12. Let $B = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$. Show that there exists a subspace V such that B is the matrix of the orthogonal projection to the subspace V in the natural basis! What is V?

B SYMMETRIC ✓, $B \cdot B = B$ ✓

B IS A PROJECTION THAT PROJECTS ONTO COL(B)

$$\mathrm{col}(B) = \mathcal{L}\left(\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\4\\4\end{pmatrix}, \begin{pmatrix}2\\4\\4\end{pmatrix}\right) = \mathcal{L}\left(\begin{pmatrix}1\\2\\2\end{pmatrix}\right)$$

B PROJECTS ONTO THE LINE GOING THROUGH THE ORIGIN AND HAVING NORMAL VECTOR $\underline{v}$

14. Let $C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix}$. Moreover, let P_r be the matrix of the orthogonal projection from $\mathbb{R}^4$ to row(A) and let P_c be the matrix of the orthogonal projection from $\mathbb{R}^3$ to col(A)! Find P_r and P_c!

$$\begin{pmatrix} \boxed{1} & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{pmatrix} \sim \begin{array}{cccc} 1 & 1 & & \\ \hline \boxed{1} & 1 & -1 & -1 \\ 0 & \boxed{-1} & 2 & 4 \\ \cancel{0} & \cancel{-1} & \cancel{2} & \cancel{4} \end{array} \qquad \mathrm{col}(C) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right)$$

$$\mathrm{row}(C) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix} \right) \overset{\dim = 2}{=} \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right)$$

$$M_{COL} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}, \quad P_{COL} = M_{COL}\left(M_{COL}^{T} M_{COL}\right)^{-1} \cdot M_{COL}^{T}$$

$$P_{COL} \overset{HW}{=} \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

$$M_{row} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \\ -1 & 3 \end{pmatrix}, \quad P_{row} = \underbrace{M_{row} \overbrace{\left(\underbrace{M_{row}^T M_{row}}\right)^{-1}}^{\frac{1}{35}\begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix}} M_{row}^T}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -3 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 14 & 7 \\ 11 & 3 \\ -8 & 1 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} 21 & 14 & -7 & 7 \\ 14 & 11 & -8 & -2 \\ -7 & -8 & 9 & 11 \\ 7 & -2 & 11 & 29 \end{pmatrix} \text{APPLE}$$

SOLUTION:

$$\frac{\text{APPLE}}{35}$$

EXTRA EXERCISE

$$x^2 + 11y^2 + 10\sqrt{3}\, x\cdot y - 16 = 0$$

SOLUTION:

$$\underbrace{\begin{pmatrix} 1 & 5\sqrt{3} \\ 5\sqrt{3} & 11 \end{pmatrix}}^{A}, \quad 0 = \begin{vmatrix} 1-\lambda & 5\sqrt{3} \\ 5\sqrt{3} & 11-\lambda \end{vmatrix} = \underbrace{(1-\lambda)(11-\lambda) - 75}^{\lambda^2 - 12\lambda - 64 = 0}$$

$$\lambda_{1,2} = \frac{12 \pm \sqrt{144 + 256}}{2} \to 16, \; -4$$

$$\lambda_1 = 16 \quad \left(\begin{array}{cc|c} \boxed{-15} & 5\sqrt{3} & 0 \\ \cancel{5\sqrt{3}} & \cancel{-5} & \cancel{0} \end{array}\right) \quad v_2 = t$$

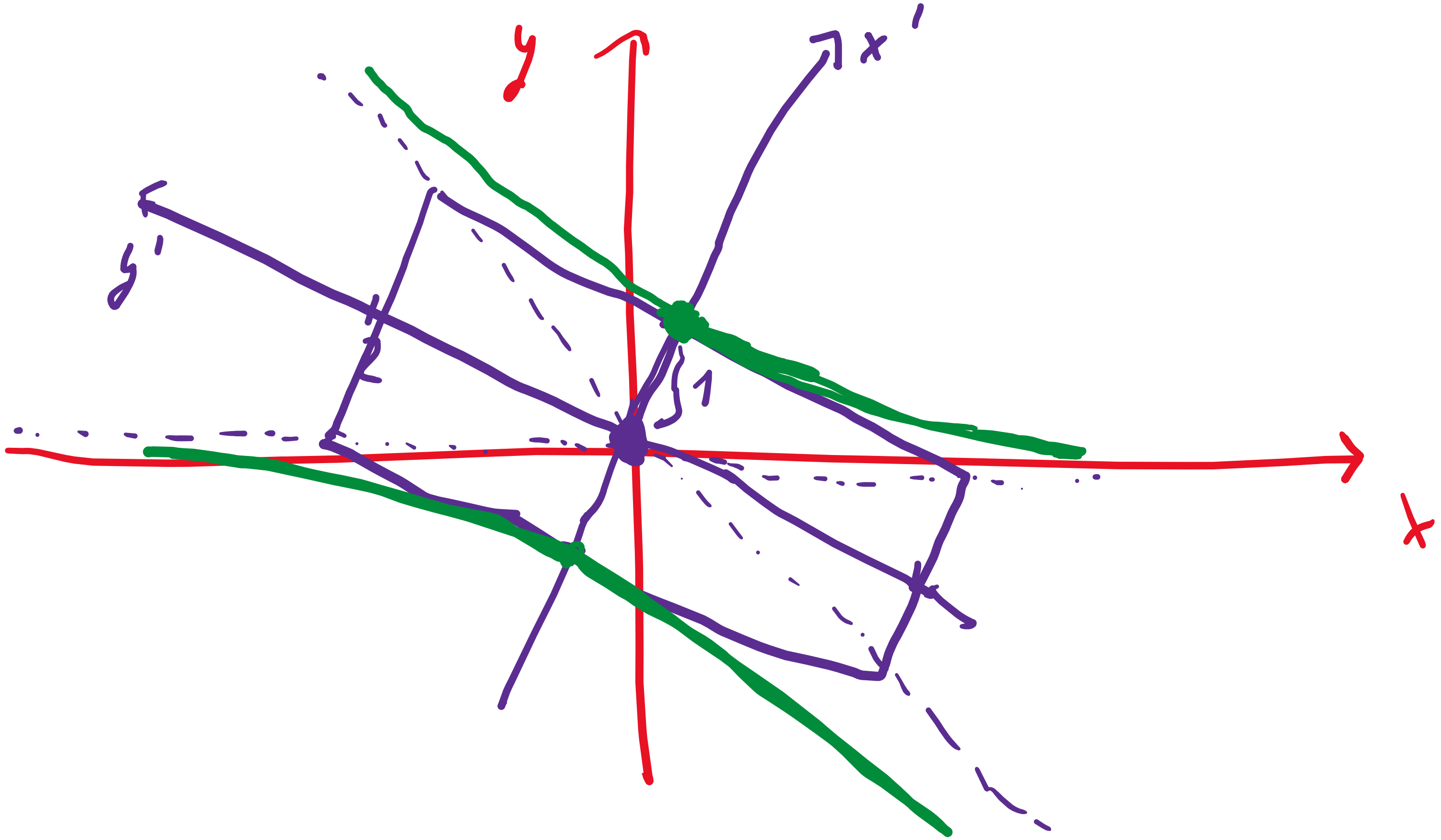
$$-15v_1 + 5\sqrt{3}\, v_2 = 0 \Rightarrow v_1 = \frac{\sqrt{3}}{3} t$$

$$\left\{ \begin{pmatrix} \frac{\sqrt{3}}{3} t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}, \quad \begin{pmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{pmatrix} \cdot \frac{3}{2\sqrt{3}} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \lambda_2 = -4, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$16(x')^2 - 4\cdot(y')^2 = 16$$

$$\boxed{\frac{(x')^2}{1^2} - \frac{(y')^2}{2^2} = 1}$$

$$, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$



16. Find the solution with the method of smallest squares of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

THEORETICAL BACKGROUND

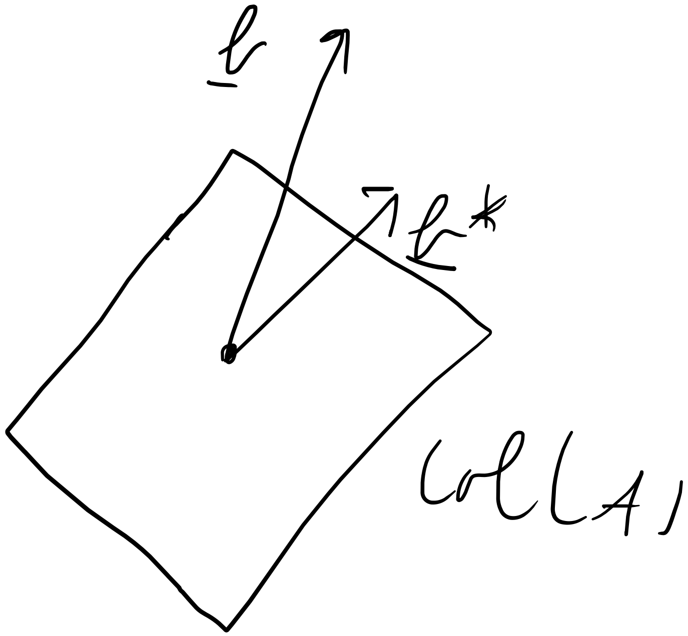
$$A = \left(\left(\underline{a_1}\right), \left(\underline{a_2}\right), \dots, \left(\underline{a_k}\right) \right)$$

$$A\underline{x} = \underline{b}, \quad \overset{?}{x_1} \cdot \underline{a_1} + \overset{?}{x_2}\, \underline{a_2} + \dots + \overset{?}{x_k} \cdot \underline{a_k} = \underline{b}$$

THERE EXISTS SOLUTION $\iff$ $\underline{b} \in \mathrm{col}(A)$

16. Find the solution with the method of smallest squares of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$



$\boxed{\text{METHOD OF SMALLEST SQUARES}}$

WE SOLVE $A\underline{x} = \underline{b}^*$, WHERE $\underline{b}^*$ IS THE PROJECTION OF $\underline{b}$ ONTO COL(A). USING THE THEORY, THE IDEA LEADS TO $\boxed{A^T A \underline{x} = A^T \underline{b}}$

NORMAL EQUATION

16. Find the solution with the method of smallest squares of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

$$\boxed{A^T A \underline{x} = A^T \underline{b}}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 1 & 2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 24 & 8 \\ 8 & 6 \end{pmatrix}}_{A^T A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \boxed{8} & 6 & 8 \\ 24 & 8 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 8 & 6 & 8 \\ 0 & -10 & -12 \end{array}\right)$$

$$\to 8x + 6\overset{6/5}{y} = 8, \quad \underbrace{x = \frac{1}{10}}$$

$$\to -10y = -12, \quad \underbrace{y = \frac{6}{5}}$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{10} \\ \frac{12}{10} \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{14}{5} \\ 1 \end{pmatrix},$$

$$y = \overset{?}{a} + \overset{?}{b} x \qquad A^T A \underline{x} = A^T \underline{b}$$

x	y
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

$$\boxed{\begin{matrix} a + b x_1 \\ a + b x_2 \\ \vdots \\ a + b x_n \end{matrix}} \begin{matrix} \approx \\ \approx \\ \\ \approx \end{matrix} \boxed{\begin{matrix} y_1 \\ y_2 \\ \\ y_n \end{matrix}}$$

$$\underbrace{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\underline{x}} \approx \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\underline{b}}$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \quad \underbrace{\begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}}_{A^T A} \qquad \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

15. With the method of smallest squares, find the line, which approximates the points $(2,1)$, $(3,2)$, $(5,3)$ and $(6,4)$ the best!

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$\boxed{y = -\frac{3}{10} + \frac{7}{10} \cdot x}$$

$$\begin{pmatrix} 4 & 16 \\ 16 & 74 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 47 \end{pmatrix}, \quad \left(\begin{array}{cc|c} 4 & 16 & 10 \\ 16 & 74 & 47 \end{array}\right) \sim$$

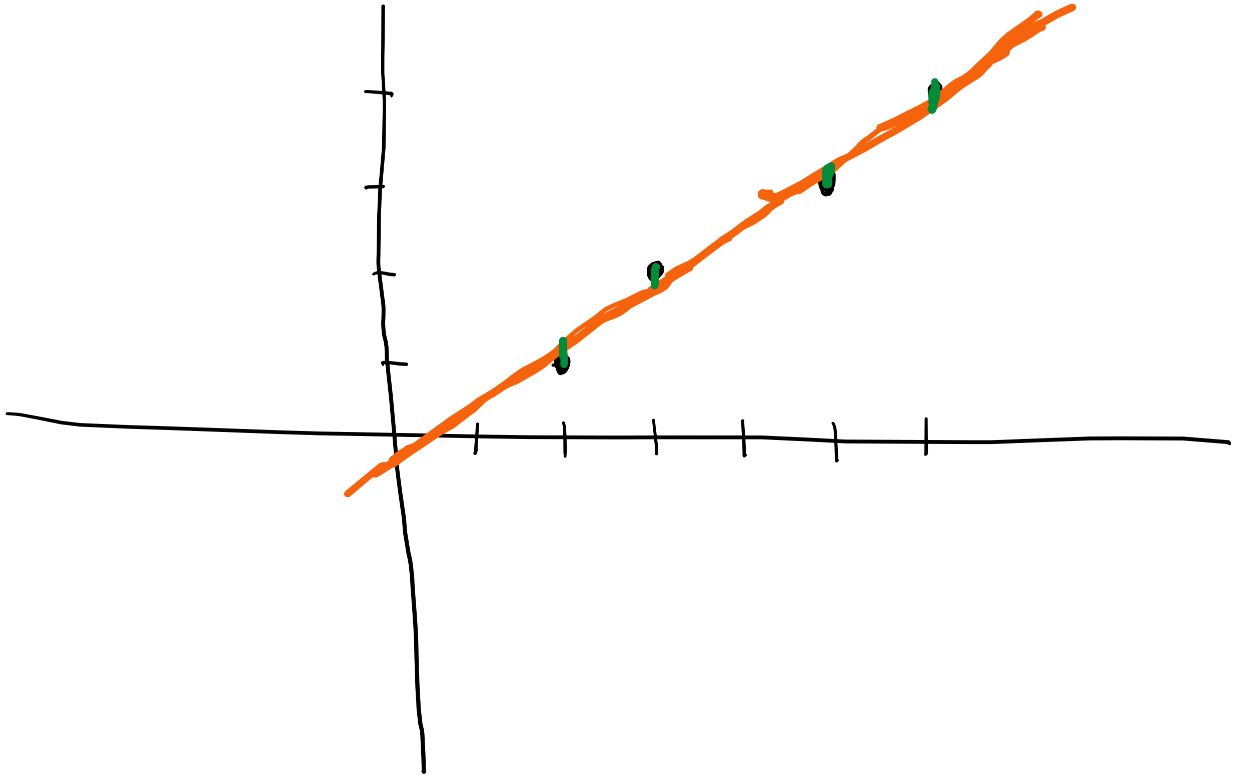
$$\sim \left(\begin{array}{cc|c} 4 & 16 & 10 \\ 0 & 10 & 7 \end{array}\right)$$

$$\to 4a + 16 \cdot b = 10, \quad 4a = -\frac{6}{5}$$

$$\to 10b = 7 \Rightarrow b = \frac{7}{10} \qquad \Downarrow \quad a = -\frac{3}{10}$$

15. With the method of smallest squares, find the line, which approximates the points $(2,1),(3,2),(5,3)$ and $(6,4)$ the best!

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$



17. Let $B = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}$. Show that B is positive definite, and find a matrix C such that $C^2 = B$.

In case of a symmtetric matrix, positive definitness is equivalent to the nonnegativity of the eigenvalues

EIGENVALUES

$$\boxed{0 = \begin{vmatrix} 9-\lambda & 6 \\ 6 & 9-\lambda \end{vmatrix} \Rightarrow \lambda_1 = 15,\ \lambda_2 = 3}$$

$\lambda_1 = 15 \geq 0$, $\lambda_2 = 3 \geq 0$

POS DEF

$\lambda_1 = 15$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -6 & 6 & 0 \\ \cancel{6} & \cancel{-6} & \cancel{0} \end{array}\right) \quad v_1 = t,\ v_2 = t$$

$$\left\{ \begin{pmatrix} t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$$

$\lambda_2 = 3$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 9 & 6 \\ 6 & 9 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 15 & 0 \\ 0 & 3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q^T}$$

$$Q^T \cdot C \cdot Q = \sqrt{D}$$

$$\underbrace{C = Q \begin{pmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \cdot Q^T}$$

$$C \cdot C = Q(\sqrt{D}) \underbrace{Q^T \cdot Q}_{I} \cdot \sqrt{D} \cdot Q^T = Q \cdot D \cdot Q^T = B \checkmark$$

IT IS GOOD CHOICE FOR C

$$Q \cdot \sqrt{D} \cdot Q^{-1} \overset{HW}{=} \begin{pmatrix} \frac{\sqrt{15}+\sqrt{3}}{2} & \frac{\sqrt{15}-\sqrt{3}}{2} \\ \frac{\sqrt{15}-\sqrt{3}}{2} & \frac{\sqrt{15}+\sqrt{3}}{2} \end{pmatrix} = C$$

EXTRA EXERCISE : LET $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

FIND THE SINGULAR VALUE DECOMPOSITION OF A!

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}}_{A^T} \qquad \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \sim A^T A$$

$$0 = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 \qquad \lambda_{1,2} = \frac{10 \pm \sqrt{100 - \cancel{64}}}{2}, \; \frac{8}{2}$$

$$\lambda_1 = 8 \qquad \left(\begin{array}{cc|c} -3 & 3 & 0 \\ \cancel{3} & \cancel{-3} & \cancel{0} \end{array}\right) \qquad v_2 = t \qquad v_1 = t \qquad \lambda_2 = 2$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \underline{v_1}, \qquad \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \underline{v_2}$$

$$\alpha_1 = \sqrt{8}, \quad \alpha_2 = \sqrt{2}, \quad \underline{u_1} = \frac{1}{\sqrt{8}} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{u_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A = U \cdot D \cdot V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T =$$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}}$$