1. Solve the following system of linear equations with Gauss elimination!

$$
\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}
$$

$$
\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right) \quad (*)
$$

We arrange the coefficients into a matrix. The matrix uniquely determines the system of linear equations and vice versa.

We will systematically perform three possible manipulations:

a)We can swap two rows

b)We can multiply a row with a nonzero number

c)We can add to a row a nonzero multiple of another row

These operations do not change the solution of the system of linear equations.

(d)

$\text{II} - 2\text{I}$

$\text{IV} - 2\text{I}$

MAIN ONE

$$\sim \left(\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

VIRTUAL MAIN ONE (WITH A DIVISION WE CAN EASILY GET ONE)

BELOW THE MAIN ONE, WE HAVE TO CREATE 0-S

$\text{III} + 5\text{II}$

$\text{IV} + 4\text{II}$

$$\sim \left(\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \end{array}\right)$$

VIRTUAL MAIN ONE

$$\left(\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \end{array}\right)$$

The variables corresponding to the main one free columns are the free variables

$$\boxed{x_2 = s,\ x_4 = t,\ x_5 = v}$$

$(s, t, v \in \mathbb{R})$

The other variables are expressed from bottom to top

$$6x_6 = 2 \Rightarrow \boxed{x_6 = \frac{1}{3}}$$

$$-x_3 - 2x_4 - 3\cdot x_6 = -1 \Rightarrow \boxed{x_3 = -2t}$$

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \Rightarrow \boxed{x_1 = -3s - 4t - 2v}$$

All solutions in vector form

$$\left\{ \begin{pmatrix} -3s-4t-2v \\ s \\ -2t \\ t \\ v \\ \frac{1}{3} \end{pmatrix} \quad \begin{matrix} s \in \mathbb{R} \\ t \in \mathbb{R} \\ v \in \mathbb{R} \end{matrix} \right\}$$

Alternative viewpoints of a system of linear equations

1,

$$\underbrace{\begin{pmatrix} 1 & 3 & \cdots & 0 \\ 2 & 6 & & -3 \\ 0 & 0 & & 15 \\ 2 & 6 & & 18 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 0 \\ -1 \\ 5 \\ 6 \end{pmatrix}}_{\underline{b}}$$

We have a matrix equation

2,

$$\overset{?}{x_1} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + \overset{?}{x_2} \begin{pmatrix} 3 \\ 6 \\ 0 \\ 6 \end{pmatrix} + \cdots + \overset{?}{x_6} \begin{pmatrix} 0 \\ -3 \\ 15 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 5 \\ 6 \end{pmatrix}$$

We are looking for the weights for which it is true that the linear combination of the column vectors with these weights is equal **b**

2. Is the collection of the following vectors linearly independent?

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

Solution 1

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We know that the constant zero vector solves this equation. The question is whether it is the only solution, i.e., whether the solution is unique

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{-1} & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

The last variable is free, hence, the equation has infinitely many solutions. For this reason the vectors are not linearly independent.

2. Is the collection of the following vectors linearly independent?

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

Solution 2

When the coefficient matrix of the equation is a square matrix, then the determinant gives important information about the structure of the solution: The solution is unique if and only if the determinant is non-zero.

$$\begin{vmatrix} 1^{+} & 0^{-} & 1^{+} \\ 2 & -1 & 0 \\ 3 & 1 & 5 \end{vmatrix} = 1 \cdot \underbrace{\begin{vmatrix} -1 & 0 \\ 1 & 5 \end{vmatrix}}_{-5} - 0 \cdot \begin{vmatrix} 2 & 0 \\ 3 & 5 \end{vmatrix} + 1 \cdot \underbrace{\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}}_{5} = 0$$

The determinant is 0. Hence, theoretically there are two possibilities: no solution or infinitely many solutions. The first possibility is not possible in this case (the constant zero vector is a solution). Hence, there exist infinitely many solutions, and hence the vectors are linearly dependent.

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

Solution 1

(a) What are the coordinates of the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ in basis B? (That is, $[\mathbf{v}]_B =?$)

We are looking for the weights lambda1 and lambda2 for which it is true that the linear combination with these weights of **u1** and **u2** is equal to **v**

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -3 & 7 \end{array}\right) \Rightarrow \begin{array}{l} \lambda_1 + \lambda_2 = -1 \Rightarrow \boxed{\lambda_1 = \frac{4}{3}} \\ \Rightarrow -3\lambda_2 = 7 \Rightarrow \boxed{\lambda_2 = -\frac{7}{3}} \end{array}$$

Hence $[\underline{v}]_B = \begin{pmatrix} \frac{4}{3} \\ -\frac{7}{3} \end{pmatrix}$

Checking: $\frac{4}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ ✓

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

Solution 2

(a) What are the coordinates of the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ in basis B? (That is, $[\mathbf{v}]_B =$?)

NATURAL BASIS $N = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$(\underline{v})_N = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, SINCE $-1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Key observations: the number pair defining **v** is equal to the coordinate vector of **v** in the natural basis

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

Solution 2

(a) What are the coordinates of the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ in basis B? (That is, $[\mathbf{v}]_B =$?)

General coordinate change formula from basis B' to basis B

$B' = \{ \underline{v}_1, \underline{v}_2, \dots \underline{v}_n \}$, $B = \{ \underline{u}_1, \underline{u}_2, \dots, \underline{u}_n \}$

$$(\underline{v})_{B'} = \overbrace{\left((\underline{u}_1)_{B'}, (\underline{u}_2)_{B'}, \dots, (\underline{u}_n)_{B'} \right)}^{\text{NOTATION: } P_{B' \to B}} \cdot (\underline{v})_B$$

$$(\underline{v})_B = \underbrace{\left((\underline{v}_1)_B, (\underline{v}_2)_B, \dots, (\underline{v}_n)_B \right)}_{\text{NOTATION } P_{B \to B'}} (\underline{v})_{B'}$$

!!! $P_{B \to B'} = \left(P_{B' \to B} \right)^{-1}$!!!

CHOOSING $B' = N$ WE GET THAT $(\underline{v})_B = \left(P_{N \to B} \right)^{-1} (\underline{v})_N$

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

Solution 2

(a) What are the coordinates of the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ in basis B? (That is, $[\mathbf{v}]_B =?$)

$$(\underline{v})_B = (P_{N\to B})^{-1} (\underline{v})_N$$

$$P_{N\to B} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

SWAP

MULTIPLICATION WITH (-1)

$$(\underline{v})_N = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$P_{N\to B}^{-1} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{pmatrix}$$

DET

$$(\underline{v})_B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{-7}{3} \end{pmatrix}$$

$\boxed{\text{ADDITIONAL EXERCISE 1}}$ CONSIDER IN $\mathbb{R}^3$ THE LINEAR TRANSFORMATION A THAT PROJECTS ANY VECTOR ORTHOGONALLY ONTO THE PLANE xy. DETERMINE THE MATRIX OF A IN THE NATURAL BASIS!

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{A} \underline{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \xrightarrow{A} \underline{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \xrightarrow{A} \underline{\begin{pmatrix}0\\0\\0\end{pmatrix}}$$

$$[A]_T = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

We need to determine the images of the elements of the natural basis in natural basis, and we need to arrange the resulting coordinate vectors into a matrix column-wise.

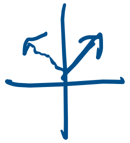
ADDITIONAL EXERCISE 2

LET A BE THE 90° POSITIVE ROTATION ON $\mathbb{R}^2$. FIND THE MATRIX A IN THE FOLLOWING BASIS: $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$

SOLUTION 1

We need to determine the images of the elements of basis B in basis B, and we need to arrange the resulting coordinate vectors into a matrix column-wise.

image



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -1 \end{matrix}}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} \mu_1 = 2 \\ \mu_2 = -1 \end{matrix}}$$

HENCE

$$(A)_B = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

ADDITIONAL EXERCISE 2

LET A BE THE 90° POSITIVE ROTATION ON $\mathbb{R}^2$. FIND THE MATRIX A IN THE FOLLOWING BASIS: $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$

You will find the explanation of the formula on the following two slides.

SOLUTION 2

$$(A)_N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$P_{N \to B} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad P_{N \to B}^{-1} = \frac{1}{-2} \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$(A)_B = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

CALCULATION

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

(b) Find the coordinate transformation $P_{N,B}$ (That is, from the natural basis to B)

(c) Find the matrix of the linear transform $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 3x_2 \end{bmatrix}$ in the basis B!

General basis change formula basis for linear transformations

$B' = \{v_1, v_2, \ldots, v_n\}$, $B = \{u_1, u_2, \ldots, u_n\}$

$$(A)_B = (P_{B' \to B})^{-1} (A)_{B'} P_{B' \to B}$$

CHOOSING $B' = N$ WE GET THAT: $(A)_B = P_{N \to B}^{-1} (A)_N P_{N \to B}$

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.

(b) Find the coordinate transformation $P_{N,B}$ (That is, from the natural basis to B)

(c) Find the matrix of the linear transform $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 3x_2 \end{bmatrix}$ in the basis B!

$$(A)_B = P_{N\to B}^{-1} \, (A)_N \, P_{N\to B}$$

$$\left.\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{aligned}\right\} \Rightarrow (A)_N = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$(A)_B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{10}{3} \end{pmatrix}$$

CALCULATION $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{5}{3} & -\frac{5}{3} \end{pmatrix}$ $\begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{10}{3} \end{pmatrix}$