10. Let $P = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Find the eigenvalues and the eigenvectors of P!

In class, we solved a computationally easier exercise. It is important that P is not symmetric; hence, it is not sure that there exist two linearly independent eigenvectors, and if there exist then they are not necessarily orthogonal

$\lambda \in \mathbb{R}$ IS AN EIGENVALUE IF THERE EXISTS $\underline{v} \neq \underline{0}$ SUCH THAT $\boxed{P\underline{v} = \lambda \underline{v}}$. IN THIS CASE, $\underline{v}$ IS AN EIGENVECTOR CORRESPONDING TO THE EIGENVALUE λ.

$\boxed{P\underline{v} = \lambda \underline{v}} \iff \underline{0} = P\underline{v} - \lambda \underline{v} = (P - \lambda I)\underline{v} \quad (*)$

λ IS AN EIGENVALUE IF $(*)$ HAS INFINITELY MANY SOLUTIONS, I.E. IF $\det(P - \lambda I) = 0$

10. Let $P = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Find the eigenvalues and the eigenvectors of P!

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-4}}{2}$$

$$0 = \det(P - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 3 = \lambda^2 - 5\lambda + 1$$

$$\lambda_1 = \frac{5+\sqrt{21}}{2}$$

WE SOLVE $(P - \lambda_1 I)\underline{v} = 0$

$$\left(\begin{array}{cc|c} \frac{2}{2} - \frac{5+\sqrt{21}}{2} & -1 & 0 \\ -3 & \frac{8}{2} - \frac{5+\sqrt{21}}{2} & 0 \end{array}\right) = \left(\begin{array}{cc|c} \cancel{\frac{-3-\sqrt{21}}{2}} & \cancel{-1} & \cancel{0} \\ -3 & \frac{3-\sqrt{21}}{2} & 0 \end{array}\right)$$

ADDING THE $\frac{2}{3-\sqrt{21}}$ MULTIPLE OF THE SECOND ROW TO THE FIRST ROW, THE FIRST ROW BECOMES $0 \quad 0 \mid 0$. OF COURSE, WE KNOW IN ADVANCE THAT IF λ_1 IS AN EIGENVALUE THEN WE HAVE INFINITELY MANY SOLUTION, HENCE IF WE ARE SURE THAT λ_1 IS CORRECT THEN WE CAN DELETE A ROW WITHOUT ANY CALCULATION

10. Let $P = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Find the eigenvalues and the eigenvectors of P!

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-4}}{2}$$

$$0 = \det(P - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 3 = \lambda^2 - 5\lambda + 1$$

$$\lambda_1 = \frac{5+\sqrt{21}}{2}$$

WE SOLVE $(P - \lambda_1 I)\underline{v} = 0$

$$\left(\begin{array}{cc|c} \frac{2}{2} - \frac{5+\sqrt{21}}{2} & -1 & 0 \\ -3 & \frac{8}{2} - \frac{5+\sqrt{21}}{2} & 0 \end{array}\right) = \left(\begin{array}{cc|c} \cancel{\frac{-3-\sqrt{21}}{2}} & \cancel{-1} & \cancel{0} \\ \boxed{-3} & \frac{3-\sqrt{21}}{2} & 0 \end{array}\right)$$

$$v_2 = t, \quad -3v_1 + \frac{3-\sqrt{21}}{2} v_2 = 0 \Rightarrow v_1 = \frac{3-\sqrt{21}}{6} t$$

ALL SOLUTIONS: $\left\{ \begin{pmatrix} \frac{3-\sqrt{21}}{6} t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$, CHOOSING FOR EXAMPLE $t=1$

WE GET THAT $\begin{pmatrix} \frac{3-\sqrt{21}}{6} \\ 1 \end{pmatrix}$ IS AN EIGENVECTOR CORRESPONDING TO λ_1

10. Let $P = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Find the eigenvalues and the eigenvectors of P!

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-4}}{2}$$

$$0 = \det(P - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 3 = \lambda^2 - 5\lambda + 1$$

$$\lambda_2 = \frac{5-\sqrt{21}}{2}, \quad \left(\begin{array}{cc|c} \cancel{\frac{-3+\sqrt{21}}{2}} & \cancel{-1} & \cancel{0} \\ \boxed{-3} & \frac{3+\sqrt{21}}{2} & 0 \end{array}\right), \quad -3v_1 + \frac{3+\sqrt{21}}{2} v_2 = 0$$

$\& \; v_2 = t$

$$v_1 = \frac{3+\sqrt{21}}{6} t$$

$$\left\{ \begin{pmatrix} \frac{3+\sqrt{21}}{6} t \\ t \end{pmatrix}, \; t \in \mathbb{R} \right\}, \quad \text{BASIS OF THE EIGENSPACE} \begin{pmatrix} \frac{3+\sqrt{21}}{6} \\ 1 \end{pmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Give a symmetric matrix S and a skew-symmetric matrix G such that $A = S + G$!

8. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$. Find $A : B$!

$$S = \frac{A + A^T}{2}, \quad G = \frac{A - A^T}{2}$$

$$G = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$F + G = A$$

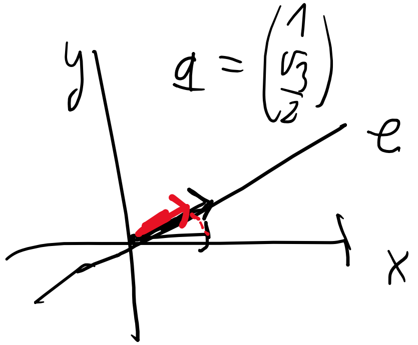
(7) $$S = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\right] = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}$$

(8) $$A : B = 5 + 12 + 32 + 45 = 94$$

IMPORTANT: $$(A+B)^T = A^T + B^T$$

$$(A \cdot B)^T = B^T A^T$$

5. Let P be the orthogonal projection to line $y = \frac{\sqrt{3}}{2}x$. Find the matrix of P in the natural basis (i.e. $N = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$).



$$a = \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$P = \frac{1}{a^T a} \cdot a a^T$$

$$a^T a = |a|^2$$

$$\left(1, \frac{\sqrt{3}}{2}\right) \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{7}{4}$$

$$\begin{pmatrix} 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{4} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{4}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{3}{7} \end{pmatrix}$$

9. Give the spectral decomposition of the matrix $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$!

If the n*n matrix M is symmetric, then all the eigenvalues are real, there exist n linearly independent eigenvectors, and their can be chosen to form an orthonormal system, i.e., to be perpendicular to each other and have length 1.

$$0 = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3, \quad \lambda_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} \begin{matrix} 3 \\ 1 \end{matrix}$$

$\lambda_1 = 3$

$$\left(\begin{array}{cc|c} \boxed{-1} & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} \end{array}\right) \to \begin{matrix} x_2 = t \\ -x_1 - x_2 = 0 \end{matrix} \Rightarrow x_1 = -t \qquad \left\{ \begin{pmatrix} -t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$$

$\cancel{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda_2 = 1$

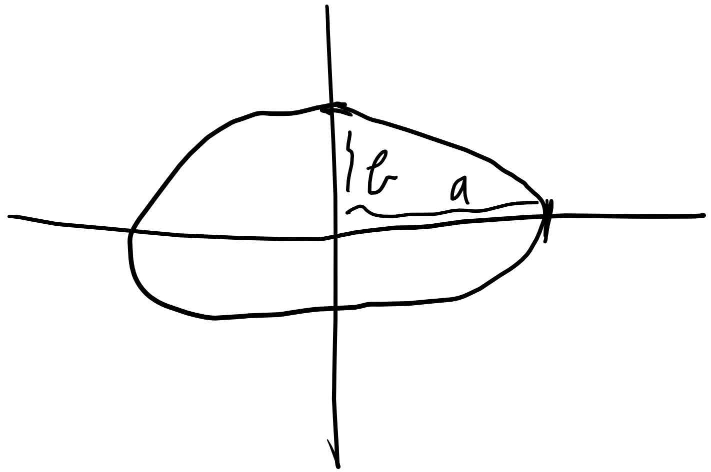
$$\left(\begin{array}{cc|c} \boxed{1} & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \to \begin{matrix} x_2 = t \\ x_1 - x_2 = 0 \end{matrix} \Rightarrow x_1 = t \qquad \left\{ \begin{pmatrix} t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$$

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q^T}$$

CANONICAL ELLIPSE

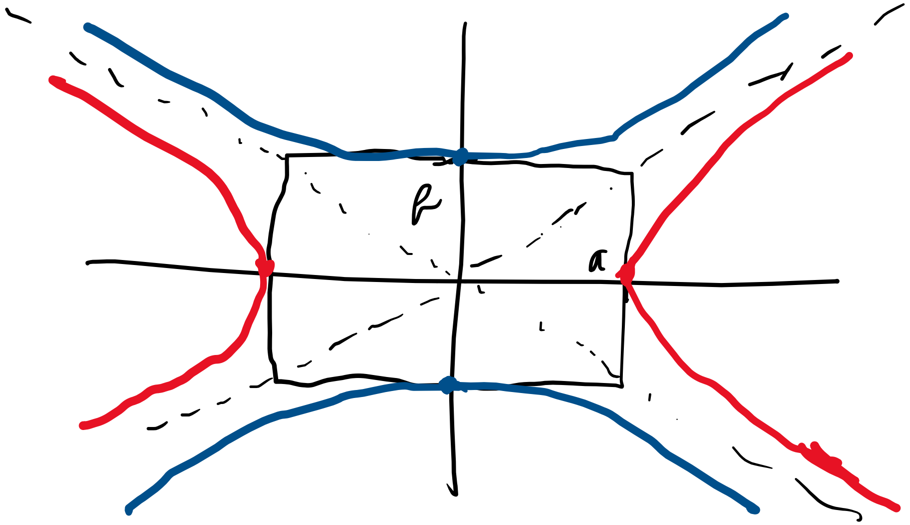
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



CANONICAL HYPERBOLAS

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



4. Draw the points on the plane, which satisfy the equation $5x^2 - 4xy + 8y^2 = 36$!

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \underbrace{\begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

$$0 = \begin{vmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} = (5-\lambda)(8-\lambda) - 4 = \lambda^2 - 13\lambda + 36 = 0$$

$$\lambda_{1,2} = \frac{13 \pm \sqrt{169-144}}{2} = \begin{matrix} 9 \\ 4 \end{matrix}$$

1. STEP DETERMINING THE SPECTRAL DECOMPOSITION OF M

$\lambda_1 = 9$

$$\left(\begin{array}{cc|c} \cancel{-4} & \cancel{-2} & \cancel{0} \\ \boxed{-2} & -1 & 0 \end{array}\right) \quad -2x_1 - x_2 = 0 \quad x_2 = t \quad x_1 = -\frac{t}{2}$$

$$\cancel{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} \quad \begin{pmatrix} -\frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{pmatrix}, \quad \left\{ \begin{pmatrix} -\frac{t}{2} \\ t \end{pmatrix}, t \in \mathbb{R} \right\}, \quad \lambda_2 = 4, \quad \cancel{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \begin{pmatrix} \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{pmatrix}$$

2. STEP

$$Q = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$\det Q = 1$

4. Draw the points on the plane, which satisfy the equation $5x^2 - 4xy + 8y^2 = 36$!

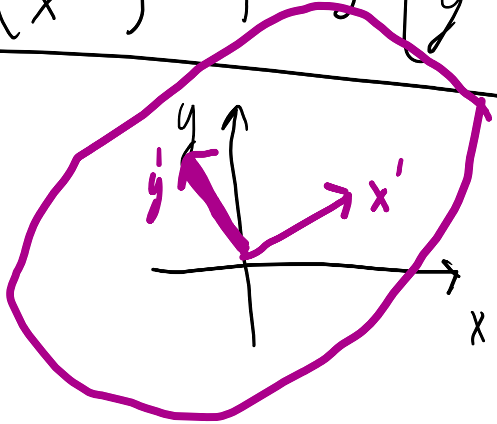
$$\begin{pmatrix} x \\ y \end{pmatrix}^T \overbrace{\begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}}^{M} \begin{pmatrix} x \\ y \end{pmatrix} = 36 \qquad Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}^T \cdot \overbrace{Q^T M Q}^{D} \begin{pmatrix} x' \\ y' \end{pmatrix} = 36, \quad \begin{pmatrix} x' \\ y' \end{pmatrix}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 36$$

$$\boxed{4(x')^2 + 9 \cdot (y')^2 = 36}$$

$$\boxed{\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1}$$



TOY EXAMPLE

$$\alpha\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-3\\-3\\0\end{pmatrix}\right) = xy \text{ PLANE}$$

$$\| $$

$$\alpha\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right)$$

6. Let L be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Find a basis of L out of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$! Give the coordinates of the remaining vectors, which are not contained in this basis, in the basis you have found!

$$\begin{pmatrix} \boxed{1} & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 1 & 0 & -1 \\ 0 & \boxed{-2} & 1 & 5 \\ 0 & 2 & 0 & 3 \\ 0 & 6 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 1 & 0 & -1 \\ 0 & \boxed{-2} & 1 & 5 \\ 0 & 0 & \boxed{1} & 8 \\ \cancel{0} & \cancel{0} & \cancel{2} & \cancel{16} \end{pmatrix} \Rightarrow \mathcal{L}(v_1, v_2, v_3, v_4) = \mathcal{L}(v_1, v_2, v_3)$$

The calculations show that the first three vectors, i.e., v1,v_2,v_3 constitute a basis since main ones are present in the first three columns

6. Let L be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Find a basis of L out of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$! Give the coordinates of the remaining vectors, which are not contained in this basis, in the basis you have found!

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 1 & 0 & -1 \\ 0 & \boxed{-2} & 1 & 5 \\ 0 & 2 & 0 & 3 \\ 0 & 6 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 0 & -1 \\ 0 & \boxed{-2} & 1 & 5 \\ 0 & 0 & \boxed{1} & 8 \\ \cancel{0} & \cancel{0} & \cancel{2} & \cancel{16} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 8 \end{array}\right)$$

$$\overset{?}{\alpha_1} \cdot \underline{v_1} + \overset{?}{\alpha_2} \cdot \underline{v_2} + \overset{?}{\alpha_3} \cdot \underline{v_3} = \underline{v_4}$$

$$1 \cdot \alpha_1 = -\frac{5}{2}$$
$$1 \cdot \alpha_2 = \frac{3}{2}$$
$$1 \cdot \alpha_3 = 8$$

(b) Find an orthonormal basis of L by using the Gram-Schmidt orthogonalisation algoritm!

EXERCISE 6b

$$\mathcal{L} = (\underline{v_1}, \underline{v_2}, \underline{v_3}) = \mathcal{L}\left(\underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}}_{\underline{v_1}}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 3 \\ 7 \end{pmatrix}}_{\underline{v_2}}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$$

$\boxed{\text{1. STEP}}$

$$\underline{b_1} = \underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathcal{L}(\underline{b_1}) = \mathcal{L}(\underline{v_1})$$

$\boxed{\text{2. STEP}}$

$$\underline{b_2} = \overset{?}{\alpha} \cdot \underline{b_1} + \underline{v_2}$$

WE WOULD LIKE TO FIND $\underline{b_2}$ IN THIS FORM

$$\mathcal{L}(\underline{b_1}, \underline{b_2}) = \mathcal{L}(\underline{v_1}, \underline{v_2})$$

WE NEED ORTHOGONALITY

$$0 = \underline{b_1} \cdot \underline{b_2} = \underline{b_1} \cdot (\alpha \underline{b_1} + \underline{v_2}) =$$

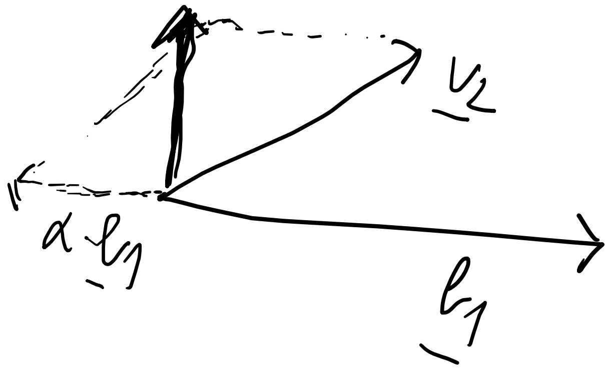
$$= \alpha \cdot \underline{b_1} \cdot \underline{b_1} + \underline{b_1} \cdot \underline{v_2}, \Rightarrow \alpha = \frac{-\underline{b_1} \cdot \underline{v_2}}{|\underline{b_1}|^2} = \frac{-11}{7}$$

$$\underline{b_2} = -\frac{11}{7}\underbrace{\begin{pmatrix}1\\2\\1\\1\end{pmatrix}}_{\underline{b_1}=\underline{v_1}} + \underbrace{\begin{pmatrix}1\\0\\3\\7\end{pmatrix}}_{\underline{v_2}} = \begin{pmatrix}-\frac{4}{7}\\-\frac{22}{7}\\\frac{10}{7}\\\frac{38}{7}\end{pmatrix}$$

COSMETICS: WE MULTIPLY THE RESULT WITH $\frac{7}{2}$

$$\underline{b_2} = \begin{pmatrix}-2\\-11\\5\\19\end{pmatrix}$$

$$\underline{b_2} = \alpha \cdot \underline{b_1} + \underline{v_2}$$



III. STEP

$$\underline{b_3} = \overset{?}{\beta_1} \cdot \underline{b_1} + \overset{?}{\beta_2}\,\underline{b_2} + \underline{v_3}, \quad L(\underline{b_1}, \underline{b_2}, \underline{b_3}) = L(\underline{v_1}, \underline{v_2}, \underline{v_3})$$

$$0 = \underline{b_1} \cdot \underline{b_3} = \beta_1 \cdot \underline{b_1} \cdot \underline{b_1} + \underline{b_1} \cdot \underline{v_3}, \quad \beta_1 = \frac{-\underline{b_1} \cdot \underline{v_3}}{|\underline{b_1}|^2}$$

$$0 = \underline{b}_2 \cdot \underline{b}_3 \Rightarrow \beta_2 = -\frac{\underline{b}_2 \cdot \underline{v}_3}{|\underline{b}_2|^2} = \frac{30}{511} \qquad \underline{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{b}_2 = \begin{pmatrix} -2 \\ -11 \\ 5 \\ 19 \end{pmatrix},$$

$$\boxed{\underline{b}_3 = \beta_1 \cdot \underline{b}_1 + \beta_2 \cdot \underline{b}_2 + \underline{v}_3}, \quad \beta_1 = -\frac{\underline{b}_1 \cdot \underline{v}_3}{|\underline{b}_1|^2} = -\frac{1}{7} \qquad \underline{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\underline{b}_3 = -\frac{1}{7}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + \frac{30}{511} \cdot \begin{pmatrix} -2 \\ -11 \\ 5 \\ 19 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{511}\begin{pmatrix} -133 \\ 35 \\ 27 \\ -14 \end{pmatrix}$$

COSMETICS

$$\underline{b}_3 = \begin{pmatrix} -19 \\ 5 \\ 11 \\ -2 \end{pmatrix}$$

NORMALIZATION

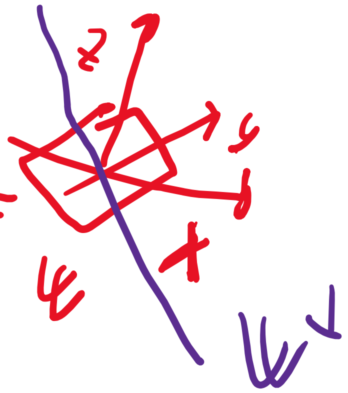
$$\underline{o}_1 = \begin{pmatrix} \frac{1}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \\ \frac{1}{\sqrt{7}} \\ \frac{1}{\sqrt{7}} \end{pmatrix}, \quad \underline{o}_2 = \begin{pmatrix} \frac{-2}{\sqrt{511}} \\ \frac{-11}{\sqrt{511}} \\ \frac{5}{\sqrt{511}} \\ \frac{19}{\sqrt{511}} \end{pmatrix}, \quad \underline{o}_3 = \begin{pmatrix} \frac{-19}{\sqrt{511}} \\ 5/\sqrt{511} \\ 11/\sqrt{511} \\ -2/\sqrt{511} \end{pmatrix}$$

$W \subseteq \mathbb{R}^n$ SUBSPACE , $W^{\perp} = \{ \underline{v} : \ \underline{v} \cdot \underline{w} = 0 \ \ \forall \underline{w} \in W \}$

EXAMPLE

$W \subseteq \mathbb{R}^3$ PLANE THAT GOES THROUGH THE ORIGIN , $W^{\perp} = \mathcal{L}(\underline{n})$

WHERE $\underline{n}$ IS THE NORMAL VECTOR OF THE PLANE



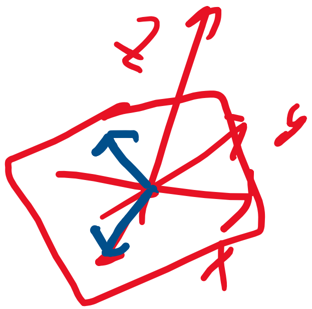
THEOREM

$$\dim(W) + \dim(W^{\perp}) = n$$

PROP

$$(W^{\perp})^{\perp} = W$$

13. Find the matrix in the natural basis of the orthogonal projection to the plane $V = \{(x, y, z) : 2x - y + 3z = 0\}$! By using this matrix, decompose the vector $\mathbf{v} = (2, 4, -1)$ into perpendicular and paralell components with respect to V!



$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\} \text{ BASIS IN } W$$

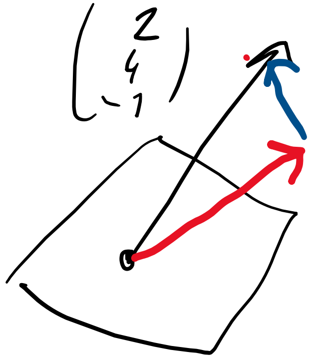
$$M = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 6 & 10 \end{pmatrix} = M^T M$$

$$(M^T M)^{-1} = \frac{1}{14} \begin{pmatrix} 10 & -6 \\ -6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & -6 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -6 \\ 2 & 3 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 2 & -6 \\ 2 & 13 & 3 \\ -6 & 3 & 5 \end{pmatrix}$$



$$P = \frac{1}{14}\begin{pmatrix} 10 & 2 & -6 \\ 2 & 13 & 3 \\ -6 & 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} \frac{17}{7} \\ \frac{53}{14} \\ \frac{-5}{14} \end{pmatrix}$$

THE PARALELL COMPONENT IS THE PROJECTION

PERPENDICULAR COMPONENT IS THE DIFFERENCE

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{17}{7} \\ \frac{53}{14} \\ \frac{-5}{14} \end{pmatrix} = \begin{pmatrix} \frac{-6}{14} \\ \frac{3}{14} \\ \frac{-9}{14} \end{pmatrix}$$

11. Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{bmatrix}$

(a) Find the subspaces row(A), col(A), null(A), null(A^TA), null(A^T)!

(b) Find nullity(A), rank(A) and rank(A^TA)!

(c) Check that row$(A)^\perp$ = null(A) and col$(A)^\perp$ = null(A^T)!

$\{ \underline{v} : A\underline{v} = \underline{0} \}$

nullity$(A) = \dim(\text{null}(A)) = 1$

$\text{col}(A) = \mathcal{L}\left\{ \begin{pmatrix} 1\\2\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\3\\7 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix}, \begin{pmatrix} -1\\3\\2\\0 \end{pmatrix} \right\}$

N OF COLUMNS = rank(A) + nullity(A)

rank$(A) = 3$

$$\left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 & 0 \\ 1 & 3 & 0 & 2 & 0 \\ 1 & 7 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & -1 & 0 \\ 0 & \boxed{-2} & 1 & 5 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 6 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & -1 & 0 \\ 0 & \boxed{-2} & 1 & 5 & 0 \\ 0 & 0 & \boxed{1} & 8 & 0 \\ \cancel{0} & \cancel{0} & \cancel{2} & \cancel{16} & \cancel{0} \end{array}\right)$$

$v_1 + v_2 - v_4 = 0 \Rightarrow v_1 = \frac{5}{2}t$

$v_4 = t$

$-2v_2 + v_3 + 5v_4 = 0$

$v_3 + 8v_4 = 0$

$v_2 = -\frac{3}{2}t$

$v_3 = -8t$

$\text{col}(A) = \mathcal{L}\left\{ \begin{pmatrix} 1\\2\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\3\\7 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix} \right\}$

$\text{null}(A) = \left\{ \begin{pmatrix} \frac{5}{2}t \\ -\frac{3}{2}t \\ -8t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$

$\text{null}(A) = \mathcal{L}\left(\begin{pmatrix} 5\\-3\\-16\\2 \end{pmatrix} \right)$

$\text{row}(A) = \mathcal{L}\left\{ \begin{pmatrix} 1\\1\\0\\-1 \end{pmatrix}, \begin{pmatrix} 2\\0\\1\\3 \end{pmatrix}, \begin{pmatrix} 1\\3\\0\\2 \end{pmatrix}, \begin{pmatrix} 1\\7\\-1\\0 \end{pmatrix} \right\} = \mathcal{L}\left\{ \begin{pmatrix} 1\\1\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\-2\\1\\5 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\8 \end{pmatrix} \right\}$

1. Legyen $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{bmatrix}$. Határozzuk meg a következő altereket: $\mathrm{row}(A), \mathrm{col}(A), \mathrm{null}(A), \mathrm{null}(A^T \cdot A)$. Továbbá határozzuk meg $\mathrm{nullity}(A) =?$ és és $\mathrm{rank}(A^T \cdot A) =?$ (Számoljuk ki direktbe és NE az órán tanult tétellel az $A^T \cdot A$-ra vonatkozó kérdéseket.)

$$A \left\{ \begin{pmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{pmatrix} \right.$$

$$A^T \left\{ \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 7 \\ 0 & 1 & 0 & -1 \\ -1 & 3 & 2 & 0 \end{pmatrix} \right. \begin{pmatrix} 7 & 11 & 1 & 7 \\ 11 & 59 & -7 & 5 \\ 1 & -7 & 2 & 3 \\ 7 & 5 & 3 & 14 \end{pmatrix}$$

$$\mathrm{null}(A^T A)$$

$$\mathrm{null}(A^T) = \quad \text{HW}$$

$$= \alpha \left(\begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 7 & 11 & 1 & 7 \\ 11 & 59 & -7 & 5 \\ 1 & -7 & 2 & 3 \\ 7 & 5 & 3 & 14 \end{pmatrix} \underset{\sim}{\overset{\text{III. sor FEL}}{}} \begin{pmatrix} \boxed{1} & -7 & 2 & 3 \\ 7 & 11 & 1 & 7 \\ 11 & 59 & -7 & 5 \\ 7 & 5 & 3 & 14 \end{pmatrix} \underset{\text{III}-\text{IV}}{\overset{\text{II}-\text{IV}}{\sim}} \begin{pmatrix} \boxed{1} & -7 & 2 & 3 \\ 0 & 6 & -2 & -7 \\ 4 & 54 & -10 & -9 \\ 7 & 5 & 3 & 14 \end{pmatrix} \overset{\text{III}-4\text{I},\ \text{IV}-7\text{I}}{\sim} \begin{pmatrix} \boxed{1} & -7 & 2 & 3 \\ 0 & \boxed{6} & -2 & -7 \\ 0 & 82 & -18 & -21 \\ 0 & 54 & -11 & -7 \end{pmatrix}$$

$$\overset{\text{III}-\text{IV}}{\sim} \begin{pmatrix} 1 & -7 & 2 & 3 \\ 0 & 6 & -2 & -7 \\ 0 & 28 & -7 & -14 \\ 0 & 54 & -11 & -7 \end{pmatrix} \overset{\text{IV}-2\text{III}}{\sim} \begin{pmatrix} 1 & -7 & 2 & 3 \\ 0 & 6 & -2 & -7 \\ 0 & 28 & -7 & -14 \\ 0 & -2 & 3 & 21 \end{pmatrix} \underset{\sim}{\overset{\text{III}/7,\ \text{II} \leftrightarrow \text{IV}}{}} \begin{pmatrix} \boxed{1} & -7 & 2 & 3 \\ 0 & \boxed{-2} & 3 & 21 \\ 0 & 4 & -1 & -2 \\ 0 & 6 & -2 & -7 \end{pmatrix}$$

$$\underset{\sim}{\overset{\text{III}+2\text{II},\ \text{IV}+3\text{II}}{}} \begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \quad \circ \\ \begin{pmatrix} \boxed{1} & -7 & 2 & 3 \\ 0 & \boxed{-2} & 3 & 21 \\ 0 & 0 & \boxed{5} & 40 \\ \cancel{0} & \cancel{0} & \cancel{7} & \cancel{56} \end{pmatrix} \end{array}$$

$v_4 = t$

$v_1 - 7\underset{-\frac{3}{2}t}{v_2} + 2\underset{-8t}{v_3} + 3\underset{t}{v_4} = 0$

$v_1 = \frac{5}{2}t$

$5v_3 + 40v_4 = 0$

$v_3 = -8t$

$-2v_2 + 3\underset{-8t}{v_3} + 21\underset{t}{v_4} = 0$

$v_2 = -\frac{3}{2}t$

$$\text{null}(A^TA) = \left\{ \begin{pmatrix} \frac{5}{2}t \\ -\frac{3}{2}t \\ -8t \\ t \end{pmatrix},\ t \in \mathbb{R} \right\} = \alpha \begin{pmatrix} 5 \\ -3 \\ -16 \\ 2 \end{pmatrix}$$

(c) Check that $\text{row}(A)^\perp = \text{null}(A)$ and $\text{col}(A)^\perp = \text{null}(A^T)$!

ORTHOGONALITY

$$\text{row}(A) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 8 \end{pmatrix} \right), \quad \text{null}(A) = \mathcal{L}\left(\begin{pmatrix} 5 \\ -3 \\ -16 \\ 2 \end{pmatrix} \right)$$

$$\text{null}(A) \subseteq \text{row}(A)^\perp$$

since $\dim(\text{null}(A)) + \dim(\text{row}(A)) = 4$

$\text{null}(A) = \text{row}(A)^\perp$ FOLLOWS