(b) $\begin{cases} u_{tt} = 2u_{xx}, & 0 < x < 5,\ 0 < t \\ u(x,0) = 3x, & 0 \le x \le 5 \\ u_t(x,0) = \cos(\frac{3\pi}{5}x)\sin(\pi x), & 0 < x < 5 \\ u(0,t) = u(5,t) = 0, & 0 \le t \end{cases}$

$L = 5$

$c = \sqrt{2}$

FINITE VIBRATING STRING

PARTIAL DIFFERENTIAL EQ.

$$\boxed{\begin{aligned} &g(x) = \cos\left(\frac{3\pi}{5}x\right)\sin(\pi x) = \frac{1}{2}\sin\left(\frac{2\pi}{5}x\right) + \frac{1}{2}\sin\left(\frac{8\pi}{5}x\right), \text{ HENCE} \\ &\frac{2\sqrt{2}\pi}{5}B_2 = \frac{1}{2}, \quad \frac{8\sqrt{2}\pi}{5}B_8 = \frac{1}{2}. \text{ ALL THE OTHER } B_i\text{'s ARE } 0 \end{aligned}}$$

INITIAL VELOCITY $f(x) = 3x,\ 0 \le x \le 5$ (USING THE CHEATING SHEET)

$$F(x) = 3 \cdot \frac{5}{\pi} f\left(\frac{\pi}{5}x\right) = \frac{15}{\pi} 2\left[\sin\left(\frac{\pi}{5}x\right) - \frac{\sin\left(2\cdot\frac{\pi}{5}x\right)}{2} + \frac{\sin\left(3\cdot\frac{\pi}{5}x\right)}{3} \dots\right]$$

$$\Rightarrow A_k = \frac{30}{\pi \cdot k}(-1)^{k-1}$$

HENCE THE SOLUTION IS:

$$u(x,t) = \frac{5}{4\sqrt{2}\pi} \sin\left(\frac{2\cdot\pi}{5}x\right) \sin\left(\frac{2\cdot\sqrt{2}\,\pi}{5}t\right)$$

$$+ \frac{5}{16\sqrt{2}\,\pi} \sin\left(\frac{8\cdot\pi}{5}x\right)\cdot\sin\left(\frac{8\cdot\sqrt{2}\cdot\pi}{5}\cdot t\right)$$

$$+ \sum_{k=1}^{\infty}\left[\frac{30}{\pi\cdot k}(-1)^{k-1}\sin\left(\frac{k\cdot\pi}{5}x\right)\cdot\cos\left(\frac{k\cdot\sqrt{2}\cdot\pi}{5}t\right)\right]$$

D'ALEMBERT FORMULA FOR VIBRATING STRING

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

IT ALSO WORKS FOR INFINITE STRING

36. The motion of an infinite string is given by:

$$\begin{cases} u_{tt} = 4u_{xx} \\ u(x,0) = \cos(3x) \\ u_t(x,0) = \frac{6x}{x^2+1} \end{cases}$$

INFINITE STRING

$\Rightarrow c = 2$

$: f(x)$

$: g(x)$

Determine its shape at t.

$c=2$

D'ALAMBERT

$$u(x,t) \overset{\text{D'ALAMBERT}}{=\!=} \frac{1}{2}\left(f(x+2t) + f(x-2t)\right) + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} g(u)\,du$$

$$= \frac{1}{2}\cos(3x+6t) + \frac{1}{2}\cos(3x-6t) + \underbrace{\frac{1}{4}\int_{x-2t}^{x+2t} \frac{6u}{u^2+1}\,du}_{\left[3\ln(u^2+1)\right]_{u=x-2t}^{u=x+2t}}$$

$$= \frac{1}{2}\cos(3x+6t) + \frac{1}{2}\cos(3x-6t) + \frac{3}{4}\ln\left[(x+2t)^2+1\right] - \frac{3}{4}\ln\left[(x-2t)^2+1\right]$$

5. On the interval $[0,2]$, there is a string. The initial form is given by $f \equiv 0$ function and the movement is described by the equation $u''_{tt} = u''_{xx}$. We hit this string with a hammer on the interval $[1/2, 3/2]$ from below with velocity 1. Give the shape of the string at time $t = 3.5$ with the method of D'Alambert. $f \equiv 0$

(Hint: the general formula $u(x,t) = \frac{1}{2}(\tilde{f}(x+ct) + \tilde{f}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(\tau)d\tau$, where $u''_{tt} = c^2 u''_{xx}$, $u(x,0) = f(x)$, $u'_t(x,0) = g(x)$.)

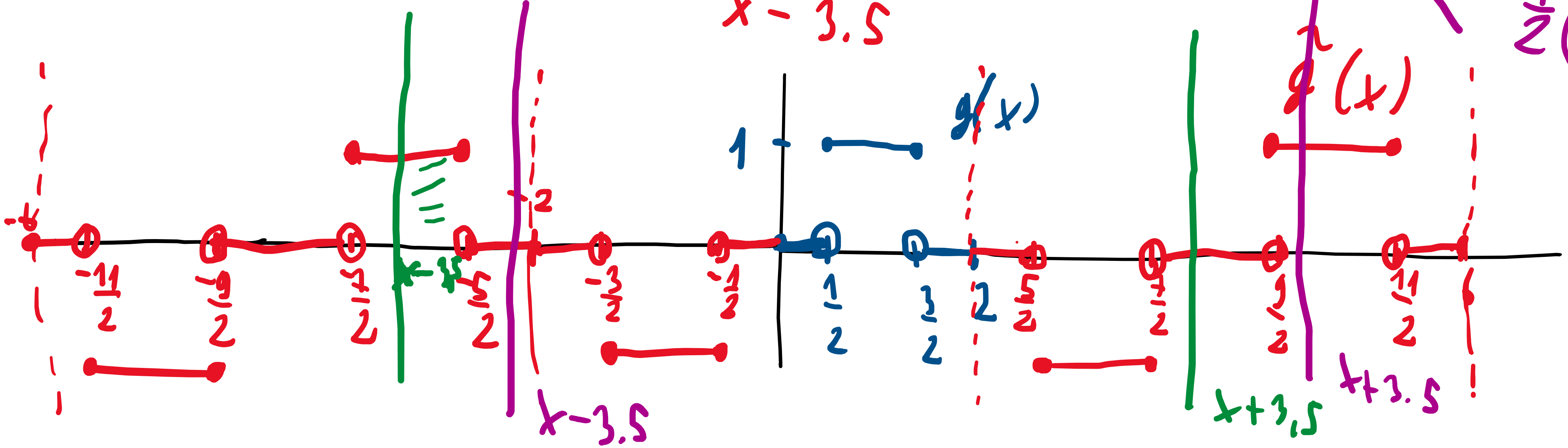
FOR FINITE STRING WE NEED THE NATURAL EXTENSIONS $\tilde{f}, \tilde{g}$

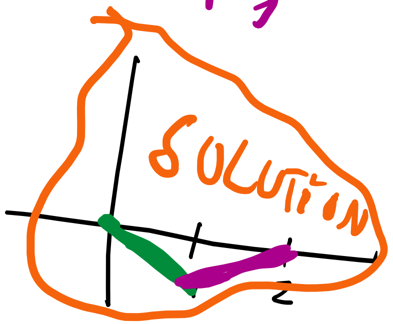
$0 < x < 2$ $c = 1$

$$u(x, 3.5) = \frac{1}{2}\int_{x-3.5}^{x+3.5} \tilde{g}(\tau)\, d\tau =$$

$0 < x < 1$: $\frac{1}{2}\left(\int_{-2.5}^{x-2.5} 1\, dx - 1\right) = \frac{x}{2}$ ... $1-x$

$1 < x < 2$: $\frac{1}{2}\left(-1 + \int_{4.5}^{x+3.5} 1\, dx\right) = \frac{x}{2} - 1$ ... $x-1$



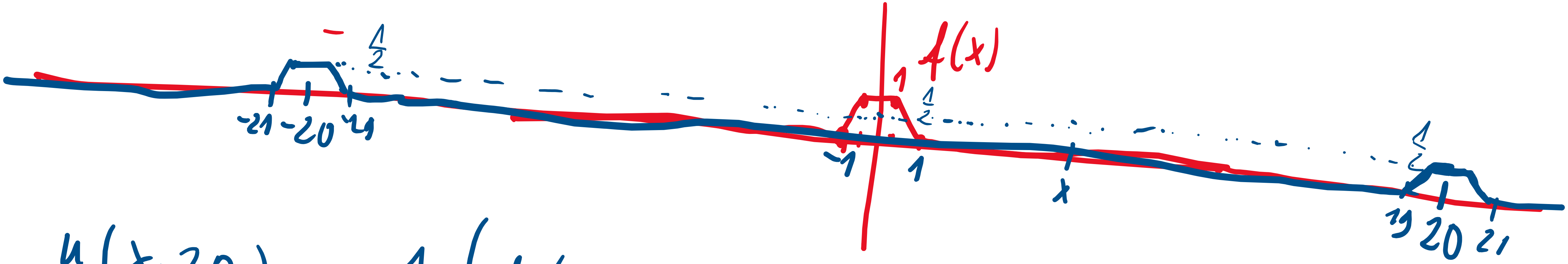


$t = 3.5$

6. Let us consider a string with infinite length. Hold down on the points $(-1,0)$, $(-1/2,1)$, $(1/2,1)$ and $(1,0)$ and release it. The movement of the string is described by the equation $u''_{tt} = u''_{xx}$. Describe the shape of the string at time $t = 20$!

$c = 1$, $g \equiv 0$ (INITIAL VELOCITY IS 0 EVERYWHERE)

RED CURVE: INITIAL SHAPE



$$u(x,20) = \frac{1}{2}\left(f(x+20) + f(x-20)\right)$$