Master thesis abstract

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In this paper we parameterized the Bures and Hellinger distances on C^*-algebras by defining d_p^τ as the square root of image of the difference of the arithmetic mean and a κ_p operation under a faithful tracial positive linear functional τ. This way the Bures and Hellinger distances are d_2^τ and d_1^τ respectively. We examined κ_p and d_p^τ. First, we studied whether there is an inequality between the arithmetic mean and our κ_p operation on C^*-algebras (whether the difference of the arithmetic mean and κ_p is non-negative) which would trivially imply that d_p^τ is well-defined. In Proposition 1 we showed however, that this happens only in special cases, precisely on commutative algebras. Following this, we referred to [1] where the author proves that d_p is well-defined on $M_n[\mathbb{C}]^+$ for all $p \geq 0$, and that d_p^τ is also well-defined on the positive cone of a general C^*-algebra in the case of $p \leq 2$. (The $p > 2$ case seems to hold quite a challenge.) After this, we studied the metric properties of d_p. In Proposition 2 we showed that d_p is a true metric on the set $P_1(H)$ of the rank one projections of a Hilbert space H if and only if $p \leq 2$. This implies that for any $p > 2$, d_p fails to be a metric even on the positive cone of the algebra of 2 by 2 matrices. The question, whether d_p^τ is a true metric in these cases, is certainly a hard problem, the further research of which is required. In Theorem 3 we proved that if there exists a φ positive linear functional on a C^*-algebra such that d_p^φ is well defined, then it is necessarily tracial. We note, that the $p = 0$ case of both this result and Proposition 1 tell us about how much the exponential function fails to be operator convex. Motivated by the symmetric Stein divergence, we defined $\delta_{S,p}^\tau$ on the positive definite cone of an arbitrary C^*-algebra as the image of the difference of the logarithm of the arithmetic mean and the logarithm of a κ_p under a faithful tracial positive linear functional τ. It is natural to ask whether these functions coincide for all p mimicking the behavior on $M_n[\mathbb{C}]^{++}$. Our last result shows that this happens to a bounded linear functional of a von Neumann algebra precisely when it is tracial. We note that one can prove a similar result in the case of a C^*-algebra and a positive linear functional. Whether the square root of δ_S^τ is a true metric in the latter context, is a deep problem as well, which we would like to examine in the future.

References

[1] Á. Komálovics and L. Molnár, *On a parametric family of distance measures that includes the Hellinger and the Bures distances,* J. Math. Anal. Appl, 2023.