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Budapest University of Technology and Economics

Institute of Mathematics

Lachyn Jumakova

The Prime Number Theorem

BACHELOR THESIS

Supervisor: Sándor Kiss

Let $\pi(x)$ be the number of primes less than or equal to x, for every $x \in \mathbb{R}^+$. The Prime Number Theorem states that $\pi(x) \sim \frac{x}{\log(x)}$ as $x \to \infty$. That is, $\lim\limits_{x\to\infty} \dfrac{\pi(x)\log(x)}{x} = 1$. This theorem was independently proved by Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896 using ideas introduced by Bernhard Riemann.

The **Riemann zeta function** is defined for real $s > 1$ by the convergent series $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$. $\zeta(s)$ has no zeros on the line $\Re(s) = 1$. Upper bounds on $|\zeta(s)|, |\zeta'(s)|, \frac{1}{|\zeta(s)|}$ for every $\epsilon > 0$:

$|\zeta(s)| \leq c_\epsilon |t|^{1-\sigma_0+\epsilon}$, if $\sigma_0 \leq \sigma$ and $|t| \geq 1$.

$|\zeta'(s)| \leq c_\epsilon |t|^\epsilon$, if $1 \leq \sigma_0$; and $|t| \geq 1$.

$1/|\zeta(s)| \leq c_\epsilon |t|^\epsilon$ if $\sigma \geq 1$ and $|t| \geq 1$.

The **von Mangoldt function** is defined as $\Lambda(n) = \log p$ if $n = p^m$ for some prime p and $m \geq 1$, and $\Lambda(n) = 0$ otherwise. **The Tchebyceff ψ-function** is defined as: $\psi(x) = \displaystyle\sum_{n\leq x} \Lambda(n)$. **The Tchebycheff ψ_1-function** is defined as: $\psi_1(x) = \int_0^x \psi(t)\,dt$. Now we state the connection between Tchebycheff ψ_1-function and Riemann zeta function: $\psi_1(x) = -\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta'(s)}{\zeta(s)}\,ds$, where the path of integration is the straight line $\sigma = c > 1$ and $x > 0$.

As $x \to \infty$, we have: $\psi_1(x) \sim \dfrac{1}{2}x^2 \iff \psi(x) \sim x \iff \pi(x) \sim \dfrac{x}{log(x)}$. Hence, to prove the Prime Number Theorem, it is sufficient to prove $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$. The main factors in the argument are: the formula connecting ψ_1 to ζ, $\zeta(s)$ has no zeros on the line $\Re(s) = 1$, and the estimates for ζ near the line.