Collection of formulas for the exam
Advanced Mathematics for civil engineers

$\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$, $\qquad \cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$, $\qquad \sinh^2 x = \frac{\cosh 2x - 1}{2}$

Notable derivatives

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\log_a x)' = \frac{1}{x \ln a}$

$(x^\alpha)' = \alpha x^{\alpha - 1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln(a)$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \frac{1}{\cos^2 x}$

$(\cot x)' = -\frac{1}{\sin^2 x}$

$(\ln x)' = \frac{1}{x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\operatorname{ar\,sinh} x)' = \frac{1}{\sqrt{1+x^2}}$

$(\operatorname{ar\,cosh} x)' = \frac{1}{\sqrt{x^2-1}}$

$(\operatorname{ar\,tanh} x)' = \frac{1}{1-x^2}$

$(\operatorname{ar\,coth} x)' = \frac{1}{1-x^2}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\operatorname{arc\,cot} x)' = -\frac{1}{1+x^2}$

Differentiation rules

$(cu)' = cu'$ $\qquad$ (c constant)

$(u+v)' = u' + v'$

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}$

Rules of Integration

$\int cf \, dx = c \int f \, dx$ $\qquad$ (c constant)

$\int (f+g) \, dx = \int f \, dx + \int g \, dx$

$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + c$,
where F is the primitive function of f

$\int f(g(x)) g'(x) \, dx = F(g(x)) + c$,
where F is the primitive function of f

$\int f^\alpha f' \, dx = \frac{f^{\alpha+1}}{\alpha+1} + c$, ha $\alpha \neq -1$

$\int \frac{f'}{f} \, dx = \ln |f| + c$

$\int uv' \, dx = uv - \int u'v \, dx$

Notable substitution of variables

$R(e^x)$	$e^x = t$
$R(\sqrt{ax+b})$	$\sqrt{ax+b} = t$
$R\left(\frac{\sqrt{ax+b}}{\sqrt{cx+d}}\right)$	$\frac{\sqrt{ax+b}}{\sqrt{cx+d}} = t$
$R(\sin x, \cos x)$	$\sin x, \cos x, \tan x, \tan \frac{x}{2} = t$
$R(x, \sqrt{a^2 - x^2})$	$x = a \sin t,\ x = a \cos t$
$R(x, \sqrt{a^2 + x^2})$	$x = a \sinh t$
$R(x, \sqrt{x^2 - a^2})$	$x = a \cosh t$

Notable integrals

$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + c$ $\qquad$ $(\alpha \neq -1)$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$

$\int a^x \, dx = \frac{a^x}{\ln a} + c$

$\int \cos x \, dx = \sin x + c$

$\int \sin x \, dx = -\cos x + c$

$\int \frac{1}{\cos^2 x} \, dx = \tan x + c$

$\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$

$\int \frac{1}{x} \, dx = \ln |x| + c$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar\,sinh} \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{ar\,cosh} \frac{x}{a} + c$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar\,tanh} \frac{x}{a} + c$, $\quad$ if $\left|\frac{x}{a}\right| < 1$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar\,coth} \frac{x}{a} + c$, $\quad$ if $\left|\frac{x}{a}\right| > 1$

$\int \tan x \, dx = -\ln |\cos x| + c$

$\int \cot x \, dx = \ln |\sin x| + c$

1. Linear algebra

1. Gram-Schmidt orthogonalization: Let $\{\underline{w}_1, \ldots, \underline{w}_k\}$ be a basis of the subspace $W \subset \mathbb{R}^d$. Then $\{\underline{v}_1, \ldots, \underline{v}_k\}$ forms an orthonormal basis of W, where

$$\underline{v}_1 = \frac{\underline{w}_1}{\|\underline{w}_1\|} \text{ and for } i = 2, \ldots, k \ \underline{v}_i = \frac{\underline{w}_i - \sum_{j=1}^{i-1}(\underline{v}_j \cdot \underline{w}_j)\underline{v}_j}{\|\underline{w}_i - \sum_{j=1}^{i-1}(\underline{v}_j \cdot \underline{w}_j)\underline{v}_j\|}.$$

2. Partial Differential equations

1. The sine-Fourier series of a function $f\colon [0, L] \mapsto \mathbb{R}$ is:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} \cdot x\right), \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L} \cdot x\right) dx.$$

2. $\int x\sin(ax)dx = \dfrac{\sin(ax) - ax\cos(ax)}{a^2} + c$ and $\int x^2\sin(ax)dx = \dfrac{2\cos(ax) + 2ax\sin(ax) - a^2x^2\cos(ax)}{a^3} + c$

3. Bernoulli's solution for the vibrating string problem:

$$\begin{cases} u''_{tt} = c^2 u''_{xx} & 0 < x < L \text{ and } 0 < t \\ u(0,t) = u(L,t) \equiv 0 & 0 < t \\ u(x,0) = f(x) & 0 < x < L \\ u'_t(x,0) = g(x) & 0 < x < L \end{cases}$$

then $u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right) \cdot \left(A_k \cos\left(\frac{kc\pi}{L}t\right) + B_k \sin\left(\frac{kc\pi}{L}t\right)\right)$, where A_k are the coefficients of the Fourier-sine series of $f(x)$ and $\frac{kc\pi}{L}B_k$ are the coefficients of the Fourier-sine series of $g(x)$.

4. Heat equation for finite rod:

$$\begin{cases} u'_t = \alpha u''_{xx} & 0 < x < L \text{ and } 0 < t \\ u(0,t) = u(L,t) \equiv 0 & 0 < t \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

then $u(x,t) = \sum_{k=1}^{\infty} A_k e^{-\left(\frac{k\pi}{L}\right)^2 \alpha t} \sin\left(\frac{k\pi}{L}x\right)$, where A_k are the coefficients of the Fourier-sine series of $f(x)$.

5. The Fourier-sine series of some functions:

   (a) $f(x) = x$ if $0 \le x < \pi$: $f(x) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots\right)$ for $0 \le x < \pi$

   (b) $g(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$: $g(x) = \frac{4}{\pi}\left(\sin(x) - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \cdots\right)$ for $0 \le x \le \pi$.

   (c) $h(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pi \end{cases}$: $h(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots\right)$ for $0 \le x \le \pi$

   (d) $\varphi(x) = x(\pi - x)$: $\varphi(x) = \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \frac{\sin(7x)}{7^3} + \cdots\right)$ for $0 \le x \le \pi$

3. Vectoranalysis

1. Let $\mathcal{A}$ be an orientable surface with parametrization $\mathbf{r}(u,v)$, where $(u,v) \in T$ for some domain T and $\vec{F}\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a vectorfield. Then

$$\iint_{\mathcal{A}} \vec{F} d\vec{A} = \pm \iint_T \vec{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}'_u \times \mathbf{r}'_v) du dv,$$

   where we choose $+$ if the orientation of $\mathcal{A}$ corresponds to $\mathbf{r}'_u \times \mathbf{r}'_v$, otherwise $-$.

2. **Gauss' Theorem**: Let $K \subset \mathbb{R}^3$ be a body with boundary ∂K oriented pointing outwards. If all the second partial derivatives of the vectorfield $\vec{F}$ exist and continuous on K then

$$\iint_{\partial K} \vec{F} d\vec{A} = \iiint_K \operatorname{div}(\vec{F}) dx dy dz.$$

3. **Stokes' Theorem**: Let $\mathcal{F}$ be an orientable surface and $\partial\mathcal{F}$ its boundary with coherent orientation. If all the partial derivatives of the vectorfield $\vec{F}$ exist and continuous on $\mathcal{F}$ then

$$\int_{\partial\mathcal{F}} \vec{F} d\mathbf{r} = \iint_{\mathcal{F}} \operatorname{curl}(\vec{F}) d\vec{A}.$$

4. **Green's Theorem**: Let T be a domain on the plane such that its boundary is a γ simple closed curve. If all the partial derivatives of the vectorfield $\vec{F}$ exist and continuous on T then

$$\int_\gamma \vec{F} d\mathbf{r} = \iint_T Q'_x - P'_y dx dy, \text{ where } \vec{F}(x,y) = (P(x,y), Q(x,y)).$$

5. Cylindrical substitution:

$$\begin{aligned} x &= r\cos(\varphi) \\ y &= r\sin(\varphi) \\ z &= z \end{aligned}$$

   with Jacobian determinant: r.

6. Spherical substitution:

$$\begin{aligned} x &= r\sin(u)\cos(v) \\ y &= r\sin(u)\sin(v) \\ z &= r\cos(u) \end{aligned}$$

   with Jacobian determinant: $r^2\sin(u)$.

7. Polar substitution on the plane:

$$\begin{aligned} x &= r\cos(v) \\ y &= r\sin(v) \end{aligned}$$

   with Jacobian determinant: r.