Abstract

Szűcs Tamás

December 15, 2023

In this thesis we investigate regular graphs with even girth. In particular we obtain a lower bound (called the Moore bound) on how many vertices such a graph can have, and then we want to find out with which parameters is it possible to reach this lower bound. Also we investigate the case if we do not quite reach this lower bound but, we exceed it with only a few vertices.

For the case when we reach the Moore bound, we first show a strong relation between these minimal graphs and generalized n-gons, namely, if such a graph exist then, it has to be the incidence graph of a generalized n-gon. Then the main part of the thesis is the proof of the Feit-Higman theorem, a strong result about generalized n-gons. We approach the proof from a graph theoretic perspective, also we try to present a more detailed version compared to what we find in the Feit-Higman article.

The proof uses a lot of different techniques such as recursion, a certain polynomial representation, linear algebra and some number theory. The main idea of the proof is that we form $M = QQ^T$, where Q is the incidence matrix of the graph, and notice that the powers of M can be represented by the number of walks of certain length between the vertices of the graph, and we give a formula for that. With these techniques in hand we determine an annihilating polynomial of M and an eigenvalue of M, and we calculate the multiplicity of this eigenvalue. We get that the formula we obtain for the multiplicity is rational only in some cases, and since the multiplicity has to be rational, this excludes a lot of cases.

In the case when we exceed this lower bound by a few vertices, we obtain around the beginning that we have to exceed the bound with at least 2 vertices, and we will focus on

that case, when the excess is 2. To be more exact, we narrow down our focus on the case when the girth of the graph is 6, since for bigger even girths Biggs and Ito showed in the article that there are no such graphs, so for the excess 2 case the problem is open only for girth 6. In this case using Payne's approach (that is very similar to the techniques of Feit and Higman), we can calculate the minimal polynomial of M directly and we obtain all the eigenvalues of M, from that we calculate the determinant of M, and we get that it has to be a perfect square and with this information, we can obtain quite strong restrictions.

Lastly together with some number theoretic criteria we write a computer program that checks the possible valency values in the excess 2 case and we obtain that up to k less than $\sqrt{k}$ possible values remain.