Morse Theory and its applications

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Abstract

One of the powerful tools for studying the topology of a smooth manifold is Morse theory, the aim of which is to examine the properties of smooth functions defined on it. This field was established by Marston Morse in the 1920s and 30s, who built upon ideas of Arthur Cayley and James Clerk Maxwell, and showed that there is a fairly direct connection between the topological properties of the manifold and the critical points and the gradient vector field of a smooth function defined on it. Since then, Morse theory, and its variants and extensions, has been applied in a large number of areas inside and outside mathematics, from artifical intelligence to biology to astronomy to topological data analysis. It plays a very important role in certain areas of mathematical physics, in particular in string theory. For a detailed introduction to Morse theory, the interested reader is referred to the famous book Morse Theory by J.Milnor.

Due to the fact that the original Morse theory smooth functions defined on smooth manifolds, its applications are limited by differentiability. In the second half of the 20th century, with the appearance of computers and the widespread use of algorithms in science, the need emerged to apply the results

of Morse theory to discrete structures. The first ideas regarding the solution of this problem appeared in the works of J.H.C Whitehead. Nevertheless, the systematic investigation of a combinatorial variant of Morse's theory was started in 1998 by Robert Forman, who also coined the name 'discrete Morse theory' for this area. An important application of discrete Morse theory involves the investigation of the properties of piecewise linear functions, and thus, it plays an important role in computer graphics.

This thesis has three main parts. In the first chapter, we give a brief introduction to classical Morse theory describing the critical points of a smooth function defined on a smooth manifold. Our main object here will the definition of a certain CW-decomposition of the manifold, called the *Morse-Smale complex*. In the second chapter we adapt the tools from the first part to study piecewise linear functions on simplicial complexes, and define the Morse-Smale complexes generated by them. Finally, in the third chapter we present an algorithm, introduced by Herbert Edelsbrunner et al. in 2003, that computes the Morse-Smale complex generated by a piecewiselinear function. The main paradigm of the algorithm is the simulation of differentiability that allows to simulate smoothness on a simplicial complex.

The project and visualization were made in the Blender software. The algorithmic part was implemented in a python script with a built-in python API in Blender, which is included as the bpy library.