Reference: Thomas' Calculus Ch 15.7-8

Def! $S \subset \mathbb{R}^d \quad f: \mathbb{R}^d \to \mathbb{R}$

$\int_S f \, dV$ = mass/weight of S with density given by f

In particular, if $f \equiv 1 \Rightarrow$ volume of S

if S has special shape $\Rightarrow$ change of coordinates

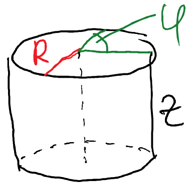
Cylindrical coordinates, | Spherical coordinates,

parameters

(r, φ, z)

$\varphi \in [0, 2\pi)$

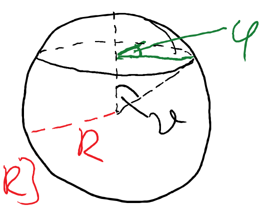
$r \in [0, R]$



(r, φ, ϑ)

$\varphi \in [0, 2\pi]$

$\vartheta \in [0, \pi)$ $r \in [0, R]$



substitutions

$x = r\cos\varphi$ } polar-coord.
$y = r\sin\varphi$
$z = z$

$x^2 + y^2 = r^2$

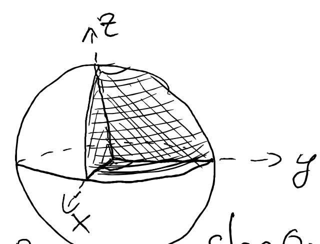
$x = r\cos\varphi \sin\vartheta$
$y = r\sin\varphi \sin\vartheta$
$z = r\cos\vartheta$

$x^2 + y^2 + z^2 = r^2$

$\boxed{r}\, dr\, d\varphi\, dz \quad \leftarrow \quad dx\,dy\,dz$ Jacobi-det. $\quad \rightarrow \quad \boxed{r^2 \sin\vartheta}\, dr\, d\varphi\, dz$

Ex1 $S := \{(x, y, z) : \underbrace{x^2 + y^2 + z^2 \le 1}_{\text{unit sphere}}, \quad x, y, z > 0\}$

$f(x, y, z) := \sqrt{x^2 + y^2 + x^2}$ $\qquad \iiint_S f(x,y,z)\,dx\,dy\,dz = ?$



step 0: draw

step1: determine which coordinates to use, determine range of parameters

here: spherical $\quad 0 \le r \le 1 \quad 0 \le \varphi \le \pi/2 \quad 0 \le \vartheta \le \pi/2$

step2: write $f(x,y,z)$ in new coordinates

here: $f(x, y, z) = \sqrt{r^2} = r$

step3: integrate with substitution

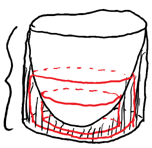
$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \int^1 \int^{\pi/2} \int^{\pi/2} r \cdot r^2 \sin\vartheta \, d\vartheta\, d\varphi\, dr$$

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \int\limits_{\substack{0\\(r)}}^{1}\int\limits_{\substack{0\\(\varphi)}}^{\pi/2}\int\limits_{\substack{0\\(\vartheta)}}^{\pi/2} r\cdot r^2\sin\vartheta\,d\vartheta\,d\varphi\,dr$$

$$= \underbrace{\left(\int_0^1 r^3\,dr\right)}_{1/4}\cdot\underbrace{\left(\int_0^{\pi/2} 1\cdot d\varphi\right)}_{\pi/2}\cdot\underbrace{\left(\int_0^{\pi/2}\sin\vartheta\,d\vartheta\right)}_{\left[-\cos\vartheta\right]_0^{\pi/2}} = \underline{\underline{\pi/8}}$$

<u>Ex2</u>

$$T := \left\{(x,y,z) : \begin{array}{l} x^2+y^2\le 1 \\ 0\le z\le x^2+y^2 \end{array}\right\}$$

1  $\Longrightarrow$ cylindrical coords

$f(x,y,z) := x^2+y^2 = r^2$ in cylindrical

z depends on x,y
$\Longrightarrow$ inner most in integration

$$\iiint_T f\,dx\,dy\,dz = \int\limits_{\substack{0\\(r)}}^{1}\int\limits_{\substack{0\\(\varphi)}}^{2\pi}\int\limits_{\substack{0\\(z)}}^{r^2} r^2\cdot r\,dz\,d\varphi\,dr$$

$$= \left(\int_0^{2\pi} 1\,d\varphi\right)\cdot\int_0^1 r^3\left(\int_0^{r^2} 1\cdot dz\right)dr = 2\pi\cdot\int_0^1 r^5\,dr = \underline{\underline{\pi/3}}$$

Reference: Thomas' Calculus Ch. 16.2

Def. vector field $\vec{F} = \vec{F}(x,y,z) = \big(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\big)$ is an $\mathbb{R}^3 \to \mathbb{R}^3$ mapping, i.e. a vector (F_1, F_2, F_3) is assigned to each point of 3d space

Notation position vector $\underline{r} = (x,y,z)$ $\quad r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$

Ex. ↳ $f(x,y,z) := x^2 y \sin z$ is NOT a vector field, since f assigns a number/scalar to each point of space ⇝ scalar field

But $\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xy\sin z, x^2 \sin z, x^2 y \cos x)$ is a vector field ⇝ gradient field (important later)

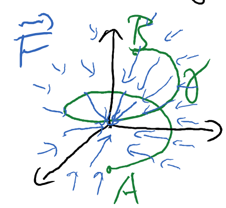
↳ physical interpretations:

force field: $\vec{F}(\underline{r})$ gives the force in each point $\underline{r}$

e.g. electromagnetic field or gravitational force, for some $k>0$

$$\vec{F}(\underline{r}) = -k\frac{\underline{r}}{r^3} \quad \text{or} \quad \vec{F}(x,y,z) = \left(\frac{-kx}{(x^2+y^2+z^2)^{3/2}}, \frac{-ky}{(x^2+y^2+z^2)^{3/2}}, \frac{-kz}{(x^2+y^2+z^2)^{3/2}}\right)$$

Question given a curve γ in vector field $\vec{F}$, how much work is needed to get from one endpoint A of γ to the other B?



line integral

velocity field: $\vec{F}(\underline{r})$ gives the velocity at each point $\underline{r}$

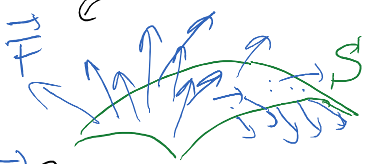
e.g. flow of air in a wind tunnel or water in a pipeline

spin/rotation around z-axis (parallel to xy-axis) at constant speed ω (angular velocity)

$\vec{F}(x,y,z) = (-\omega y, \omega x, 0)$

$F(x,y,z) = (-\omega y, \omega x, 0)$ velocity

Question How much fluid leaves/enters the surface of S if the fluid is flowing according to the velocity field $\vec{F}$?



surface integral

2. Line Integrals of Vector Fields

Tuesday, October 31, 2017 1:14 PM

Reference: Thomas' Calculus Ch. 16.2-3

Def. line integral γ is a directed, parameterized curve in space given by

$$\underline{r}(t) = (x(t), y(t), z(t)) \quad a \leq t \leq b \qquad A = \underline{r}(a) \text{ starting point}, \quad B = \underline{r}(b) \text{ endpoint}$$

$\vec{F} = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ is a vector field (interpret as force field)

allways assume γ is smooth, depends "nicely" on t, similarly all components of $\vec{F}$ and partial derivates depend "nicely" on x, y, z

$$\int_\gamma \vec{F}(\underline{r})\, d\underline{r} \overset{\text{other notation}}{=} \int_\gamma F_1\, dx + F_2\, dy + F_3\, dz = \text{work done in moving an object from } A \text{ to } B \text{ along } \gamma \text{ against } \vec{F}$$

Thm. evaluating a line integral according to parametrization $\underline{r}(t)$

$$\boxed{\int_\gamma \vec{F}(\underline{r})\, d\underline{r} = \int_a^b \vec{F}(\underline{r}(t)) \cdot \dot{\underline{r}}(t)\, dt}$$

localize $\vec{F}$ to γ

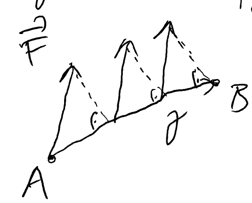
scalar product of the two vectors

tangent/velocity vector $= \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right)$

explanation

↳ first assume $\begin{cases} \gamma \text{ is a line segment between } A \text{ and } B \quad \& \\ \vec{F} \equiv (F_1, F_2, F_3) \text{ is constant, i.e. independent of } x, y, z \end{cases}$

amount of force in the direction of the movement = orthogonal projection of $\vec{F}$ onto $\overrightarrow{AB}$



$= \vec{F} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ (recall from lin. algebra)

$\vec{F}$ is constant $\Rightarrow$ this is true for every point on γ

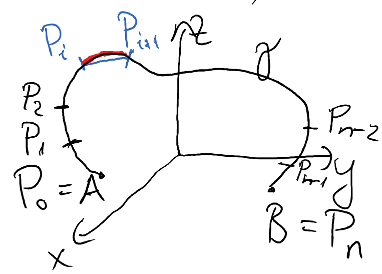
total movement is $|\overrightarrow{AB}| \Longrightarrow$ work $= \vec{F} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} |\overrightarrow{AB}| = \vec{F} \cdot \overrightarrow{AB}$ velocity

↳ for general curve γ and vector field $\vec{F}$:

idea is to cut up γ into small pieces which are "almost" line segments and $\vec{F}$ changes only "little"; apply previous point

continue refining this subdivision to get from $\sum_i$ to integration
a bit more formally:

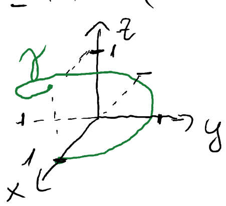
- subdivide the parameter interval $[a,b]$ into n pieces
  $$a =: t_0 < t_1 < \ldots < t_{n-1} < t_n := b \quad \text{let } P_i := \underline{r}(t_i)$$
- replace $(\underline{r}(t_i), \underline{r}(t_{i+1}))$ on γ with the line segment $\overrightarrow{P_i P_{i+1}}$
- the work on this small segment is approx
  $$\approx \vec{F}(\underline{r}(t_i)) \cdot \overrightarrow{P_i P_{i+1}}$$
  according to the previous point if we take $\vec{F}$ constant on $\overrightarrow{P_i P_{i+1}}$
  but $\overrightarrow{P_i P_{i+1}} = \underline{r}(t_{i+1}) - \underline{r}(t_i) \approx \dot{\underline{r}}(t_i) \cdot (t_{i+1} - t_i)$



- Thus the work on the whole curve is approx.
  $$\approx \sum_{i=0}^{n-1} \vec{F}(\underline{r}(t_i)) \cdot \dot{\underline{r}}(t_i) \cdot \underbrace{(t_{i+1} - t_i)}_{\sim dt} \xrightarrow[\text{refining the subdivision}]{n \to \infty} \int_a^b \vec{F}(\underline{r}(t)) \cdot \dot{r}(t)\, dt$$

'□'

Ex:
$$\left.\begin{array}{l} \vec{F}(x,y,z) = (-y, z, 2x) \\ \underline{r}(t) = (\cos t, \sin t, t) \quad t \in [0, 2\pi] \end{array}\right\} \Rightarrow \begin{array}{l} \vec{F}(\underline{r}(t)) = (-\sin t, t, 2\cos t) \\ \dot{\underline{r}}(t) = (-\sin t, \cos t, 1) \end{array}$$

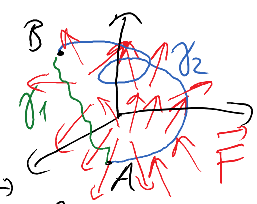


$$\int_\gamma \vec{F}(\underline{r})\, d\underline{r} = \int_\gamma -y\,dx + z\,dy + 2x\,dz$$

$$= \int_0^{2\pi} \Big(\underbrace{\sin^2 t}_{\frac{1-\cos 2x}{2}} + \underbrace{t\cos t}_{\text{int. by parts}} + 2\cos t\Big)\, dt = \left[\frac{t}{2} - \frac{\sin 2t}{4} + t\sin t + \cos t + 2\sin t\right]_0^{2\pi} = \underline{\underline{\pi}}$$

Path independence

Question: Given two different curves γ_1 and γ_2 between the same endpoints A and B, for what $\vec{F}$ will the work be the same along the two different curves?

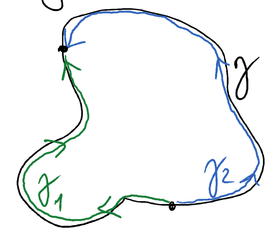


Def^{n}: The line integral is independent of the path in $\vec{F}$ if for any two curves γ_1 and γ_2 (within the domain of $\vec{F}$) with the same starting point A and endpoint B $\quad \int_{\gamma_1} \vec{F}(\underline{r})\,d\underline{r} = \int_{\gamma_2} \vec{F}(\underline{r})\,d\underline{r}$. (1)

In this case we call $\vec{F}$ a conservative field.

- Assume $\vec{F}$ is a gradient field, i.e. $\vec{F} = \operatorname{grad} f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$.
  We call f the potential function for $\vec{F}$.

Remark 1. Condition (1) holds if and only if for any closed loop γ
$\oint_\gamma \vec{F}(\underline{r})\,d\underline{r} = 0$. Indeed, any closed γ
can be written as the union of γ_1 and γ_2
so that $\oint_\gamma \vec{F}\,d\underline{r} = \int_{\gamma_1} \vec{F}\,d\underline{r} - \int_{\gamma_2} \vec{F}\,d\underline{r}$



- In a conservative field $\int_\gamma \vec{F}\,d\underline{r}$ only depends on the endpoints of γ.

Thm 2. Assume $\vec{F}$ is a vector field on some domain of space with continuous components. Then

$$\boxed{\vec{F} \text{ is conservative} \iff \vec{F} \text{ is a gradient field for some differentiable potential } f.}$$

In this case, for any smooth curve γ in the domain of $\vec{F}$
parameterized by $\underline{r}(t)$, $a \le t \le b$,
joining points $A = \underline{r}(a)$ with $B = \underline{r}(b)$

$$\boxed{\int_\gamma \vec{F}(\underline{r})\,d\underline{r} = f(B) - f(A)} \quad (2)$$

Remark 2. If $\vec{F}$ is a gradient field, then (2) shows that the line integral only depends on the value of the potential function at the endpoints.
Thus independent of the path itself $\Rightarrow$ conservative.
Other direction (conservative $\Rightarrow$ gradient field) slightly more involved.

proof (of (2)) Simple application of the Newton-Leibnitz formula
$\vec{F} = \operatorname{grad} f$ for some potential, γ parameterized by $\underline{r}(t)$ $a \le t \le b$
Define $g : [a,b] \to \mathbb{R}$ $\quad g(t) := f(\underline{r}(t))$. By the chain rule

$$\frac{d}{dt} g(t) = (\operatorname{grad} f)(\underline{r}(t)) \cdot \dot{\underline{r}}(t) = \vec{F}(\underline{r}(t)) \cdot \dot{\underline{r}}(t).$$

Define $g:[a,b]\to\mathbb{R}$, $g(t)=f(\underline{r}(t))$, by the chain rule

$$\frac{d}{dt}\,g(t)=(\operatorname{grad} f)(\underline{r}(t))\cdot\dot{\underline{r}}(t)=\vec{F}(\underline{r}(t))\cdot\dot{\underline{r}}(t).$$

Then by the Newton-Leibniz formula

$$f(B)-f(A)\overset{\text{def. of } g}{=}g(b)-g(a)=\int_a^b g'(t)\,dt=\int_a^b \vec{F}(\underline{r}(t))\cdot\dot{\underline{r}}(t)\,dt\overset{\text{Thm}}{=}\int_\gamma \vec{F}\,d\underline{r}$$

$\square$

Question: How to determine whether $\vec{F}$ is conservative and if it is, then how to determine the potential function of $\vec{F}$?

3. Curl-test and finding a potential

Wednesday, November 1, 2017 11:20 AM

Def 1. curl / rotation of a vector field $\vec{F}$ notation $\text{curl}(\vec{F})$, $\text{rot}(\vec{F})$

$\nabla \vec{F} = \text{grad}\, \vec{F}$

If $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is defined in the plane

$\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$ then $\boxed{\text{curl}(\vec{F}) = \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1}$ (is a real number)

If $\vec{F}(x,y,z) = (F_1, F_2, F_3)$ is defined in $\mathbb{R}^3$, then $\text{rot}(\vec{F})$ is the vector

$$\boxed{\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2 ,\ \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3,\ \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right)}$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \text{"symbolic determinant"} \quad = \nabla \times \vec{F}$$

∂_x is shorthand for $\frac{\partial}{\partial x}$

Remark 1. $\text{curl}\, \vec{F}$ has to do with how the fluid circulates close to the point (x,y,z)

Observe rot in the plane is the same as the third component of rot in space.

If it is zero (number or vector), then there is no rotation.

In the plane if $\text{rot}(\vec{F}) > 0 \rightsquigarrow$ rotation counter clockwise

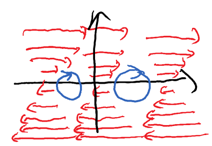
$< 0 \rightsquigarrow$ rotation clockwise

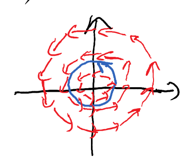
also called circulation density

Ex 1. • In the plane

$\vec{F}(x,y) = (-\omega y, \omega x)$ rotation, $\omega > 0$ $\text{curl}(\vec{F}) = \omega - (-\omega) = 2\omega > 0$

$\vec{F}(x,y) = (-y, 0)$ shear, $\text{curl}(\vec{F}) = -1 < 0$





• In space

$\vec{F}$ = gravitational field $= -k \cdot \frac{\underline{r}}{r^3}$

$\text{curl}\, \vec{F} = 3(x^2+y^2+z^2)^{5/2} \cdot k\,(yz - yz, -xz + xz, xy - xy) \equiv (0,0,0)$ no rotation

in general let $f: \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable, $\vec{F} := \text{grad} f$

then $\text{curl}\, \vec{F} = \nabla \times \nabla f = (f''_{zy} - f''_{yz}, f''_{xz} - f''_{zx}, f''_{yx} - f''_{xy}) \equiv (0,0,0)$

$\Longrightarrow$ If $\vec{F}$ is conservative, then $\text{rot}\, \vec{F}$ must be $\equiv (0,0,0)$.

Curl test In the plane and space gives sufficient (but not necessary) conditions for a vector field to be conservative.

1) We just saw: If $\operatorname{rot}(\vec{F}) \neq 0$ or $(0,0,0)$ in at least one point then $\vec{F}$ is <u>NOT</u> conservative.

(ex. rotation and shear are not conservative as we expect)

2) $\operatorname{rot}\vec{F} \equiv 0$ or $(0,0,0)$ and

(a) in the plane: $\vec{F}$ and the partial derivatives $\partial_x F_1, \partial_y F_1, \partial_x F_2, \partial_y F_2$ are defined at <u>every</u> point of the plane $\Longrightarrow \vec{F}$ <u>IS</u> conservative

If this fails (at least one of the partial derivatives isn't defined in at least one point) $\Longrightarrow$ test is <u>INCONCLUSIVE</u>

(b) in space: $\vec{F}$ and all nine partial derivates are defined in all, but except perhaps a <u>finite</u> number of points of space $\Longrightarrow \vec{F}$ <u>IS</u> conservative
ex. just the origin

If this fails (ex. a line or plane going through the origin) $\Longrightarrow$ test is <u>INCONCLUSIVE</u>

Ex 2. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 2 \right)$ is it conservative?

First calculate $\operatorname{rot}\vec{F}$:

$$\operatorname{rot}(\vec{F}) = \Big(\underbrace{\partial_y F_3 - \partial_z F_2}_{=0}, \underbrace{\partial_z F_1 - \partial_x F_3}_{=0}, \frac{\overbrace{x^2+y^2-2x^2-(-(x^2+y^2)-(-y)2y)}^{=0}}{(x^2+y^2)^2} \Big) = (0,0,0) \checkmark$$

<u>But</u> $\vec{F}$ is not defined where $x^2+y^2=0$, i.e. on the z-axis
$\Rightarrow$ curl test is inconclusive

First two coordinates $(-y, x)$ resemble the spin/rotation field $\rightsquigarrow$ guess: not cons.

From Remark 2.1 we know that $\vec{F}$ not cons. $\Longleftrightarrow \exists$ closed γ : $\oint_\gamma \vec{F}\,d\underline{r} \neq 0$.

take the curve γ parameterized by $\underline{r}(t) = (\cos t, \sin t, 0)$ $0 \le t \le 2\pi$

Then $\vec{F}(\underline{r}(t)) = \left(\frac{-\sin t}{1}, \frac{\cos t}{1}, 2 \right)$, $\dot{\underline{r}}(t) = (-\sin t, \cos t, 0)$

Thus $\oint_\gamma \vec{F}\, dr = \int_0^{2\pi} (\underbrace{\sin^2 t + \cos^2 t}_{=1})\, dt = 2\pi \neq 0 \Longrightarrow \vec{F}$ is NOT conservative

Finding a potential function

$\vec{F} = (F_1, F_2, F_3)$ is given

$$\vec{F} = (f'_x, f'_y, f'_z) \quad f = ? \qquad f(x,y,z) = \begin{cases} \int F_1(x,y,z)\, dx + g_1(y,z) \\ \int F_2(x,y,z)\, dy + g_2(x,z) \\ \int F_3(x,y,z)\, dz + g_3(x,y) \end{cases}$$

combining these we can determine g_1, g_2, g_3 thus obtain $f(x,y,z)$

Ex3. $\vec{F}(x,y,z) = (1+4y+5z,\ 2+4x,\ 3+5x)$ is it cons? If yes, give a potential!

$\text{rot}(\vec{F}) = (0-0,\ 5-5,\ 4-4) \equiv (0,0,0)$, everything is well-defined $\Rightarrow$ YES cons.

potential:

$$\begin{aligned} \int F_1\, dx &= x + 4xy + 5xz + g_1(y,z) \\ \int F_2\, dy &= 2y + 4xy \qquad\quad + g_2(x,z) \\ \int F_3\, dz &= 3z + 5xz \qquad\quad + g_3(x,y) \end{aligned}$$

$g_1(y,z) = 2y + 3z$

$g_2(x,z) = 5xz + x + 3z$

$g_3(x,y) = 4xy + x + 2y$

$$\Rightarrow \underline{\underline{f(x,y,z)}} = x + 2y + 3z + 4xy + 5xz + \text{const.}$$

Surface Integrals

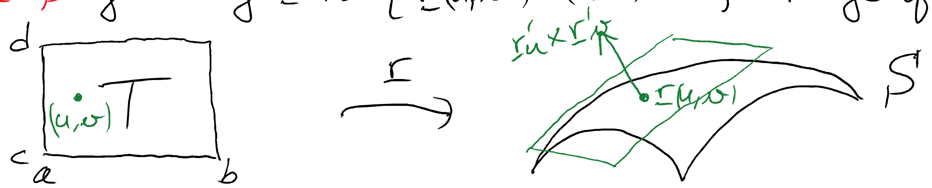
Reference Thomas' Calculus Ch 16.5-6

Surface given by parametrization

parameters $a \le u \le b$, $c \le v \le d$ $\quad T = [a,b] \times [c,d]$ = parameter domain

parametrization $\underline{r} : T \to \mathbb{R}^3$ $\quad \underline{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

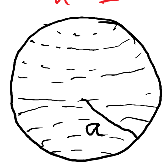
surface S given by $\underline{r}$ is $\{ \underline{r}(u,v) : (u,v) \in T \}$ = range of $\underline{r}$



tangent plane at point $\underline{r}(u,v)$ is spanned by $\begin{cases} \underline{r}'_u = (\partial_u x, \partial_u y, \partial_u z) \\ \underline{r}'_v = (\partial_v x, \partial_v y, \partial_v z) \end{cases}$

normal vector at $\underline{r}(u,v)$ is $\underline{r}'_u \times \underline{r}'_v$

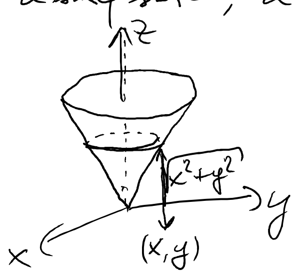
Ex sphere $x^2+y^2+z^2=a^2$



with the spherical coords. $r = a$ is fixed
$0 \le \varphi \le 2\pi \quad 0 \le \vartheta \le \pi$

Thus $\underline{r}(\varphi, \vartheta) = (a \cos\varphi \sin\vartheta, \ a \sin\varphi \sin\vartheta, \ a \cos\vartheta)$

cone $\quad z = \sqrt{x^2+y^2} \quad 0 \le z \le 1$



cylindrical coords.
$0 \le r \le 1 \quad 0 \le \varphi \le 2\pi$
$z = r$

Thus $\underline{r}(r,\varphi) = (r\cos\varphi, r\sin\varphi, r)$

Def surface integral Assume a surface $S \subset \mathbb{R}^3$ is given by the parametrization $\underline{r}(u,v)$ on some domain $T \subset \mathbb{R}^2$. S can be oriented by either taking $\underline{n}(u,v) = \underline{r}'_u \times \underline{r}'_v$ or $\underline{n}(u,v) = -\underline{r}'_u \times \underline{r}'_v$ at every point. $\vec{F}$ is a "nice" vector field (at least twice differentiable).

Then the surface integral $\equiv$ flux

$\iint_S \vec{F}\, d\vec{A}$ = the volume of fluid which flows through S in unit time according to the velocity field $\vec{F}$ in the direction of the orientation.

$$\text{flux} \begin{cases} > 0 \Rightarrow \text{more fluid flows in the direction of orientation.} \\ < 0 \Rightarrow \text{more in the opposite direction} \\ = 0 \Rightarrow \text{it is even both directions} \end{cases}$$

Thm Depending on the orientation of S

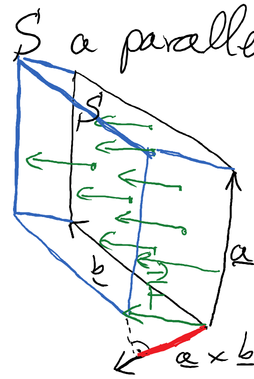
$$\boxed{\iint_S \vec{F}\, d\vec{A} = \iint_T \underbrace{\vec{F}(\underline{r}(u,v)) \overset{\text{scalar product}}{\cdot} \underline{n}(u,v)}_{(*)}\, du\, dv} \quad (1)$$

In partical, if $(*)$ is the same constant q for every $(u,v) \in T$ (that is, the length of the orthogonal projection of $\vec{F}$ onto the normal vector (w.r.t. the orientation) $\underline{n}$ is the same)

then $$\boxed{\iint_S \vec{F}\, d\vec{A} = q \cdot \iint_T 1 \cdot du\, dv = q \cdot \text{surface}(S)} \quad (2)$$

explanation very similar to the line integral

First consider $\vec{F}$ constant everywhere and S a parallelogram spanned by the vectors $\underline{a}$ and $\underline{b}$.

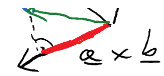


Flux = volume of blue paralelepipedon

$= \text{area}(S) \cdot \underbrace{\text{height}}_{\text{red line}} = |\underline{a} \times \underline{b}| \cdot \vec{F} \cdot \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$\Rightarrow \iint_S \vec{F}\, d\vec{A} = \vec{F} \cdot (\underline{a} \times \underline{b})$

$$\Rightarrow \iint_S \vec{F}\, d\vec{A} = \vec{F} \cdot (\underline{a} \times \underline{b})$$



In general the usual method. Split the surface S into many very small pieces. Instead of the piece, consider the corresponding parallelogram in the tangent space. On this small parallelogram we consider $\vec{F}$ constant. Hence we are in the previous setting. Summing up over all small pieces and refining the subdivision we get

$$\iint_S \vec{F}\, d\vec{A} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum \vec{F}(\underline{r}(u_0, v_0)) \cdot (\underline{r}_u' \times \underline{r}_v')\, \Delta u \Delta v = \iint_T \vec{F}(\underline{r}(u,v)) \cdot (\underline{r}_u' \times \underline{r}_v')\, du\, dv$$

If the orientation points towards $-(\underline{r}_u' \times \underline{r}_v')$, the flux $= (-1) \cdot$ this

'(1)'

Ex1 (to use (1)) Let S be the surface given by the parametrization

$\underline{r}(u,v) = (u+2v, -v, u^2+3v)$ with domain T: $0 \le u \le 3$ & $0 \le v \le 1$.

S is oriented so that $\underline{n}$ points "upwards" (i.e. its third component is positive)

$\vec{F}(x,y,z) = (xy, 2x+y, z)$. Determine $\iint_S \vec{F}\, d\vec{A}$!

↳ step1: lokalize $\vec{F}$ to the surface, i.e. $x = u+2v$, $y = -v$, $z = u^2+3v$

$$\vec{F}(\underline{r}(u,v)) = ((u+2v)(-v),\ 2(u+2v)-v,\ u^2+3v)$$

↳ step2: determine the normal vector according to the orientation given

$$\underline{r}_u' \times \underline{r}_v' = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 2u \\ 2 & -1 & 3 \end{vmatrix} = (+2u, 4u-3, -1) \xrightarrow[\text{other direction}]{\text{need}} \underline{n}(u,v) = (-2u, 3-4u, 1)$$

↳ step3: calculate the scalar product

$$\vec{F}(\underline{r}(u,v)) \cdot \underline{n}(u,v) = (2u^2v + 4uv^2) + (6u + 9v - 8u^2 - 12uv) + (u^2 + 3v)$$
$$= 6u + 12v - 7u^2 + 2u^2v + 4uv^2 - 12uv$$

↳ step4: use formula (1)

$$\iint_S \vec{F}\, d\vec{A} = \int_0^3 \int_0^1 (6u + 12v - 7u^2 + 2u^2v + 4uv^2 - 12uv)\, dv\, du$$

↳ step 4: use formula (1)

$$\iint_S \vec{F}\, d\vec{A} = \int_{0\,(u)}^{3}\int_{0\,(v)}^{1} \left(6u+12v-7u^2+2u^2v+4uv^2-12uv\right) dv\, du$$

$$= 6\cdot\frac{9}{2} + 12\cdot\frac{3}{2} - 7\frac{3^3}{3} + 2\frac{3^3}{3}\cdot\frac{1}{2} + 4\frac{3^2}{2}\cdot\frac{1}{3} - 12\frac{3^2}{2}\cdot\frac{1}{2} = \underline{\underline{-30}}$$

Ex2 when you can use (2) If $\vec{F}$ is constant & S is a plane figure

$A=(1,0,1)$ S is the triangle determined by A, B, C,

$B=(1,1,1)$ oriented "downwards" $\vec{F}(x,y,z)\equiv(5,4,3)$

$C=(2,0,3)$

According to (2) we need area(S) & $\perp$proj. of $\vec{F}$ onto $\underline{n}$.

we need two sides of the triangle $\begin{cases} \underline{b} := \vec{AB} = (0,1,0) \\ \underline{c} := \vec{AC} = (1,0,2) \end{cases}$

$$\left.\begin{aligned} &\text{then } \operatorname{area}(S) = \tfrac{1}{2}\,|\underline{b}\times\underline{c}|, \\ &|\perp\text{proj}| = \vec{F}\bullet\frac{\underline{b}\times\underline{c}}{|\underline{b}\times\underline{c}|} \end{aligned}\right\} \Longrightarrow \frac{1}{2}\cdot\cancel{|\underline{b}\times\underline{c}|}\cdot\vec{F}\bullet\frac{\underline{b}\times\underline{c}}{\cancel{|\underline{b}\times\underline{c}|}}$$

$$= \frac{1}{2}\,\underbrace{\vec{F}\bullet(\underline{b}\times\underline{c})}_{\text{triple product}}$$

$$\begin{vmatrix} F_1 & F_2 & F_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

check the orientation!

$$\underline{b}\times\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = (2,0,-1)$$ ✓ ok since 3rd component <0.

Thus $\text{flux} = \frac{1}{2}\cdot\begin{vmatrix} 5 & 4 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = \frac{1}{2}\cdot(10-3) = \underline{\underline{\frac{7}{2}}}$

Reference: Thomas's Calculus Ch. 16.8

Gauss' divergence theorem

Gives an alternative way to calculate flux through a closed surface.

Def 1. divergence of $\vec{F} = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ is the scalar function

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3 \quad \left(\overset{\text{notation}}{=\!=} \operatorname{grad}\cdot\vec{F} = \nabla\cdot\vec{F}\right)$$

Thm 1. Gauss / divergence theorem

Let ∂K be a piecewise smooth closed surface in space, which encloses the body K. ∂K is oriented outwards. $\vec{F}$ is a vector field defined at every point of K (and whose components have continuous partial derivatives). Then

$$\boxed{\iint_{\partial K} \vec{F}\, d\vec{A} = \iiint_K \operatorname{div} F\, dV.}$$

Ex 1. physical interpretation of divergence

assume K is a ball around a point (x,y,z) with very tiny radius

then we can say that $\operatorname{div} F$ is approximately constant

The divergence thm states

$$\operatorname{div}\vec{F}(x,y,z) \approx \frac{\iint_{\partial K} \vec{F}\, d\vec{A}}{\text{volume}(K)} = \frac{\text{flux "outwards"}}{\text{volume}}$$

$$\text{If } \operatorname{div} F(x,y,z) \begin{cases} > 0 & \Longrightarrow (x,y,z) \text{ is a source} \\ < 0 & \Longrightarrow (x,y,z) \text{ is a sink} \\ = 0 & \Longrightarrow \text{same amount flows in and out} \end{cases}$$

Another way to look at it: $\vec{F}$ is the velocity field of flowing gas then $\operatorname{div}\vec{F}$ gives the rate at which the gas is compressing (<0) or expanding (>0) at point (x,y,z).

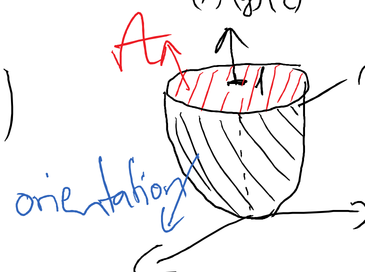
Ex2 Let K be the unit cube with surface ∂K pointing "outwards". $\vec{F}(x,y,z) = (xy, yz, xz)$ Determine $\iint_{\partial K} \vec{F}\, d\vec{A}$!

We can use the divergence thm ($\vec{F}$ is nice) instead of having to calculate the flux for all 6 sides of ∂K.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}xy + \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}xz = x+y+z \quad \text{so}$$

$$\iint_{\partial K} \vec{F}\, d\vec{A} \overset{\text{Gauss}}{=} \iiint_K (x+y+z)\, dV = \underset{(x)}{\int_0^1} \underset{(y)}{\int_0^1} \underset{(z)}{\int_0^1} (x+y+z)\, dz\, dy\, dx = 3\int_0^1 x\, dx = \underline{\underline{\frac{3}{2}}}$$

Ex3. $\vec{F} = (yz, xz, x^2+y^2)$



$\mathcal{F}$: rotate $z = x^2$ around z axis $(0 \le x \le 1)$

Use the divergence thm to calculate $\iint_{\mathcal{F}} \vec{F}\, d\vec{A}$!

$\mathcal{F}$ is NOT a closed surface, but $\mathcal{F} \cup A$ IS! We can use the divergence thm. for $\partial K = \mathcal{F} \cup A$, with A oriented "upward".

$$\iiint_K \operatorname{div}\vec{F}\, dV = \iint_{\mathcal{F}} \vec{F}\, d\vec{A} + \iint_A \vec{F}\, d\vec{A}$$

$$\operatorname{div}\vec{F} \equiv 0 \implies \iint_{\mathcal{F}} \vec{F}\, d\vec{A} = -\int_A \vec{F}\, d\vec{A}$$ enough to calculate this!

- $A = \{(x,y,z) : x^2+y^2 \le 1, z = 1\}$ parametric: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$

$0 \le r \le 1$, $0 \le \varphi \le 2\pi$

so $\underline{r}(r,\varphi) = (r\cos\varphi, r\sin\varphi, 1) \implies \vec{F}(\underline{r}(r,\varphi)) = (r\sin\varphi, r\cos\varphi, r^2)$

$\left.\begin{array}{l}\underline{r}'_r = (\cos\varphi, \sin\varphi, 0) \\ \end{array}\right\} \implies \underline{r}'_r \times \underline{r}'_\varphi = (0,0,r)$

$$\left.\begin{array}{l} \underline{r}'_r = (\cos\varphi,\ \sin\varphi,\ 0) \\ \underline{r}'_\varphi = (-r\sin\varphi,\ r\cos\varphi,\ 0) \end{array}\right\} \Longrightarrow \underline{r}'_r \times \underline{r}'_\varphi = (0,0,r)$$

this is in the direction of the orientation

$$\Downarrow$$

$$\iint_A \vec{F}\, d\vec{A} = \int_{\underset{(r)}{0}}^{1}\int_{\underset{(\varphi)}{0}}^{2\pi} \vec{F}(\underline{r})\cdot \underline{n}\, d\varphi\, dr = \int_0^1\int_0^{2\pi} r^2\cdot r\, d\varphi\, dr = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}$$

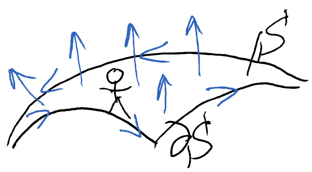
$$\text{Thus } \iint_{\tilde{F}} \vec{F}\, d\vec{A} = \underline{\underline{-\frac{\pi}{2}}}$$

Stokes' theorem

Reference: Thomas' Calulus Ch. 16.7

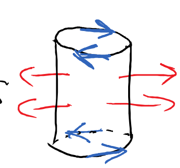
Relates the surface integral of rot $\vec{F}$ to the line integral around the boundary of S in counterclockwise direction.

Def2 A surface S and its boundary ∂S are directed coherently, if someone walks around ∂S so that he/she is standing according to the orientation of S, then S is allways on the left-hand side of the person.



Another way to say: if the thumb of a right-hand points in the direction of the orientation of S, then the fingers curl in the direction of ∂S.

Ex4 S = surface of a cylinder, orientation outwards



How to orient ∂S so that it is coherent with orientation of S? on top clockwise
on bottom counterclockwise

Thm2 Stokes' theorem $\vec{F}$ is a vector field.

Assume S and ∂S are directed coherently, furthermore, the partial derivatives of $\vec{F}$ are defined in <u>every</u> point of S.

Then $$\boxed{\iint_S \operatorname{curl}(\vec{F})\, d\vec{A} = \oint_{\partial S} \vec{F}\, d\underline{r}}$$

<u>Remark</u> Most of the time ∂S consists of a single curve. But more than one is possible, see Ex6. Even ∂S = empty set is possible: think S = surface of a sphere. Then $\oint_{\partial S} \vec{F}\, d\underline{r} = 0$, so $\iint_S \operatorname{curl}(\vec{F})\, d\vec{A} = 0$ for any nice $\vec{F}$.

<u>Ex5</u> $S = \{(x,y,z) : x^2+y^2 \le 1,\ z=0\}$ unit disk on xy-plane oriented with normal vector pointing "upward"

∂S = unit circle oriented counterclockwise

$\vec{F} = \frac{1}{\sqrt{x^2+y^2}}(y, -x, 3z)$ $\quad \oint_{\partial S} \vec{F}(\underline{r})\, d\underline{r} = ?$

Can we use Stokes' thm? <u>NO</u>, because $\vec{F}$ is <u>not</u> defined at the origin. $\operatorname{rot}(\vec{F}) = (\text{something}, \text{something}, 0)$

$\underline{n} = (0, 0, \text{something})$

$\Rightarrow \operatorname{rot}(\vec{F}) \cdot \underline{n} = 0 \Rightarrow \iint_S \operatorname{rot}(\vec{F})\, d\vec{A} = 0$

But $\oint_{\partial S} \vec{F}(\underline{r})\, d\underline{r}$ can be computed directly. $= -2\pi$

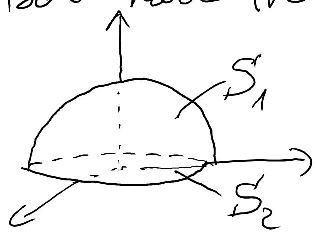
exercise for home

<u>Ex6</u> $S_1 := \{(x,y,z) : x^2+y^2+z^2 = 9,\ z \ge 0\}$ hemisphere

$S_2 := \{(x,y,z) : x^2+y^2 \le 9,\ z = 0\}$ disk on xy-plane, radius $=3$

Both have the <u>same</u> boundary $\partial S = \{(x,y,z) : x^2+y^2 = 9,\ z=0\}$

Both have the <u>same</u> boundary $\partial S = \{(x,y,z) : x^2+y^2=9, z=0\}$



$\vec{F} := (y, -x, 0)$ Assume S_1, S_2, and ∂S are oriented coherently.

$$\overset{\text{Stokes}}{\Longrightarrow} \iint_{S_1} \operatorname{rot}\vec{F}\, d\vec{A} = \iint_{S_2} \operatorname{rot}\vec{F}\, d\vec{A} = \oint_{\partial S} \vec{F}\, d\underline{r}$$

Calculation of $\oint_{\partial S} \vec{F}\, dr$: ∂S $\quad \underline{r}(t) = (3\cos t, 3\sin t, 0)$

$$\dot{\underline{r}}(t) = (-3\sin t, 3\cos t, 0)$$

$$\oint_{\partial S} \vec{F}\, d\underline{r} = \int_0^{2\pi} 3\sin t(-3\sin t) + (-3\cos t)3\cos t\, dt = \int_0^{2\pi} -9\, dt = \underline{\underline{-18\pi}}$$

<u>Remark</u> physical interpretation of $\operatorname{rot}(\vec{F})$

Let P be a point in space, S_ε a disk around P with a $\underbrace{\text{very tiny}}$ radius ε and unit normalvector $\underline{n}$.

$\rightarrow$ can take $\operatorname{rot}(\vec{F}) \approx \operatorname{rot}(\vec{F}(P))$ constant on S_ε

$$\overset{\text{Stokes}}{\Longrightarrow} \underbrace{\frac{\underline{n}}{|\underline{n}|} \cdot \operatorname{rot}(\vec{F}(P))}_{\text{orthogonal proj. of } \operatorname{rot}(\vec{F}(P)) \text{ onto } \underline{n}} \approx \frac{\oint_{\partial S_\varepsilon} \vec{F}\, dr}{\operatorname{area}(S_\varepsilon)}$$