Probability Theory 2

Sample exam A long time ago in a galaxy far, far away

Ex.1 *(20 points)* State and prove both Borel-Cantelli lemmas.

Ex.2 *(20 points)* State and prove Kolmogorov's two series theorem. (In the proof, you might use Kolmogorov's inequality without proving it.)

Ex.3 *(20 points)* Let X be a random variable and let $\varphi\colon \mathbb{R} \to \mathbb{C}$ be its characteristic function ($\varphi(u) = \mathbb{E}e^{iuX}$).

(a) Prove that φ is uniformly continuous on $\mathbb{R}$.

(b) Show that φ is positive definit, that is, for every $n \geq 1$, $u_1, \ldots, u_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in \mathbb{C}$

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} v_k \overline{v_\ell} \varphi(u_k - u_\ell) \geq 0.$$

Ex.4 *(20 points)* Let X, Y, Z be independent random variables such that X and Y have distribution $EXP(1)$ and let Z be such that $\mathbb{P}(Z = -1) = \mathbb{P}(Z = 1) = 1/2$. Show that the random variables $U := X - Z$ and $V := ZX$ have the same distribution.

Ex.5 *(20 points)* Let $Z_1, Z_2, \ldots$ be independent identically distributed random variables with standard Cauchy distribution $CAU(0,1)$. (The characteristic function of $CAU(0,1)$ is $\mathbb{E}(e^{iuZ_1}) = e^{-|u|}$. Show that for every $\varepsilon > 0$

(a) $\dfrac{Z_1 + \cdots + Z_n}{n^{1+\varepsilon}} \xrightarrow{\mathbb{P}} 0$ and (b) $\dfrac{Z_1 + \cdots + Z_n}{n^{2+\varepsilon}} \xrightarrow{\textbf{a.s.}} 0.$