# Probability Theory 2

## I. Midterm test

**MT.1** *(10 points)* Thanos killed exactly half of the population with his magic ring in a second. Suppose that there were $2N$ people and exactly $N$ men and $N$ women. The ring choose between people uniformly, that is, it killed every subset of the population with $N$ element with equal probabilities. Let $X_N$ the ratio of men in the survivor population. Show that $X_N$ converges to $1/2$ in probability as $N \to \infty$.

**MT.2** *(10 pont)* We put an amoebae into a Petri dish. The amoebaes divide in every minute according to the following rule: at the end of every odd minute an amoebae divides into $k$ many amoebaes with probability $p_1(k)$, $k \geq 0$ (so it can die), and at the end of every even minute, an amoebae divides with probability $p_2(k)$ to $k$ many amoebaes ($k \geq 0$). So after $2n+1$ minutes every amoebaes divides independently with prob. $p_1(k)$ into $k$ amoebaes and the same happens independently with probability $p_2(k)$ after $2n$ minutes.
(a) Denote $G_1(x)$ and $G_2(x)$ the probability generating functions of the distributions $p_1$ and $p_2$ respectively. Find the prob. generating function of the $n$th generation! (We put the amoebae into the dish in the 0th minute.)
(b) Let $p_1$ be $GEO(1/3)$ (i.e. $p_1(k) = \frac{1}{3}\left(\frac{2}{3}\right)^k$ for $k \geq 0$), and let $p_2$ be $GEO(3/4)$ ($p_2(k) = \frac{3}{4}\left(\frac{1}{4}\right)^k$ for $k \geq 0$). Find the probability that the amoebaes eventually die out.

**MT.3** *(10 pont)* Let $X$ be the uniform random variable on the set $\{0, 1, \ldots, n-1\}$. Show that if $n$ is not a prime then there exist independent $\mathbb{N}$-valued random variables $Y, Z$ such that $Y + Z = X$. (Hint: Decompose with respect to the remaining classes.)