1. Számítsuk ki a következő határozatlan és határozott integrálokat!

a) $\int x^4 - 3x^2 + 2\,dx$ b) $\int \sqrt[3]{x^2}\,dx$ c) $\int\limits_0^1 \sqrt{x\sqrt{x\sqrt{x}}}\,dx$

d) $\int \frac{x^4 + 2x - 1}{x}\,dx$ e) $\int\limits_1^4 \sqrt{x}(7x^2 + 10x - 3)\,dx$

2. A következő integrálok kiszámításához alkalmazzuk a láncszabály megfordítását:
$\int f(g(x))g'(x)\,dx = F(g(x)) + C$.

a) $\int_0^1 xe^{x^2}\,dx$ b) $\int \frac{\sin(\ln x)}{x}\,dx$ c) $\int \mathrm{tg}\,x\,dx$

d) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ e) $\int_0^{\frac{\pi}{2}} \cos x \sin^3 x\,dx$

3. Az integrandus megfelelő átalakítása után alkalmazzuk a láncszabályból adódó
$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$ összefüggést!

a) $\int \sqrt[4]{2-3x}\,dx$ b) $\int \frac{e^x + 1}{e^{2x}}\,dx$ c) $\int \cos^2 x\,dx$ d) $\int_0^{\frac{\pi}{4}} \cos^4 x\,dx$

e) $\int \frac{1}{x^2+4}\,dx$ f) $\int \frac{1}{\sqrt{1+3x^2}}\,dx$ g) $\int\limits_{-1}^0 (2x+1)^{10}\,dx$ h) $\int \frac{x^3-1}{x+2}\,dx$

4. Számítsuk ki az alábbi improprius integrálokat!

a) $\int_1^\infty \frac{1}{x^2+1}\,dx$ b) $\int_0^1 \frac{1}{\sqrt[3]{x^2}}\,dx$ c) $\int_e^\infty \frac{1}{x \ln x}\,dx$ d) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

5. Keressük meg azt az $f(x)$ függvényt, amelyre

a) $f'(x) = 4x + \sin 2x$, és $f(0) = 0$;

b) $f''(x) = 6x^2 + \frac{1}{x\sqrt{x}}$, $f(1) = 0$, és $f'(1) = 2$.

Megoldások

1. a) $\int x^4 - 3x^2 + 2\,dx = \frac{1}{5}x^5 - x^3 + 2x + C$

b) $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{3}{5}x^{5/3} + C$

c) $\int_0^1 \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int_0^1 x^{7/8}\,dx = \left[\frac{8}{15}x^{15/8}\right]_0^1 = \frac{8}{15}$

d) $\int \frac{x^4 + 2x - 1}{x}\,dx = \int x^3 + 2 - \frac{1}{x}\,dx = \frac{1}{4}x^4 + 2x - \ln|x| + C$

e) $\int_1^4 \sqrt{x}(7x^2+10x-3)\,dx = \int_1^4 7x^{5/2}+10x^{3/2}-3x^{1/2}\,dx = \left[2x^{7/2}+4x^{5/2}-2x^{3/2}\right]_1^4 =$
$2\cdot 128 + 4\cdot 32 - 2\cdot 8 - 4 = 364$

2. a) $\int_0^1 xe^{x^2}\,dx = \int_0^1 \frac{1}{2}e^{x^2}2x\,dx = \left[\frac{1}{2}e^{x^2}\right]_0^1 = \frac{1}{2}(e-1)$

b) $\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(\ln x)\frac{1}{x}\,dx = -\cos(\ln x) + C$

c) $\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{\cos x}\cdot(-\sin x)\,dx = -\ln|\cos x| + C$

d) $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2}(1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\cdot 2(1-x^2)^{1/2}+C = -\sqrt{1-x^2}+C$

e) $\int_0^{\frac{\pi}{2}} \cos x\sin^3 x\,dx = \left[\frac{1}{4}\sin^4 x\right]_0^{\pi/2} = \frac{1}{4}$

3. a) $\int \sqrt[4]{2-3x}\,dx = \int(-3x+2)^{1/4}\,dx = \frac{4}{5}(-\frac{1}{3})(-3x+2)^{5/4}+C = -\frac{4}{15}(-3x+2)^{5/4}+C$

b) $\int \frac{e^x+1}{e^{2x}}\,dx = \int e^{-x} + e^{-2x}\,dx = -e^{-x} - \frac{1}{2}e^{-2x} + C$

c) $\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos(2x))\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

d) $\int_0^{\pi/4} \cos^4 x\,dx = \int_0^{\pi/4} (\frac{1}{2}(1+\cos(2x)))^2\,dx = \int_0^{\pi/4} \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\,dx =$
$\int_0^{\pi/4} \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8}(1+\cos(4x))\,dx = \int_0^{\pi/4} \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,dx = \left[\frac{3}{8}x+\right.$
$\left.\frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]_0^{\pi/4} = \frac{3\pi}{32} + \frac{1}{4}$

e) $\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4}\frac{1}{(x/2)^2+1}\,dx = \frac{1}{4}\cdot 2\cdot \operatorname{arctg}(x/2) + C = \frac{1}{2}\operatorname{arctg}(x/2) + C$

f) $\int \frac{1}{\sqrt{1+3x^2}}\,dx = \int \frac{1}{\sqrt{1+(\sqrt{3}x)^2}}\,dx = \frac{1}{\sqrt{3}}\operatorname{arsh}(\sqrt{3}x) + C$

g) $\displaystyle\int_{-1}^{0}(2x+1)^{10}\,dx=\left[\frac{1}{2}\cdot\frac{1}{11}(2x+1)^{11}\right]_{-1}^{0}dx=\frac{1}{11}$

h) $\displaystyle\int\frac{x^3-1}{x+2}\,dx=\int x^2-2x+4-\frac{9}{x+2}\,dx=\frac{1}{3}x^3-x^2+4x-9\ln|x+2|+C$

4. a) $\displaystyle\int_1^{\infty}\frac{1}{x^2+1}\,dx=\lim_{b\to\infty}\Big[\operatorname{arctg}x\Big]_1^b=\lim_{b\to\infty}\operatorname{arctg}b-\operatorname{arctg}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

b) $\displaystyle\int_0^1\frac{1}{\sqrt[3]{x^2}}\,dx=\lim_{b\to 0^+}\int_b^1 x^{-2/3}\,dx=\lim_{b\to 0^+}\Big[3x^{1/3}\Big]_b^1=\lim_{b\to 0^+}3-3\sqrt[3]{b}=3$

c) $\displaystyle\int_e^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\int_e^b\frac{1}{\ln x}\cdot\frac{1}{x}\,dx=\lim_{b\to\infty}\Big[\ln\ln x\Big]_e^b=\lim_{b\to\infty}\ln\ln b-\ln\ln e=\infty$

d) $\displaystyle\int_e^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{b\to\infty}\int_e^b(\ln x)^{-2}\frac{1}{x}\,dx=\lim_{b\to\infty}\Big[-\frac{1}{\ln x}\Big]_e^b=\lim_{b\to\infty}-\frac{1}{\ln b}+\frac{1}{\ln e}=1$

5. a) $\displaystyle f(x)=\int 4x+\sin(2x)\,dx=2x^2-\frac{1}{2}\cos(2x)+C$, és $0=f(0)=-\frac{1}{2}+C$, tehát $C=\dfrac{1}{2}$, és $f(x)=2x^2-\frac{1}{2}\cos(2x)+\frac{1}{2}$.

b) $\displaystyle f'(x)=\int 6x^2+x^{-3/2}\,dx=2x^3-2x^{-1/2}+C$, és $2=f'(1)=2-2+C$, így $C=2$, és $f'(x)=2x^3-2x^{-1/2}+2$. Ebből $\displaystyle f(x)=\int 2x^3-2x^{-1/2}+2\,dx=\frac{1}{2}x^4-4x^{1/2}+2x+D$, és $0=f(1)=\dfrac{1}{2}-4+2+D$, tehát $D=\frac{3}{2}$, és $f(x)=\dfrac{1}{2}x^4-4\sqrt{x}+2x+\dfrac{3}{2}$.