1. Solve the following system of linear equations with Gauss elimination!

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6 \end{array}$$

2. Is the collection of the following vectors linearly independent?

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$.
   (a) What are the coordinates of the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ in basis B? (That is, $[\mathbf{v}]_B = ?$)
   (b) Find the coordinate transformation $P_{N,B}$ (That is, from the natural basis to B)
   (c) Find the matrix of the linear transform $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 3x_2 \end{bmatrix}$ in the basis B!
4. Draw the points on the plane, which satisfy the equation $5x^2 - 4xy + 8y^2 = 36$!
5. Let P be the orthogonal projection to line $y = \frac{\sqrt{3}}{2}x$. Find the matrix of P in the natural basis (i.e. $N = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$).

6. Let L be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Find a basis of L out of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$! Give the coordinates of the remaining vectors, which are not contained in this basis, in the basis you have found!

(b) Find an orthonormal basis of L by using the Gram-Schmidt orthogonalisation algoritm!

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Give a symmetric matrix S and a skew-symmetric matrix G such that $A = S + G$!

8. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$. Find $A : B$!

9. Give the spectral decomposition of the matrix $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$!

10. Let $P = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Find the eigenvalues and the eigenvectors of P!

11. Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 7 & -1 & 0 \end{bmatrix}$

(a) Find the subspaces $\text{row}(A), \text{col}(A), \text{null}(A), \text{null}(A^T A), \text{null}(A^T)$!

(b) Find $\text{nullity}(A), \text{rank}(A)$ and $\text{rank}(A^T A)$!

(c) Check that $\text{row}(A)^\perp = \text{null}(A)$ and $\text{col}(A)^\perp = \text{null}(A^T)$!

12. Let $B = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$. Show that there exists a subspace V such that B is the matrix of the orthogonal projection to the subspace V in the natural basis! What is V?
13. Find the matrix in the natural basis of the orthogonal projection to the plane $V = \{(x, y, z) : 2x - y + 3z = 0\}$! By using this matrix, decompose the vector $\mathbf{v} = (2, 4, -1)$ into perpendicular and paralell components with respect to V!
14. Let $C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix}$. Moreover, let P_r be the matrix of the orthogonal projection from $\mathbb{R}^4$ to $\mathrm{row}(A)$ and let P_c be the matrix of the orthogonal projection from $\mathbb{R}^3$ to $\mathrm{col}(A)$! Find P_r and P_c!
15. With the method of smallest squares, find the line, which approximates the points $(2, 1), (3, 2), (5, 3)$ and $(6, 4)$ the best!
16. Find the solution with the method of smallest squares of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

17. Let $B = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}$. Show that B is positive definite, and find a matrix C such that $C^2 = B$.
18. Let $C = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$. Find the singular value decomposition of C!