x < 3. \end{cases}$$

Hence

$$u\left(x, \frac{1}{4}\right) = \begin{cases} \frac{1}{8}\left[\int_{x-1}^{0}(3u+2)\,\mathrm{d}u + \int_{0}^{x+1}(3u-2)\,\mathrm{d}u\right] = \frac{1}{4}x, & \text{if } 0 \le x \le 1, \\ \frac{1}{8}\left[\int_{x-1}^{2}(3u-2)\,\mathrm{d}u + \int_{2}^{x+1}(3u-10)\,\mathrm{d}u\right] = \frac{1}{2} - \frac{1}{4}x, & \text{if } 1 \le x \le 2. \end{cases}$$

35. Initially the shape of the string is given by

$$f(x) = \begin{cases} 0, & \text{if } x \le -5 \text{ or } x \ge 2, \\ x+5, & \text{if -5} \le x \le -3, \\ 2, & \text{if } -3 \le x \le 1, \\ -2x+4, & \text{if } 1 \le x \le 2. \end{cases}$$

According to d'Alembert formula $u(x,t) = \frac{1}{2}(f(x+t) + f(x-t))$, so at $t = 5$:

$$u(x,5) = \begin{cases} 0, & \text{if } x \le -10 \text{ or } -3 \le x \le 0 \text{ or } x \ge 7, \\ \frac{1}{2}x + 5, & \text{if } -10 \le x \le -8, \\ 1, & \text{if } -8 \le x \le -4 \text{ or } 2 \le x \le 6, \\ -x-3, & \text{if } -4 \le x \le -3, \\ \frac{1}{2}x, & \text{if } 0 \le x \le 2, \\ -x+7, & \text{if } 6 \le x \le 7. \end{cases}$$

36. By d'Alembert formula:

$$u(x,t) = \frac{1}{2}\left[\cos(3(x+2t)) + \cos(3(x-2t))\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \frac{6x}{x^2+1}\,\mathrm{d}t$$
$$= \cos(3x)\cos(6t) + \frac{3}{4}\left[\ln((x+2t)^2+1) + \ln((x-2t)^2+1)\right].$$

37. (a) $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{2}\right)^2 2t} \sin\left(\frac{n\pi}{2}x\right)$, where the A_n are the Fourier sine series coefficients of $\sin(4\pi x)\cos\left(\frac{5\pi}{2}x\right)$ considered on the interval $[0,2]$.

$$\sin(4\pi x)\cos\left(\frac{5\pi}{2}x\right) = \frac{1}{2}\left[\sin\left(\frac{13\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right)\right].$$

Hence $A_3 = A_{13} = \frac{1}{2}$, and $A_n = 0$, if $n \notin \{3, 13\}$. Thus

$$u(x,t) = \frac{1}{2}e^{-\left(\frac{3\pi}{2}\right)^2 2t}\sin\left(\frac{3\pi}{2}x\right) + \frac{1}{2}e^{-\left(\frac{13\pi}{2}\right)^2 2t}\sin\left(\frac{13\pi}{2}x\right).$$

(b) $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{3}\right)^2 \sqrt{3}t} \sin\left(\frac{n\pi}{3}x\right)$, where the A_n are the Fourier sine series coefficients of $x(3-x)$ considered on $[0,3]$, that is

$$A_n = \frac{2}{3}\int_0^3 x(3-x)\sin\left(\frac{n\pi}{3}x\right)\,\mathrm{d}x = \frac{2}{3}\frac{3}{\pi}\int_0^{\pi} \frac{3y}{\pi}\left(3 - \frac{3y}{\pi}\right)\sin(ny)\,\mathrm{d}y = \frac{2}{\pi}\frac{9}{\pi^2}\int_0^{\pi} y(\pi-y)\sin(ny)\,\mathrm{d}y = \frac{9}{\pi^2}B_n,$$

where the B_n are the Fourier sine series coefficients of $y(\pi - y)$ considered on $[0,\pi]$, which are, as discussed in class

$$B_n = \begin{cases} \frac{8}{n^3\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Hence

$$A_n = \begin{cases} \frac{72}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

3. Vector calculus

38. (a) $-\frac{17}{105}$
    (b) -4π
    (c) 0
    (d) $\frac{10}{3}$
    (e) $0,5$
    (f) -2π

39. (a) $\frac{64}{3}\pi$
    (b) -30
    (c) $9\pi^2$

40. (a) $\operatorname{div}\mathbf{F}(x,y,z) = 2y$, $\operatorname{rot}\mathbf{F}(x,y,z) = (z-x, 0, z-x)$
    (b) $\operatorname{div}\mathbf{F}(x,y,z) = x + \frac{1}{y} + \frac{1}{z}$, $\operatorname{rot}\mathbf{F}(x,y,z) = (-\frac{y}{z^2}, -z, -\frac{x}{y^2})$
    (c) $\operatorname{div}\mathbf{F}(x,y,z) = 2xyz[\cos(x^2yz) + \cos(xy^2z) + \cos(xyz^2)]$,
    $\operatorname{rot}\mathbf{F}(x,y,z) = (xz^2\cos(xyz^2) - xy^2\cos(xy^2z), x^2y\cos(x^2yz) - yz^2\cos(xyz^2), y^2z\cos(xy^2z) - x^2z\cos(x^2yz))$

(d) $\operatorname{div}\mathbf{F}(\mathbf{r}) = \frac{\mathbf{ar}}{|\mathbf{r}|}$, $\operatorname{rot}\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}\times\mathbf{a}}{|\mathbf{r}|}$

41. (a) As the vector field is undefined on the plane $x+y+z=0$, the curl test is inconclusive.

    (b) $\operatorname{rot}\mathbf{F}(x,y,z) = (0,0,0)$ in every point of the three dimensional space, hence there exists some potential, in particular: $u(x,y,z) = \sin(x+y) + \cos(x-y) + C$.

    (c) $\operatorname{rot}\mathbf{F}(x,y,z) = (0,0,0)$ in every point of the three dimensional space, hence there exists some potential, in particular: $u(x,y,z) = x^2y + yz^2 - xz^2 - xyz + C$.

42. The potential is $u(x,y,z) = xy + xz + yz$, the two endpoints of the curve are $(0,0,0)$ and $(\frac{a}{2}, a, \frac{a}{2})$, thus the value of the line integral (irrespective of the path) is $\frac{5}{4}a^2$.

43. (a) $\frac{12}{5}a^5\pi$

    (b) $\operatorname{div}\mathbf{F}(x,y,z) = x+y+z$, which when integrated on the interior of the domain gives $\frac{189}{2}$. The surface integral of $\mathbf{F}$ on the face lying inside the plane $x=-2$ (with outward pointing normal) is 49. Hence the surface integral of $\mathbf{F}$ on the remaining faces is $\frac{189}{2} - 49 = \frac{91}{2}$ by Gauss theorem.

44. $\operatorname{rot}\mathbf{F}(x,y,z) = (xz\cos(xyz) + ye^{yz}, -yz\cos(xyz) + e^x\cos z, 0)$. According to Stokes theorem the line integral along the circle can be obtained as the surface integral of $\operatorname{rot}\mathbf{F}$ on the circular disc. As the disc lies within the plane xy, and the z component of $\operatorname{rot}\mathbf{F}$ is everywhere 0, the surface integral is 0.

45. To be integrated is the function $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y$ on the semi-circular disc, which gives $\frac{2}{3}$.

46. As we have seen in class, by Green theorem the area can be obtained as the line integral of $\mathbf{F}(x,y) = (0,x)$ along the boundary curve (oriented counterclockwise). On the segment connecting $(0,0)$ to $(2\pi,0)$ the integral of $(0,x)$ is 0. Along the cycloid arc the line integral is (note the parameter runs from 2π to 0 this time):

$$-\int_0^{2\pi}(0, R(t-\sin t))\cdot(R(1-\cos t), R\sin t)\,\mathrm{d}t = -\int_0^{2\pi} R^2\sin t(t-\sin t)\,\mathrm{d}t =$$
$$-R^2\left(\int_0^{2\pi} t\sin t\,\mathrm{d}t - \int_0^{2\pi}\sin^2 t\right) = -R^2\left([-t\cos t]_0^{2\pi} + \int_0^{2\pi}\cos t\,\mathrm{d}t - \int_0^{2\pi}\frac{1-\cos 2t}{2}\,\mathrm{d}t\right) =$$
$$-R^2(-2\pi-\pi) = 3R^2\pi.$$

Hence the enclosed area is $3R^2\pi$.