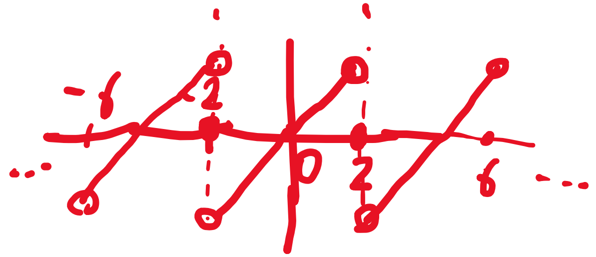
LET $f(x) = \begin{cases} x & \text{if } 0 \le x < 2 \\ 0 & \text{if } x = 2 \end{cases}$



$L=2$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\cdot x\right),$$

ADDITIONAL EXERCISE

GIVE THE FOURIER-SINE SERIES ON $[0,2]$!

$$b_n = \frac{2}{L}\cdot\int_{t=0}^{L} f(t)\sin\left(\frac{n\pi}{L}\cdot t\right)dt.$$

$$b_n = \frac{2}{2}\cdot\int_0^2 x\cdot\sin\left(\frac{n\cdot\pi}{2}\cdot x\right)dx = 2\left(-\cos(n\cdot\pi)\right)\cdot\frac{2}{\pi\cdot n} =$$

$$= (-1)^{n+1}\cdot\frac{4}{\pi\cdot n},$$

$$s(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{\pi\cdot n}\cdot\sin\left(\frac{n\cdot\pi}{2}x\right) = \frac{4}{\pi}\cdot\sin\left(\frac{\pi}{2}x\right) - \frac{2}{\pi}\cdot\sin\left(2\cdot\frac{\pi}{2}x\right) + \frac{4}{\pi\cdot 3}\cdot\sin\left(3\cdot\frac{\pi}{2}\cdot x\right) - \dots$$

$$\int x\cdot\sin\left(n\cdot\frac{\pi}{2}\cdot x\right)dx = x\cdot\left[-\cos\left(n\cdot\frac{\pi}{2}\cdot x\right)\right]\frac{2}{\pi\cdot n} + \underbrace{\int 1\cdot\cos\left(n\cdot\frac{\pi}{2}\cdot x\right)\frac{2}{\pi\cdot n}}_{\sin\left(n\cdot\frac{\pi}{2}\cdot x\right)\cdot\frac{4}{\pi^2\cdot n^2}} +$$

CONSIDER $f(x) = \cos\left(\frac{3\pi}{5}x\right)\cdot \sin(\pi x)$ WITH DOMAIN $[0,5]$.
DETERMINE ITS FOURIER-SINE SERIES! ADDITIONAL EXERCISE

$$f(x) = \frac{1}{2}\sin\left(\frac{2\pi}{5}x\right) + \frac{1}{2}\cdot\sin\left(\frac{8\pi}{5}\cdot x\right)$$

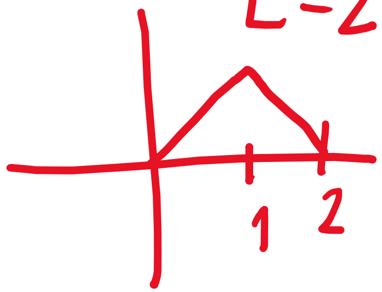
$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$\boxed{b_2 = b_8 = \frac{1}{2}}$, $b_i = 0$ OTHERWISE

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\cdot x\right), \qquad b_n = \frac{2}{L}\cdot \int_{t=0}^{L} f(t)\sin\left(\frac{n\pi}{L}\cdot t\right) dt.$$

2. Using the cheating sheet below, give the Fourier-sine series of the function

$$G(x)=\begin{cases} x & \text{if } 0\le x\le 1,\\ 2-x & \text{if } 1\le x\le 2.\end{cases}$$



$x \in [0,2]$ $\quad \frac{\pi}{2}\cdot x \in [0,\pi]$

$g(x) = \frac{2}{\pi} g\left(\frac{\pi}{2}\cdot x\right)$

$0\cdot f(t)$

$$b_n = \frac{2}{L}\cdot\int_{t=0}^{L} f(t)\sin\left(\frac{n\pi}{L}\cdot t\right)dt.$$

$g\left(\frac{\pi}{2}x\right) = \begin{cases} \frac{\pi}{2}x, & 0<x<1 \\ \pi - \frac{\pi}{2}x = \frac{\pi}{2}(2-x) & 1<x<2 \end{cases}$

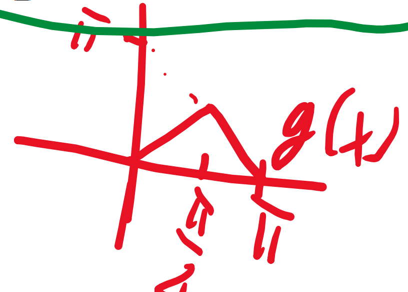
$x \in [0,2]$

$g(x) = \frac{2}{\pi}\cdot g\left(\frac{\pi}{2}x\right) = \frac{2}{\pi}\left(\frac{4}{\pi}\left(\sin\frac{\pi}{2}x - \frac{\sin\left(3\cdot\frac{\pi}{2}x\right)}{3^2} + \frac{\sin\left(5\cdot\frac{\pi}{2}x\right)}{5^2} - \dots\right)\right)$

$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$

$L=2$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\cdot x\right),$$



(b) $g(x)=\begin{cases} x & 0\le x\le \pi/2 \\ \pi - x & \pi/2\le x\le\pi \end{cases}$: $g(x) = \frac{4}{\pi}\left(\sin(x) - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \cdots\right)$

for $0\le x\le\pi$.

$x \in [0,2]$

$$y(x) = \frac{2}{\pi} \cdot g\left(\frac{\pi}{2}x\right) = \frac{2}{\pi}\left(\frac{4}{\pi}\left(\sin\frac{\pi}{2}x - \frac{\sin\left(3\cdot\frac{\pi}{2}x\right)}{3^2} + \frac{\sin\left(5\cdot\frac{\pi}{2}x\right)}{5^2} - \dots\right)\right)$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{\substack{n=1 \\ n:\ \text{odd}}}^{\infty} \frac{8}{\pi^2 \cdot n^2} (-1)^{\frac{n-1}{2}} \cdot \sin\left(n \cdot \frac{\pi}{2} \cdot x\right) =$$

$$= \sum_{k=1}^{\infty} \frac{8}{\pi^2 \cdot (2k-1)^2} (-1)^{k-1} \cdot \sin\left((2k-1) \cdot \frac{\pi}{2} \cdot x\right)$$

3. Using the cheating sheet below, give the Fourier-sine series of the function $\Phi(x) = x(2-x)$ on the interval $[0, 2]$.

$$\frac{\pi}{2} x \left(\pi - \frac{\pi}{2} x\right) = \left(\frac{\pi}{2}\right)^2 \cdot x \cdot (2-x)$$

$$\boxed{\Phi(x) = \frac{4}{\pi^2} \varphi\left(\frac{\pi}{2} \cdot x\right)} = \frac{32}{\pi^3} \cdot \left(\sin\left(\frac{\pi}{2} x\right) + \frac{\sin\left(3 \cdot \frac{\pi}{2} \cdot x\right)}{3^3} + \cdots \right)$$

$$b_n = \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ \dfrac{32}{\pi^3 \cdot n^3} & \text{IF } n \text{ IS ODD} \end{cases}$$

$$= \sum_{\ell=1}^{\infty} \frac{32}{\pi^3 \cdot (2\ell-1)^3} \cdot \sin\left((2\ell-1) \cdot \frac{\pi}{2} x\right)$$

$$= \sum_{\substack{n=1 \\ n:\,\text{ODD}}}^{\infty} \frac{32}{\pi^3 \cdot n^3} \sin\left(n \cdot \frac{\pi}{2} \cdot x\right)$$

(d) $\varphi(x) = x(\pi - x)$: $\varphi(x) = \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \frac{\sin(7x)}{7^3} + \cdots\right)$ for $0 \le x \le \pi$

$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} \cdot x\right)$, $\to L = 2$

$$\begin{cases} u_{tt}(x,t) = 3.1^2 u_{xx}(x,t), & 0 < t, 0 < x < 2 \\ u(x,0) = \frac{1}{40}\left(1 - |x-1|\right), & 0 \le x \le 2 \\ u_t(x,0) = \frac{1}{80} x(2-x), & 0 < x < 2 \\ u(0,t) = 0, \quad u(2,t) = 0, & 0 \le t \end{cases}$$

$L = 2$, $c = 3{,}1$

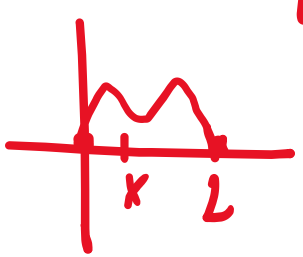
INITIAL POSITION:

$$f(x) = \frac{1}{40}\cdot\left(1 - |x-1|\right) = \begin{cases} \frac{1}{40}(x), & 0 < x < 1 \\ \frac{1}{40}(2-x), & 1 < x < 2 \end{cases}$$

$$A_k = \begin{cases} 0 & \text{if } k \text{ is EVEN} \\ \frac{1}{40}\cdot\frac{8}{\pi^2\cdot k^2}(-1)^{\frac{k-1}{2}} & \text{if } k \text{ is ODD} \end{cases}$$

3. Wibrating string:

$$\begin{cases} u''_{tt} = c^2 u''_{xx} & 0 < t, 0 \le x \le L \\ u(x,0) = f(x) & 0 \le x \le L \\ u'_t(x,0) = g(x) & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & 0 < t \end{cases}$$



$u(x,t)$: POSITION OF THE STRING AT x AFTER t TIME UNIT

Solution: $u(x,t) = \sum_{k=1}^{\infty} \sin(\frac{k\pi}{L}x)\left(A_k \cos(\frac{kc\pi}{L}t) + B_k \sin(\frac{kc\pi}{L}t)\right)$, where $\{A_k\}$ are the coefficients of the Fourier-sine series of $f(x)$ and $\{\frac{kc\pi}{L}B_k\}$ are the coefficients of the Fourier-sine series of $g(x)$.

$$\begin{cases} u_{tt}(x,t) = 3.1^2 u_{xx}(x,t), & 0 < t, 0 < x < 2 \\ u(x,0) = \frac{1}{40}\left(1 - |x-1|\right), & 0 \le x \le 2 \\ u_t(x,0) = \frac{1}{80} x(2-x), & 0 < x < 2 \\ u(0,t) = 0, \ u(2,t) = 0, & 0 \le t \end{cases}$$

INITIAL VELOCITY

$0 \le x \le 2$

$$g(x) = \frac{1}{80} \cdot x(2-x)$$

$$\frac{k \cdot \pi \cdot 3.1}{2} \cdot B_k = \begin{cases} 0 & \text{if } k \text{ is EVEN} \\ \frac{1}{80} \frac{32}{\pi^3 \cdot k^3} & \text{if } k \text{ is ODD} \end{cases}$$

$\Rightarrow$

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k \cdot \pi}{2} \cdot x\right)\left[\frac{1}{40} \cdot \frac{8}{\pi^2 \cdot k^2} (-1)^{\frac{k-1}{2}} \cdot \cos\left(\frac{k \cdot 3.1 \cdot \pi}{2} \cdot t\right) + \right.$$

k: ODD

$$\left. + \frac{64}{\pi^4 \cdot k^4 \cdot 3.1} \frac{1}{80} \sin\left(\frac{k \cdot 3.1 \cdot \pi}{2} \cdot t\right)\right]$$

$\frac{1}{80}$

3. Wibrating string:

$$\begin{cases} u''_{tt} = c^2 u''_{xx} & 0 < t, 0 \le x \le L \\ u(x,0) = f(x) & 0 \le x \le L \\ u'_t(x,0) = g(x) & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & 0 < t \end{cases}$$

Solution: $u(x,t) = \sum_{k=1}^{\infty} \sin(\frac{k\pi}{L}x)\left(A_k \cos(\frac{kc\pi}{L}t) + B_k \sin(\frac{kc\pi}{L}t)\right)$, where $\{A_k\}$ are the coefficients of the Fourier-sine series of $f(x)$ and $\{\frac{kc\pi}{L}B_k\}$ are the coefficients of the Fourier-sine series of $g(x)$.

$$\begin{cases} u'_t = u''_{xx} & 0 < t, 0 \le x \le 2 \\ u(x,0) = f(x) & 0 \le x \le 2 \\ u(0,t) = u(2,t) = 0 & 0 < t \end{cases} \quad \text{where } f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2. \end{cases}$$

$\alpha = 1$

$L = 2$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2 \cdot n^2} (-1)^{\frac{n-1}{2}} \cdot e^{-\left(\frac{n\cdot\pi}{2}\right)^2 1^2 \cdot t} \cdot \sin\left(\frac{n\cdot\pi}{2} \cdot x\right) =$$

n : odd

$$= \frac{8}{\pi^2} e^{-\frac{\pi^2}{4} \cdot t} \cdot \sin\left(\frac{\pi}{2} \cdot x\right) - \frac{8}{\pi^2 \cdot 9} \cdot e^{-\frac{9\pi^2}{4} \cdot t} \cdot \sin\left(3 \cdot \frac{\pi}{2} \cdot x\right) + \ldots$$

Heat conduction in a finite rod:

$$\begin{cases} u'_t = \alpha^2 u''_{xx} & 0 < t, 0 \le x \le L \\ u(x,0) = f(x) & 0 \le x \le L \\ u(0,t) = u(2,t) = 0 & 0 < t \end{cases}$$

Solution: $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \sin(\frac{n\pi}{L} x)$, where $\{A_n\}$ are the coefficients of the Fourier sine series of $f(x)$.