A1)

$$A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 2 & 1 & 1 & -4 \\ 3 & 2 & 4 & -11 \end{bmatrix} \xrightarrow[\mathrm{III}-3\mathrm{I}]{\mathrm{II}-2\mathrm{I}} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -10 \\ 0 & 2 & 10 & -20 \end{bmatrix} \xrightarrow{\mathrm{III}-2\mathrm{II}}$$

$$\to \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

a) $\mathrm{rank}(A) = 2$

$\Rightarrow \mathrm{rank}(A^T A) = 2$

$\Rightarrow \mathrm{nullity}(A) = 4 - \mathrm{rank}(A) = 2$

b) basis $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

A2) $P_1 = (1,2)$; $P_2 = (3,-1)$; $P_3 = (0,2)$; $P_4 = (-1,4)$

$y = ax + b \Rightarrow A\begin{pmatrix} a \\ b \end{pmatrix} = \underline{b} \Rightarrow \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 4 \end{pmatrix}$

$\Rightarrow A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T \underline{b} \Rightarrow \begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \dfrac{\begin{pmatrix} 4 & -3 \\ -3 & 11 \end{pmatrix}}{44-9} \Rightarrow$

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4}{35} & -\frac{3}{35} \\ \frac{-3}{35} & \frac{11}{35} \end{pmatrix}\begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} -\frac{41}{35} \\ \frac{92}{35} \end{pmatrix} \Rightarrow y = -\frac{41}{35}x + \frac{92}{35}$

A3) $B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \Rightarrow B^T B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$

$\Rightarrow \det\begin{pmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{pmatrix} = (2-\lambda)(8-\lambda) - 16 = \lambda^2 - 10\lambda = 0$

$\Rightarrow \begin{matrix} \lambda_1 = 10 \\ \lambda_2 = 0 \end{matrix} \Rightarrow \begin{matrix} \alpha_1 = \sqrt{10} \\ \alpha_2 = 0 \end{matrix} \Rightarrow \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}$

For λ_1 $\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\underline{v}_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = 2a \quad \Rightarrow \underline{v}_1 = \begin{pmatrix} a \\ 2a \end{pmatrix}$

$\Rightarrow a^2 + (2a)^2 = 1 \Rightarrow a = \frac{\sqrt{5}}{5} \Rightarrow \underline{v}_1 = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{pmatrix} \Rightarrow \underline{V}_2 = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{pmatrix}$

$\Rightarrow V = \begin{pmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix}$

$\Rightarrow \underline{u}_1 = \frac{1}{\alpha_1} A \underline{v}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \Rightarrow \underline{u}_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

$\Rightarrow U = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \Rightarrow B = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix}$

$U \Sigma V^T$

A4/ $\begin{cases} u''_{tt} = 4u''_{xx} \\ u(0,x) = 0 = f(x) \\ u'_t(0,x) = \left.\begin{cases} 1 & x \in [0,3] \\ -1 & x \in [3,6] \end{cases}\right\} = g(x) \\ u(t,0) = u(t,6) = 0 \end{cases}$

$\Rightarrow c = 2 \;\&\; L = 6 \Rightarrow$

$$u(t,x) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi}{3}t\right) + B_n \sin\left(\frac{n\pi}{3}t\right)\right) \cdot \sin\left(\frac{n\pi}{6}x\right)$$

$u(0,x) = 0 \Rightarrow A_n = 0 \quad \forall n \in \mathbb{N}$

$$B_n = \frac{2}{n \cdot 2\pi} \int_0^6 g(x) \sin\left(\frac{n\pi}{6}x\right) dx = \frac{1}{n\pi} \int_0^3 \sin\left(\frac{n\pi}{6}x\right) dx$$

$$+ \frac{1}{n\pi} \int_3^6 \sin\left(\frac{n\pi}{6}x\right) dx = \frac{1}{n\pi} \left[-\cos\left(\frac{n\pi}{6}x\right) \cdot \frac{6}{n\pi}\right]_0^3 +$$

$$\frac{1}{n\pi} \left[-\cos\left(\frac{n\pi}{6}x\right) \frac{6}{n\pi}\right]_3^6 = \frac{1}{n\pi} \left(\frac{6}{n\pi} - \frac{\cos(n\pi) \cdot 6}{n\pi} \right) =$$

$$= \frac{6}{n^2\pi^2} (1 - \cos(n\pi)) = \begin{cases} \frac{12}{(2k+1)^2\pi^2} & n = 2k+1 \\ 0 & n = 2k \end{cases}$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} \frac{6(1-\cos(n\pi))}{n^2\pi^2} \sin\left(\frac{n\pi}{3}t\right) \cdot \sin\left(\frac{n\pi}{6}x\right) =$$

$$= \sum_{k=0}^{\infty} \frac{12}{(2k+1)^2\pi^2} \cdot \sin\left(\frac{(2k+1)\pi}{3}t\right) \cdot \sin\left(\frac{(2k+1)\pi}{6}x\right).$$