2. **Gauss' Theorem:** Let $K \subset \mathbb{R}^3$ be a body with boundary ∂K oriented pointing outwards. If all the second partial derivatives of the vectorfield $\vec{F}$ exist and continuous on K then

$$\iint_{\partial K} \vec{F} d\vec{A} = \iiint_K \operatorname{div}(\vec{F}) dx dy dz.$$

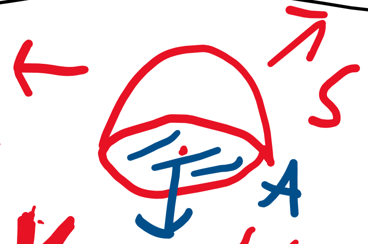
$$\left(\operatorname{div} \vec{F}(x,y,z) = \frac{\partial F_1}{\partial x}(x,y,z) + \frac{\partial F_2}{\partial y}(x,y,z) + \frac{\partial F_3}{\partial z}(x,y,z) \right)$$

EXTRA EXERCISE

$$\vec{F}(x,y,z) = (3x + y\cdot z^2,\ y + x,\ x^2+y^2)$$

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1,\ z > 0\}$$, ORIENTED OUTWARDS,

$$\iint_S \vec{F}\, d\vec{A} = ?$$



HALF SPHERE BODY ← $K = \{(x,y,z) : x^2+y^2+z^2 \le 1,\ z \ge 0\}$

$$\operatorname{div} \vec{F}(x,y,z) = 3 + 1 + 0 = 4$$

$A = \{(x,y,z) : x^2+y^2 \le 1,\ z = 0\}$, (ORIENTED DOWNWARDS)

$$\partial K = S \cup A$$

$$\boxed{\iint_S \vec{F}\, d\vec{A}} + \iint_A \vec{F}\, d\vec{A} = \iint_{\partial K} \vec{F}\, d\vec{A} = \boxed{\iiint_K \underbrace{\operatorname{div}\vec{F}}_{4}\, dx\,dy\,dz} = \underbrace{\text{VOLUME}}_{\frac{2\pi}{3}} \cdot 4 = \frac{8\pi}{3}$$

$$\vec{F}(x,y,z) = (\underbrace{3x+y\cdot z^2}, \underbrace{y+x}, \underbrace{x^2+y^2}) \quad = \quad + 1$$

$A = \{(x,y,z) : x^2+y^2 \le 1,\ z=0\}$
(ORIENTED DOWNWARDS)

$$r(u,v) = (\overbrace{u\cos v}^{x}, \overbrace{u\sin v}^{y}, \overbrace{0}^{z}), \quad \underbrace{\boxed{0 \le v \le 2\pi,\ 0 \le u \le 1}}_{T}$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\cdot\sin v & u\cdot\cos v & 0 \end{vmatrix} = i(0) - j(0) + k(u) =$$

$$= (0,0,u), \quad n(u,v) = (0,0,-u)$$

$$\iint_{\mathcal{F}} \vec{F}\,d\vec{A} = \iint_{T} \vec{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}(u,v)\,du\,dv. = \int_0^1\int_0^{2\pi} (?,?,u^2)\cdot(0,0,-u)\,dv\,du =$$

$$= \int_0^1\int_0^{2\pi} -u^3\cdot dv\,du = \int_0^1 -2\pi u^3\,du = -\frac{\pi}{2}$$

$$\boxed{\iint_S \vec{F}\, d\vec{A}} + \iint_A \vec{F}\, d\vec{A} = \iint_{\partial K} \vec{F}\, d\vec{A} = \boxed{\iiint_K \underbrace{\operatorname{div} \vec{F}}_{4}\, dx\, dy\, dz} = \frac{8\pi}{3}$$

$$\iint_A \vec{F}\, d\vec{A} = -\frac{\pi}{2}$$

(SEE THE PREVIOUS SLIDE)

HENCE

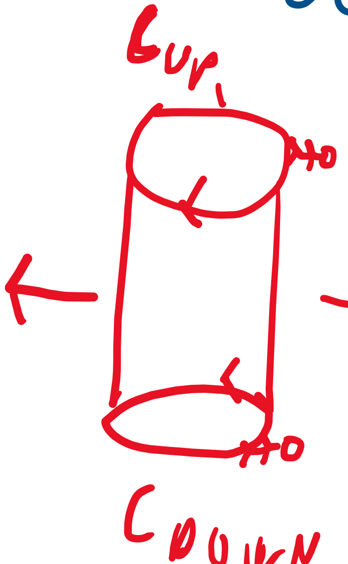
$$\iint_S \vec{F}\, d\vec{A} = \frac{8\pi}{3} + \frac{\pi}{2} = \frac{19}{6}\pi$$

3. **Stokes' Theorem**: Let $\mathcal{F}$ be an orientable surface and $\partial\mathcal{F}$ its boundary with coherent orientation. If all the partial derivatives of the vectorfield F exist and continuous on $\mathcal{F}$ then

$$\int_{\partial\mathcal{F}} \vec{F}\mathrm{dr} = \iint_{\mathcal{F}} \mathrm{curl}(\vec{F})d\vec{A}.$$

$\mathcal{F}$ AND $\partial\mathcal{F}$ ARE COHERENTLY ORIENTED: SMALL MAN WITH HEAD TOWARD THE ORIENTATION OF $\mathcal{F}$, HE GOES ON $\partial\mathcal{F}$ ACCORDING TO THE ORIENTATION $\partial\mathcal{F}$ THEN THE SURFACE IS ON THE LEFT SIDE

EXAMPLE FOR COHERENT ORI.

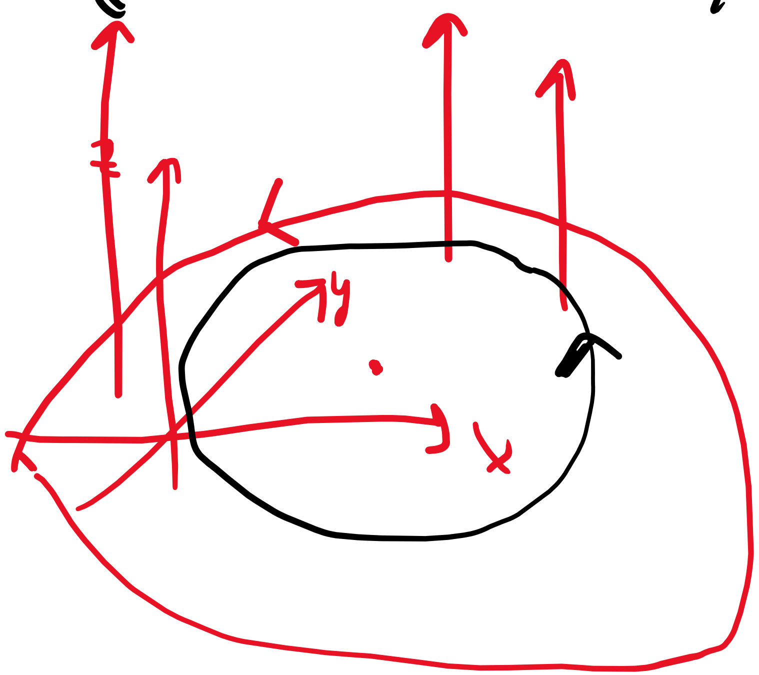


S: CYLINDRICAL SURFACE

$\partial S = C_{down} \cup C_{up}$

EXTRA EXERCISE $\vec{F}(x,y,z) = \left(x^2 y + e^x \sin z,\ \frac{x^3}{3} - e^{yz},\ \sin(xyz) \right)$

γ : CIRCLE LINE IN THE xy PLANE WITH RADIUS 6 CENTERED AT $(3,2)$ (WITH POSITIVE, i.e. ANTI-CLOCKWISE ORIENTATION)



$$\int_{\partial} \vec{F}\, d\vec{l} = \iint_{T} \operatorname{curl} \vec{F}\, d\vec{A} =$$

$$= \iint_{T} (?, ?, 0) \cdot (0, 0, m_i)\, du\, dv$$

curl $\vec{F}$

$$\iint_{\mathcal{F}} \vec{F}\, d\vec{A} = \iint_{T} \vec{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}(u,v)\, du\, dv.$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x^2 y + e^x \sin z, & \frac{x^3}{3} - e^{yz} & \sin(x \cdot y \cdot z) \end{vmatrix} = \begin{array}{l} \vec{i}\,(?) \\ -\vec{j}\,(?) \\ +\vec{k}\,(x^2 - x^2) \end{array}$$

PARAMETRIZATION OF γ:

$$r(u,v) = (3 + u \cdot \cos v,\ 2 + u \cdot \sin v,\ 0),\ \boxed{\begin{array}{l} 0 \le v \le 2\pi \\ 0 \le u \ 6 \end{array}}\ T$$

(IT WASN'T NECESSARY FOR THE SOLUTION)

4. **Green's Theorem**: Let T be a domain on the plane such that its boundary is a γ simple closed curve. If all the partial derivatives of the vectorfield $\vec{F}$ exist and continuous on T then

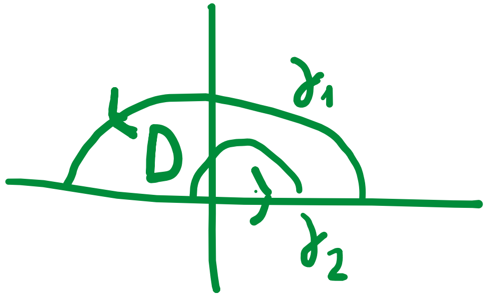
$$\int_\gamma \vec{F}\mathrm{d}\mathrm{r} = \iint_T Q'_x - P'_y dxdy, \text{ where } \vec{F}(x,y) = (P(x,y), Q(x,y)).$$

$(P(x,y), Q(x,y), 0)$

IT IS NOT A NEW THEOREM:

IF WE APPLY STOKES THM FOR PLANAR VECTOR FIELD WITH THE USUAL EXTENSION, THEN WE RECEIVE GREEN'S THM

EXTRA EXERCISE $F(x,y) = (y^2, 3xy)$, γ: boundary of the upper half circle around the origin with radius 1 (positively oriented)



$\gamma = \gamma_1 \cup \gamma_2$

$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

$$\oint_\gamma \vec{F}\, d\underline{r} = \iint_D 3y - 2y \, dx\,dy = \int_0^\pi \int_0^1 r \cdot \sin\varphi \cdot r \, dr \, d\varphi =$$

$$= \int_0^\pi \frac{\sin\varphi}{3} \, d\varphi = \boxed{\frac{2}{3}}$$