Empty triangles in geometric and topological graphs

Csenge Lili Ködmön

Supervisor: Dr. Géza Tóth

In the 1970s, Erdős asked whether for every positive integer k there exists an empty convex k-gon in every sufficiently large finite point set in general position in the plane. It has been found to be true for $3 \leq k \leq 6$, but false for $k \geq 7$. This problem can be generalized to finding the least number $h_k(n)$ of k-holes determined by any set of n points in the plane.

In Section 2, I describe the evolution of the best known upper and lower bounds for $h_3(n)$, occasionally stating some results for the bounds on $h_4(n)$, $h_5(n)$, and $h_6(n)$, as well. In Section 2.1, I present Bárány and Valtr's proof of the best known upper bound, $h_3(n) \leq 1.6195n^2+o(n^2)$, which utilizes the Minkowski sum of random Horton sets. The majority of Section 2.2 is dedicated to describing García's method and its impact on the evolution of the lower bounds on $h_3(n)$. García introduced the classification of empty triangles into two types; those that are generated by empty 5-holes, and those that cannot be obtained in this manner, and proved that the number of empty triangles of the latter type is invariant. This allowed for further improvements, including the best known lower bound, $h_3(n) \geq n^2 + \Omega(n \log^{2/3} n)$, proved by Aichholzer et al. using a lower bound on the number of empty 5-holes, $h_5(n)$.

In Section 3, Erdős' question is generalized even further, as the edges are allowed to be Jordan arcs connecting the respective endpoints with the restriction that every pair of edges meets at most once, in either a common vertex or a proper crossing. In this context, a triangle is a simple closed curve consisting of the edges between three vertices, which partitions the plane into a bounded and an unbounded connected set. A triangle is empty if either does not contain any other vertex of the point set in its relative interior. The problem is determining the minimum number $h_3^{top}(n)$ of empty triangles in complete simple topological graphs on n vertices. In Section 3.1, I briefly describe a construction, called twisted drawing, used by Harborth to show the best known upper bound, $h_3^{top}(n) \leq 2n - 4$. In Section 3.2, I discuss the evolution of the lower bounds on $h_3^{top}(n)$. First, I present Harborth's proof of the first lower bound, $h_3^{top}(n) \geq 2$. Harborth conjectured that every vertex in a complete simple topological graph is incident to at least two empty triangles, and I discuss Ruiz-Vargas' proof of this statement in detail. Section 3.2 concludes with the proof of the best known lower bound, $h_3^{top}(n) \geq n$, proved by Aichholzer et al. using Ruiz-Vargas' results.

In Section 4, I briefly summarize the bounds on $h_3(n)$ and $h_3^{top}(n)$ presented in this thesis and provide some insight regarding potential improvements of these bounds.