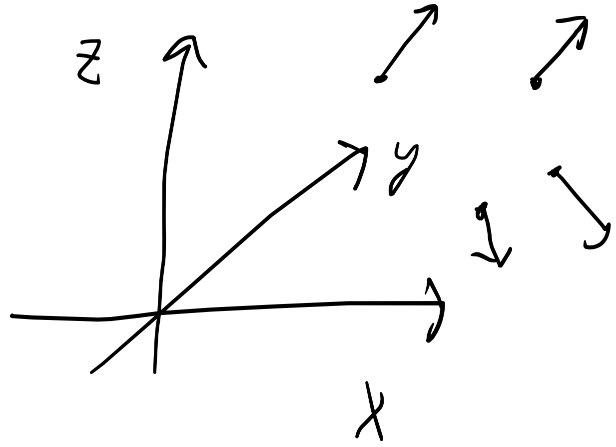
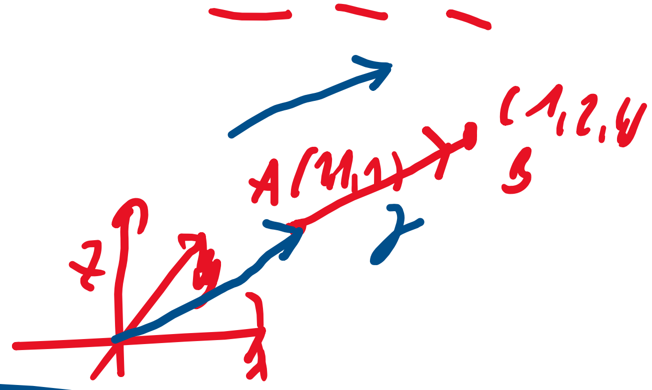
$$F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$$



$$F(x,y,z) = \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z) \right)$$

14. Let γ be the straight line connecting $A = (1,1,1)$ and $B = (1,2,4)$, and let $\vec{F}(x,y,z) = (2xyz, x^2z, x^2y)$. What is $\int_\gamma \vec{F}\,d\mathbf{r} = ?$



$\vec{F}(x,y,z) = (2xyz, x^2z, x^2y)$

$$\boxed{r(t) = A + t(B - A) = (1,1,1) + t(0,1,3) = (1, 1+t, 1+3t), \quad 0 \le t \le 1}$$

$$\int_\gamma \vec{F}(r)\,dr = \int_0^1 \vec{F}(r(t)) \cdot \dot{r}(t)\,dt =$$

$$= \int_0^1 \left(2(1+t)(1+3t),\ 1+3t,\ 1+t\right) \cdot (0, 1, 3)\,dt$$

$$= \int_0^1 \left(\overset{0}{2(1+t)(1+3t)},\ 1+3t,\ 1+t \right) \cdot (0,\ 1,\ 3)\, dt$$

$$= \int_0^1 0 + 1+3t + 3+3t \, dt = \int_0^1 6t+4 \, dt = 7$$

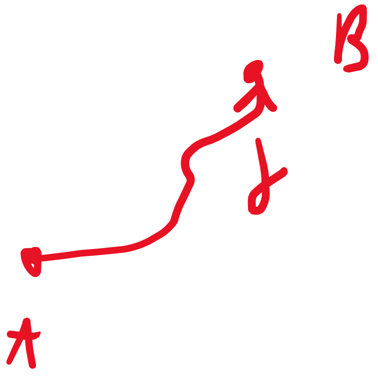
$f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\operatorname{grad} f : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{grad} f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$$

def: We call a vector field $\vec{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ conservative if there exists $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\vec{F} = \operatorname{grad} f$.
We call f the potential of $\vec{F}$.

def: We say that the line integral (work) of a vector field $\vec{F}$ is independent of the route if for every $A, B \in \mathbb{R}^3$ and for every curves γ_1 and γ_2 if both begins at A & ends at point B. Then $\int_{\gamma_1} \vec{F}(\underline{r}) d\underline{r} = \int_{\gamma_2} \vec{F}(\underline{r}) d\underline{r}$.

EQUIVALENT DEFINITIONS



IN A CONSERVATIVE F, $\vec{F} = \text{grad}\, f$

$$\int_\gamma \vec{F}\, d\vec{r} = f(B) - f(A)$$

$$\text{CURL}\ \vec{F} = \text{rot}\ \vec{F} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

CURL $\vec{F}$

IN CASE OF A 2D VECTOR SPACE

7. Let $\vec{F}(x,y,z) = (6x^2y - 4yz^3, 2x^3 - 4xz^3, -12xyz^2)$. Check that wether the vector field $\vec{F}$ is conservative or not! If it is, provide the potential function!

$A = (1,1,1)$

$B = (1,0,-1)$

AND CALCULATE THE WORK CORRESPONDING TO MOVING FROM A TO B

$\vec{F} = \text{grad } f$

$\frac{\partial}{\partial x} f(x,y,z) = 6x^2y - 4yz^3$

$f(x,y,z) = \int 6x^2y - 4y \cdot z^3 \, dx = 2x^3y - 4 \cdot y \cdot z^3 \cdot x + c_1(y,z)$

$\frac{\partial}{\partial y} f(x,y,z) = 2x^3 - 4xz^3$

$f(x,y,z) = \int 2x^3 - 4xz^3 \, dy = 2x^3 \cdot y - 4x \cdot z^3 y + c_2(x,z)$

$\frac{\partial}{\partial z} f(x,y,z) = -12x \cdot y \cdot z^2$, $f(x,y,z) = \int -12xy \cdot z^2 dz = -4 \cdot x \cdot yz^3 + c_3(x,y)$

$\int_{A \to B} \vec{F} d\vec{r} = f(B) - f(A) = \underbrace{f(1,0,-1)}_{0} - \underbrace{f(1,1,1)}_{-2} = \boxed{2}$

$f(x,y,z) = 2x^3y - 4y \cdot z^3 \cdot x$ IS A GOOD POTENTIAL FUNCTION

(HENCE $\vec{F}$ IS CONSERVATIVE)

7. Let $\vec{F}(x,y,z) = (6x^2y - 4yz^3, 2x^3 - 4xz^3, -12xyz^2)$. Check that wether the vector field $\vec{F}$ is conservative or not! If it is, provide the potential function!

$A = (1,1,1)$

$B = (1,0,-1)$

AND CALCULATE THE WORK CORRESPONDING TO MOVING FROM $\underline{A}$ TO $\underline{B}$

WE COULD HAVE TRIED TO USE THE CURL-TEST AS A FIRST STEP

Th (curl-test in $\mathbb{R}^3$)

$\vec{F}: \mathbb{R}^3 \mapsto \mathbb{R}^3 \quad \vec{F} = (F_1, F_2, F_3)$

$$\operatorname{rot}(\vec{F}) = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

1) if $\operatorname{rot}(\vec{F}) \neq \underline{0}$ at some point $\Rightarrow \vec{F}$ is <u>not</u> conservative

2) if $\operatorname{rot}(\vec{F}) = \underline{0}$ everywhere and $\forall F_i$ & and every partial derivatives are defined everywhere but ~~so~~ maybe except ~~[illegible]~~ finitely many points

SINGULARITY POINT:

THE BLUE CONDITIONS ARE NOT TRUE

IN $\mathbb{R}^3$ WE CAN HAVE FINITE SINGULARITY

AND CONTINUOUS

7. Let $\overrightarrow{F}(x, y, z) = (6x^2y - 4yz^3, 2x^3 - 4xz^3, -12xyz^2)$. Check that wether the vector field $\overrightarrow{F}$ is conservative or not! If it is, provide the potential function!

$A = (1, 1, 1)$

$B = (1, 0, -1)$

AND CALCULATE THE WORK CORRESPONDING TO MOVING FROM $\underline{A}$ TO $\underline{B}$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i^{+} & j^{-} & k^{+} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2y - 4yz^3, & 2x^3 - 4xz^3, & -12xyz^2 \end{vmatrix} = \left(-12xz^2 - (-12xz^2), -12yz^2 - (-12yz^2), 6x^2 - 4z^3 - (6x^2 - 4z^3) \right) =$$

$$= (0, 0, 0),$$

AND IT IS TRUE THAT WE HAVE ONLY FINITELY MANY SINGULARITY (NO SINGULARITY IN THIS CASE)

$\Longrightarrow$ CURL TEST CONCLUSIVE, IT REVEALS THAT $\vec{F}$ IS CONSERVATIVE

CURL TEST IN $\mathbb{R}^2$: IN THE SECOND STEP EVEN A SINGLE SINGULARITY

15. Using the planar version of the Curl-test, provide that the following vector fields are conservative or not. If it is then give the potential function. Give reasoning if the Curl-test is inconclusive

(a) $\vec{F}(x,y) = (-y, x)$, (c) $\vec{F}(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$

SHORT NOTATION $=$ IS A PREVISION $\left(\frac{+y}{r^2}, \frac{-x}{r^2}\right)$

(a) $\vec{F}(x,y) = (-y, x)$, $\operatorname{curl}\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1-(-1) = 2$

ACCORDING TO THE CURL TEST $\vec{F}$ IS NOT CONSERVATIVE

(c) $\underline{r} = (x,y)$, $r = \sqrt{x^2+y^2}$, $\vec{F}(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$

$$\operatorname{curl}\vec{F} = \frac{-1(x^2+y^2)-(-x)2x}{(x^2+y^2)^2} - \frac{1(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2} = 0$$

CURL IS 0 BUT THERE IS A SINGULARITY AT THE ORIGIN $\to$ CURL-TEST IS INCONCLUSIVE

16. Let $\overrightarrow{F}(x,y,z) = (x^{-2}, z, y)$. Is it possible to use the Curl-test? Try to guess the potential function! Is the vector field $\overrightarrow{F}$ conservative?

(a) $\text{CURL } \vec{F}(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x^2} & z & y \end{vmatrix} = (0,0,0)$

HOWEVER (x,y,z) IS A SINGULARITY IF $x = 0$, WE HAVE INFINITE SINGULARITIES $\Rightarrow$ INCONCLUSIVE

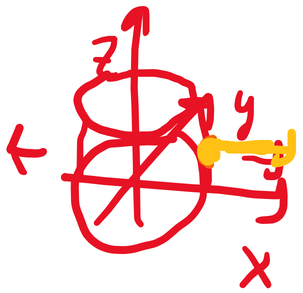
(b) $f(x,y,z) = -\frac{1}{x} + z \cdot y$: IT IS A GOOD POTENTIAL FUNCTION $\Rightarrow \vec{F}$ IS CONSERVATIVE

17. Let $\mathcal{S} = \{(x,y,z) : x^2+y^2 = 4 \text{ and } 0 \le z \le 2\}$ be cylindrical surface with normal vector pointing in the direction of the axes z. Let $\vec{F}(x,y,z) = (y+1, x, z^2)$. What is $\iint_{\mathcal{S}} \vec{F}\, d\vec{A} =$?

PARAMETRIZATION OF S

$$r(u,v) = (2\cdot\cos u,\ 2\cdot\sin u,\ v)$$

$$\boxed{0 \le u \le 2\pi,\quad 0 \le v \le 2 \ : T}$$



$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -2\cdot\sin u & 2\cdot\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \overbrace{(2\cdot\cos u,\ 2\cdot\sin u,\ 0)}^{n(u,v)}$$

$$\iint_T \vec{F}\, d\vec{A} = \int_0^{2\pi}\int_0^{2} (2\cdot\sin u + 1,\ 2\cdot\cos u,\ v^2)\cdot(2\cdot\cos u,\ 2\cdot\sin u,\ 0)\, dv\, du =$$

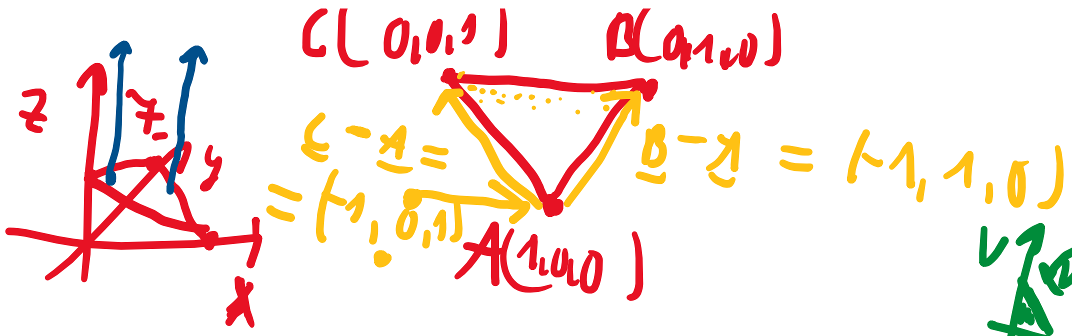
$$= \int_0^{2\pi} 2\cdot(4\cdot\sin u\cdot\cos u + 2\cos u + 4\cdot\cos u\cdot\sin u)\, du = \int_0^{2\pi} 16\cdot\sin u\cdot\cos u + 4\cos u\, du$$

$$= 0$$

$$\iint_{\mathcal{F}} \vec{F}\, d\vec{A} = \iint_T \vec{F}(\mathbf{r}(u,v))\cdot\mathbf{n}(u,v)\, du\, dv.$$

2. Let H be a triangle with vertices $A=(0,0,1), B=(1,0,0)$ and $C=(0,1,0)$. Consider the constant vector field $\overrightarrow{F}(x,y,z)=(1,2,3)$. What is $\iint_H \overrightarrow{F}\, d\overrightarrow{A} = ?$

SOLUTION 1



T: $u \geq 0,\ v \geq 0$, $u+v \leq 1$

$$\underline{r}(u,v) = \begin{pmatrix}1\\0\\0\end{pmatrix} + u\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + v\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} = (1-u-v,\ v,\ u)$$

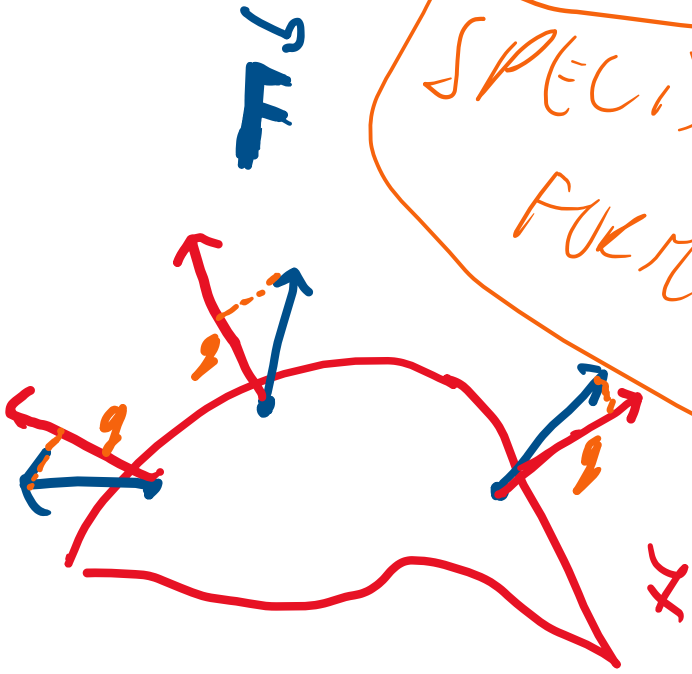
$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1,-1,-1) = \underline{r}_u \times \underline{r}_v \Rightarrow \underline{n}(u,v) = (1,1,1)$$

$$\iint \vec{F}\, d\vec{A} = \iint (1,2,3)\cdot(1,1,1)\, d\vec{A} = \int_0^1\int_0^{1-u} 6\, dv\, du = \int_0^1 6-6u\, du = \boxed{3}$$

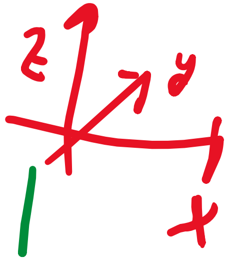
$$\iint_{\mathcal{F}} \overrightarrow{F}\, d\overrightarrow{A} = \iint_T \overrightarrow{F}(\mathbf{r}(u,v))\cdot \mathbf{n}(u,v)\, du\, dv.$$

$$\iint_{\mathcal{F}} \vec{F}\, d\vec{A} = \iint_{T} \vec{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}(u,v)\, du\, dv.$$

SPECIAL CASE FORMULA



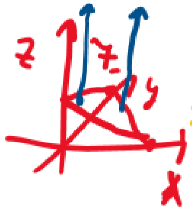
q : CONSTANT PROJECTION LENGTH



$$\iint_T \overbrace{\frac{\vec{F}(r(u,v)) \cdot n(u,v)}{|n(u,v)|}}^{q} \cdot \overbrace{|n(u,v)|}^{|r_u \times r_v|}\, du\, dv = q \cdot \text{SURFACE}(\mathcal{F})$$

2. Let H be a triangle with vertices $A=(0,0,1), B=(1,0,0)$ and $C=(0,1,0)$. Consider the constant vector field $\vec{F}(x,y,z)=(1,2,3)$. What is $\iint_H \vec{F}\,d\vec{A}=?$

$\boxed{\text{SOLUTION 1}}$



$C(0,0,1)$ $B(0,1,0)$

$A(1,0,0)$

$C-A=(-1,0,1)$ $B-A=(-1,1,0)$

$v=1-u$

T: $u\geq 0,\ v\geq 0$, $u+v\leq 1$

$$\underline{r}(u,v)=\begin{pmatrix}1\\0\\0\end{pmatrix}+u\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}+v\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}=(1-u-v,\ v,\ u)$$

$$\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & 0 & 1\\ -1 & 1 & 0\end{vmatrix}=(-1,-1,-1)=\underline{r}_u\times\underline{r}_v \Rightarrow \underline{n}(u,v)=(1,1,1)$$

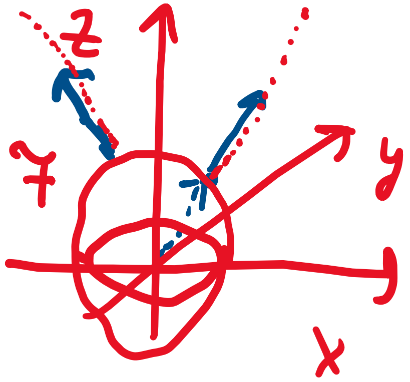
$1 \quad 1-u$

$$\text{FLUXUS} = q\cdot \underbrace{\text{TER}(\triangledown)}_{\frac{\sqrt{3}}{2}} = \boxed{3}$$

$$q=\frac{(1,1,1)\cdot(1,2,3)}{|(1,1,1)|}=\frac{6}{\sqrt{3}}$$

SOLUTION 2

18. Let $\overrightarrow{F}(\mathbf{r}) = \frac{1}{\|\mathbf{r}\|^3}\mathbf{r}$ and let $\mathcal{F}$ the surface of the unit ball with normal vector pointing out. What is $\iint_{\mathcal{F}} \overrightarrow{F}\, d\overrightarrow{A} = ?$



$$\left|\vec{F}(r)\right| = \left|\frac{1}{r^3}\cdot r\right| = \frac{1}{r^2} \overset{\text{UNIT SPHERE}}{=} \frac{1}{1^2} = 1$$

SPECIAL FORMULA $\therefore\ f = 1$

$\therefore\ f \cdot \text{SURFACE} = 1\cdot(4\cdot\pi\cdot 1^2) = 4\pi$

CALCULATION IS ALSO POSSIBLE

$r(u,v) = (\sin u\cdot\cos v,\ \sin u\cdot\sin v,\ \cos u)$

$T:\ 0\le v < 2\pi,\ 0\le u\le \pi$ .....