Linear algebraic bound methods in combinatorics

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Excerpt

In this thesis we will present linear algebraic methods used to achieve bounds of cardinality in extremal combinatorical situations. Throughout almost the entire Thesis, the general setting is that there are some sets with some conditions about their intersections. Through characteristic vectors, assigned polynomials, incidence, intersection and inclusion matrices we will present proofs for the cardinality of their family. The structure of the Thesis is as follows.

In the first chapter, we present the two main ideas: independence through orthogonality of characteristic vectors, and independence through the rank of the incidence matrix. This is illustrated with the *Oddtown-problems*, where four different variations of the same problem are solved.

In the second chapter, the key definitions and lemmas are laid out. Here we discuss orthogonality over finite fields, which is important when dealing with modular problems, since the proofs of such almost always contain a transitioning to finite fields. We give simple criteria to the independence of a system of polynomials, given their values on a set are known. The nonsingularity of the matrix $J + diag(\gamma_1, \ldots, \gamma_n)$ is proven.

In the third chapter, we present a method in which we use polynomials assigned to the elements of a set (or equivalently, to subsets in a family of subsets), and through their independence we prove a bound for the cardinality of their set. We present the trick of Blokhuis for independence of polynomials, where we show that even an extended set of polynomials is independent. We describe a proof of the Ray-Chaudhuri $-$ Wilson theorem. In the fourth chapter, we generalize the idea of an incidence matrix to the inclusion matrix. Then introducing useful statements regarding the inclusion matrix, we describe powerful lemmas applicable to set systems both uniform and nonuniform. Through the idea of $s-$independence and s^*-idenependence, we give strikingly simple proofs to theorems of great importance.