An application of Young's tower method: exponential decay of correlations in multidimensional dispersing billiards

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Abstract

We investigate multidimensional dispersing billiards with finite horizon and no corner points. We give an overview of the proof of exponential correlation decay and the central limit theorem for Hölder-continuous observables, following the lines of our paper [4]. The proof is conditional: an assumption on the complexity of the singularity structure has to be made.

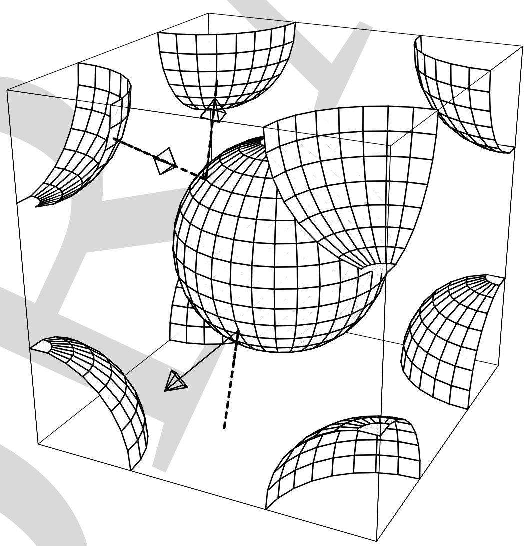

Figure 1: A 3-dimensional dispersing billiard

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Introduction

This note follows the lines of the minicourse given by the authors at the Ervin Schrödinger Institute (ESI) in Vienna between 9-13 June 2008, during the semester on Hyperbolic Dynamical Systems.

The discussion here is somewhat more detailed than the exposition at the minicourse – mainly because it's harder to cheat in print, and it's harder for the reader to ask back. However, the careful reader will notice many examples of unprecise formulation and non-mentioned technical details, even serious difficulties. Our intention was to concentrate on the main points and to keep the length limited. Readers curious for more details are welcome to ask the authors or to consult the papers used, mainly [4].

1 Lecture 1

1.1 Outlook to other minicourses of the semester

This minicourse is part of a series of minicourses during the **summer session about Hyperbolic Dynamical Systems** at ESI in June 2008, aiming to give an insight into three different approaches to statistical properties of hyperbolic systems. The three methods on the menu are

- Young tower method
- Coupling approach
- Direct functional analytic approach.

Our minicourse shows an application of the Young tower method, which was the first (in time) of the three to appear.

1.2 Statement of the result

Before the broader introduction, we start by stating the main result this minicourse is about. For the detailed definition of the notions, see the second lecture.

Theorem 1.1. *Let (M, T, μ) be a dispersing billiard map on the torus in any dimension, with the usual invariant measure.*

- *Suppose that the horizon is finite,*
- *and that there are no corner points.*
- *Assume furthermore that the singularity structure satisfies a "subexponential complexity condition".*

Then this dynamical system

- *has exponential decay of correlations,*
- *and the central limit holds*

for Hölder continuous observables.

The misterious "subexponential complexity condition" will be commented on several times.

1.3 Young towers

Our approach to the statistical properties of multidimensional dispersing billiards implements a general strategy, initiated by Lai-Sang Young in [15] and [16]. The approach is essentially that of **symbolic dynamics**. Thus the main line of argument consists of two elements:

(I) prove statistical limit theorems in the symbolic setting (the Young tower)

(II) relate the original – hyperbolic – dynamical system to the Young tower.

The coupling approach considers a suitable class of probability distributions on the phase space and shows that any two elements on this class, when pushed forward by the dynamics, can be effectively compared. See the minicourses of Nikolai Chernov and Dmitry Dolgopyat for details.

The direct functional analytic approach constructs suitable Banach spaces of functions/distributions directly on the phase space and studies the spectral properties of the Perron-Frobenius operator on them. See the minicourses of Carlangelo Liverani and Viviane Baladi for further details.

In comparision to Young towers, the common feature of the two other methods is that they work directly on the phase space, thus the above mentioned two step line of argument does not apply.

Concerning the central limit theorem (CLT) and exponential decay of correlations (EDC), step (I) has been already completed in [15], thus we only need to implement part (II) in the context of multidimensional dispersing billiards. Correspondingly, we do not concentrate on (I) here, we only collect some material on Young towers (in a very sketchy and inacurate manner) that may help the reader to understand our approach to (II).

1.3.1 Definition of Young towers

1. Let (Y, m) be a probability space. Y will be called the **base of the tower**.
2. Let $r : Y \to \mathbb{Z}^+$ integrable with respect to m. $r(x)$ will be called the **height of the tower at** x.
3. Let $\cup_i Y_i$ be a (mod m defined) partition of Y into countably many components, such that r takes constant values on each of the Y_i.
4. The **phase space of the tower** is $\Delta = \{(y, l) \,|\, y \in Y, 0 \le l < r(y)\}$.

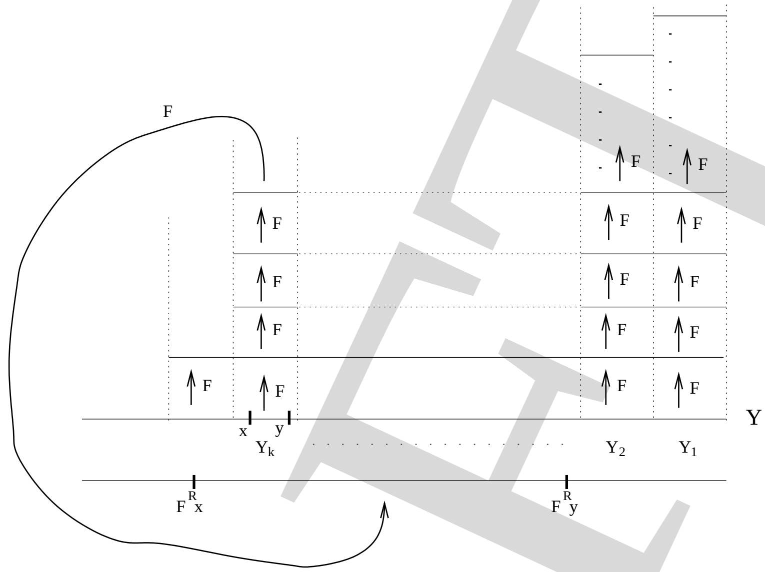


Figure 2: Young tower

5. There is a **natural probability measure** m_Δ on Δ. (Which is constant times m in the horizontal and counting measure in the vertical directions. The integrability of r is used here.)

6. The **dynamics** F on Δ is defined as follows:

   - Move up, when possible, (That is, if $l < r(y) - 1$, let $F(y, l) = (y, l + 1)$.)
   - If already on the top, map the actual component **onto** the base. (That is, from each $Y_i \times \{r(Y_i) - 1\}$ the map F is a measurable bijection onto $Y \times \{0\}$.)

7. Let $F^r : Y \to Y$ denote the return map. We need to assume certain **regularity properties of the Jacobian** of this map. (Note however, that F^r not assumed to preserve m.)

8. Regularity is understood in terms of a **symbolic distance**. For $x, y \in Y$ let their *separation time* be $s(x, y) = \{$ smallest $n \in \mathbb{Z}^+$ such that $(F^r)^n x$ and $(F^r)^n y$ lie in different partition elements $Y_i\}$. To extend to Δ, let $s((x, k), (y, k)) = s(x, y)$, and fix it as 0 for points on different levels. Now, for some fixed $\alpha < 1$, the symbolic distance of the two points is $d_s(x, y) = \alpha^{s(x,y)}$. The Jacobian of F^r is required to be Hölder continuous with respect to this distance.

1.3.2 Main results about Young towers.

- First of all, there exists an F-invariant measure μ_Δ on Δ, absolutely continuous with respect to m_Δ. (This is by intergrability of r, see eg. [15].)

- Assume now **exponential tail** for r. (That is, there exists $\gamma < 1$ such that $m(y \in Y | r(y) > n) = o(\gamma^n)$.) Then, as proved in [15], the tower dynamics (Δ, μ_{Delta}, F) enjoys **EDC**

and CLT (for a class of observables sufficiently regular – essentially, Hölder continuous – with respect to the distance $d_s(x, y)$.)

It is worth noting that, since [15], many further statistical properties have been proved for Young towers. Not aiming for completeness, let us mention large deviations ([13],[9]), stohastic stability ([1]) and various strengthenings of the CLT including local limit laws ([14]), almost sure invariance principles ([11]), and Berry-Esséen type theorems ([12]). In [16] the scheme has been generalized to return time statistics with polynomial tails, resulting in polynomial decay of correlations, which may account for non-standard limit theorems depending on the context ([3]). Decay rates for suspension flows built on (Δ, μ_Δ, F) with sufficiently regular roof functions have been also treated in [10].

Thus, subject to completing Step (II), we get all these statistical properties for free (this applies to the case of multidimensional dispersing billiards in particular). Of course, step (II) – the actual construction of the tower – may require much effort, and an approach that depends strongly on the context. Before turning to this question, let us mention that the step (I) – the proof of the statistical properties on the Young tower – may implement either coupling ideas ([16]) or spectral properties of the Perron-Frobenius operator on a suitably function space. This essentially means that, when using the tower, one/both of the other two concurrent approaches are applied, however, certain issues are handled on the tower, and not directly on the phase space. This makes the method very powerful and gives a unified line of approach for a large class of dynamical systems, in which hyperbolicity may or may not be uniform, and singularities may or may not be present. Again not aiming for completeness let us mention one dimensional maps with critical points ([15]) or neutral fixed points ([16]), Hénon maps ([15]), Lorenz attractors ([8]) and planar billiards with various mixing rates ([6]).

As mentioned above, what actually Step (II), the construction of the tower means, may depend strongly on the context. Nonetheless the different cases have certain common features. In particular, the base of the tower, Y is identified with a subset $\Lambda \subset M$ of positive Lebesgue measure in the phase space. The discrete return times $r(y)$ for points $y \in \Lambda(\cong Y)$ correspond to the number of times T should be iterated such that the orbit of y "makes a suitable return to Λ". "Suitable" essentially means the following. To ensure that there is a partition into cells Y_i such that points in a cell (i) return simulataneously (ii) $T^{r(Y_i)}$ extends fully along Λ – so that in the tower $F^{r(Y_i)}$ gives a one-to-one correspondence between Y_i and the base Y, we may need to choose $r(y)$ to be bigger than the first return time of y to Λ. One more important remark concerning the return process is that, as the tower map is highly non-invertible, to apply it to code an invertible dynamics, we can not expect that $T^{r(Y_i)}Y_i$ actually all coincide with Λ itself. Thus we require that $T^{r(Y_i)}$ fully extends along Λ in the unstable direction, however, may be much smaller in the stable direction. This roughly means that we consider returns modulo stable directions to a central unstable leaf in Λ. In particular, the construction of the tower requires an extra step which is often referred to as collapsing along stable manifolds. This is an important issue which makes things technically more complicated and requires special care, however, does not change the phenomenological description substantially. To keep the exposition limited, we disregard the problem of collapsing along stable manifolds for the rest of the section, which seems to be the appropriate choice for the semi-heuristic level of these notes.

Let us finally mention that, despite being so powerful, the tower method also has some weaknesses. One of them is that because of the two step strategy – and mainly related to the use of a symbolic metric, which results in the loss of the original geometry of the system – the results are sometimes not optimal. For example, in the context of billiard flows, to prove the optimal exponential bound on the decay of correlations, it is more or less clear that a new – more direct – approach would be needed. Another important feature is that Young towers are not really suitable for a perturbative analysis. If the system (say the billiard scatterers) are slightly perturbed, it is often an important issue how the quantities appearing in the statistical properties (eg. constants in the decay rates, asymptotic variances) change. As each system requires the construction of a separate tower, there is very little chace for such comparisions.

1.4 Tower construction for uniformly hyperbolic maps with singularities: growth lemma

We know turn to the issue of implementing Step (II), that is, the construction of the tower, for the case of multidimensional dispersing billiards. For a detailed description of these systems see Section 2.1, here we recall the two most important features:

- the dispersing billiard map is **uniformly hyperbolic**
- with *singularities*.

In the context of such systems, the problem of tower construction has been studied systematically by Nikolai Chernov in [5]. In particular, following [5], (see Theorem 2.4 below), it is enough to check that certain axioms are satisfied to implement Young's tower construction. The most important of these axioms is the a condition on the **growth of unstable manifolds**. In the rest of the section we try to visualize the essential content of Theorem 2.4, and hint on how the growth condition accounts for the construction of the Young tower.

1.4.1 Expansion versus fractioning

Let W be a short unstable manifold. (For simplicity, we can now just think of an appropriate dimensional smooth manifold with every tangent vector in the unstable cone, see the notion of hyperbolicity in the second lecture.) When T is applied, W

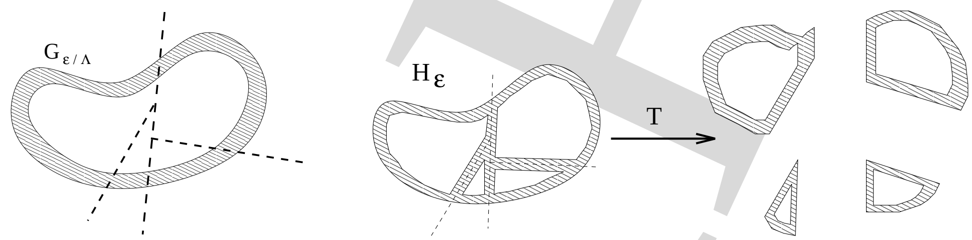
- is fractioned because of the singularities, that is, different components of W may land far from each other
- expanded because of uniform hyperbolicity, that is, $\rho_u(Tx, Ty) \geq \Lambda \rho_u(x, y)$, whenever x and y are not separated by singularities.

(here ρ_u is the inner metric of W, and $\Lambda > 1$ is the constant of least expansion.) The condition on the growth of unstable manifolds gives an effective way to ensure the fact the second one is the stronger of these two competitive phenomena ("expansion prevails fractioning").

1.4.2 Formulation of the growth property

The statement we are about to formulate can be called **growth condition** when applied, **growth lemma** when proven, or **growh property** in general. We will use all these names without special care.

Very roughly, the growth property states that when T is applied to an unstable manifold, not too much boundary is created – in comparison with the effect of expansion.



(a) old boundary (b) preimage of new boundary

Figure 3: Sets to be compared in the growth lemma.

- Let $\tilde{\Lambda}$ denote the factor of least expansion on the unstable cone.
- Let W be a sufficiently small local unstable manifold, and let m_W be the Lebesgue measure on it.
- Let G_ε be the set of points in W that **are** at most ε far from the boundary:

$$G_\varepsilon = \{x \in W \mid \rho(x, \partial W) \leq \varepsilon\}.$$

- Let H_ε be the set of points in W that **will be** at most ε far from the boundary (of the image of W) after one step of the dynamics:

$$H_\varepsilon = \left\{x \in W \mid \rho(\tilde{T}x, \partial(\tilde{T}W)) \leq \varepsilon\right\}.$$

Here ρ always denotes the distance within the given manifold (induced by the Riemannian structure). Note that like G_ε, the set H_ε is also a subset of W, and not TW. Recall also that typically $\partial(\tilde{T}W) \neq \tilde{T}(\partial W)$, since $\tilde{T}$ is not continuous on W.

If there were no singularities, then the boundary of the image would be the image of the boundary, so by uniform expansion we would have $H_\varepsilon \subset G_{\varepsilon/\Lambda}$, which clearly implies $m_W(H_\varepsilon) \leq m_W(G_{\varepsilon/\Lambda})$ The growth lemma expresses that the presence of singularities can make this relation only slightly worse. Namely,

Proposition 1.2. *There exists a constant $\tilde{\lambda}$ strictly less than $\tilde{\Lambda}$, independent of W, such that*

$$m_W(H_\varepsilon) \leq \tilde{\lambda}\, m_W(G_{\varepsilon/\Lambda}).$$

1.4.3 Role of the growth lemma in the tower construction

To give a hint on how the growth lemma ensures the possibility for tower construction, first imagine that the phase space is "densely covered" by finitely many small unstable manifolds, in the sense that any unstable manifold, which has at least ε inner diameter, covers one of them (up to small differences along stable directions). Picture these as "the bases of finitely many towers".

No let W be one of these unstable manifolds. Introduce H_ε^n as the set of points that will lie in the ε neighborhood of the boundary (of the relevant component of T^nW) after T^n is applied. It is not hard to believe – though requires extra work and technical complications in the notation – that the growth lemma can be iterated giving

$$m_W(H_\varepsilon^n) \leq \tilde{\lambda}^n \cdot m_W(G_{\varepsilon/\tilde{\Lambda}^n}).$$

Now since W is a fixed manifold with some given geometry (say, an appropriate dimensional sphere), $m_W(G_{\varepsilon/\tilde{\Lambda}^n})$ can be estimated in terms of the surface area of ∂W, giving

$$m(H_\varepsilon^n) \leq C \left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^n \varepsilon \to 0$$

as n increases, for any (small enough) $\varepsilon > 0$. Note that $x \notin H_\varepsilon^n$ implies that the component of T^nW containing T^nx, also **contains a sphere of radius ε around T^nx**. Thus the growth lemma implies the following statement: there is a fixed $q > 0$ and $n_1 \in \mathbb{Z}^+$, such that after n_1 iterations at least q percent (in terms of the measure m_W) of the points of W have a sphere of radius ε around them in their relevant component. This applies, in particular, to $\varepsilon = \mathrm{diam}(W)$. So after n_1 steps, a positive q fraction of points can make a "proper return" near the base of one of the tower, defining a partition element, and n_1 will be the number of levels above it. The points that did not manage to make a proper return will go on, and some q fraction of them is expected to return after another n_2 iterations. If one can ensure that the "not yet returned" part has a boundary not much worse than the original, then n_2 is not much greater than n_1, and the procedure can be continued to give a tower with exponential tail for the return time.

2 Lecture 2

Before we can precisely state our main theorem, we need to define several notions.

2.1 Dispersing billiards

2.1.1 Definition of the dynamical system

A billiard is a dynamical system describing the motion of a point particle which flies freely in space, except for events of momentary collisions with certain obstacles, when it bounces back from the obstacles according to the rules of elastic collision.

For our purposes, the "space" will be the d-torus Π^d, and there will be finitely many obstacles $O_1, \dots O_L$. So the allowed set of points in space – which we call the *billiard table* is

$$\mathbb{Q} = \Pi^d \setminus \left(\cup_{i=1}^{L} O_i \right).$$

That is, $\cup_{i=1}^{L} O_i$ is the forbidden region of Π^d, which the particle must stay outside.

We require each obstacle to be strictly convex with C^3 smooth boundary∂O_i, and we also call them *scatterers*, referring to the phenomemon that if a bunch of particles with parallel velocities reaches such a strictly convex obstacle, it is "scattered" into different directions, with the particles diverging from each other.

We also need a property which is somewhat stronger than strict convexity: we require that the curvature of the boundary ∂O_i is always positive – more precisely, that the second fundamental form of ∂O_i is positive definite everywhere. This is what we call the "strictly dispersing" property of the billiard table.

Now we describe the usual Poincaré section phase space M. We only consider collision moments. Since kinetic energy is preserved in the system, we also fix the speed to be 1. We choose to describe the motion of the particle at a collision time by recording its velocity just *after* the collision (we use the 'outgoing' Poincaré section).

So a possible state of the particle is described by giving a boundary point $q \in \partial\mathbb{Q}$ and a unit velocity $v \in S^{d-1}$, which is often written roughly as $M = \partial\mathbb{Q} \times S_+^{d-1}$, where the $+$ indicates that only velocity vectors pointing inward $\mathbb{Q}$ are allowed.

The dynamics $T : M \to M$ gives the state of the particle at the next collision as a function of the present state.

This dynamics has a natural invariant (SRB) measure μ, which is absolutely continuous w.r.t. the Lebesgue measure on M. So in billiards, the existence of the SRB measure is no problem, we are interested in its mixing properties.

Definition 2.1. *The dynamical system (M, T, μ) is called the dispersing billiard map.*

2.1.2 Singularity structure

A key feature of billiads dynamics is that the map T is not continuous, but has *singularities.* Indeed, let $\mathcal{S}^0 = \partial M$ denote the boundary of the phase space M, which is in our case exactly the set of tangent (or grazing) collision points, where the particle touches a scatterer, but is not diverted.

We suppose that the billiard table has no corner points – that is, the scatterers O_i are disjoint.

The dynamics T is singular exactly at the points of

$$\mathcal{S} := \mathcal{S}^1 := T^{-1}\mathcal{S}^0 = T^{-1}\partial M.$$

Similarly, for $n \geq 1$ the iterate T^n of the dynamics is singular exactly at

$$\mathcal{S}^{(n)} := \bigcup_{i=1}^{n} T^{-i}\partial M.$$

We call

$$\mathcal{S}^i = T^{-i}\partial M$$

the i-step singularity set for $i = 0, 1, 2, \ldots$.

The singularity set of billiards features a *continuation property.* This means that the singularities are one-codimensional submanifolds that can only terminate on each other, or on the boundary of M. See Figure 4.

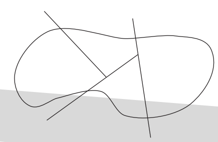


Figure 4: A piece of the phase space cut by singularities.

As a consequence, singularities do cut the phase space: if a small open subset U of M is intersected by a singularity manifold, then it is indeed cut into two (not necessarily connected) components.

Definition 2.2. ***Complexity.*** *We define the complexity of the singularity set $\mathcal{S}^{(n)}$ of T^n to be the maximum number of components, into which $\mathcal{S}^{(n)}$ can cut a sufficiently small piece of the phase space. We denote this by K_n.*

This is one more than the maximum number of at most n-step singularities that can meet at a single point.

More precisely: For every n and any set $U \subset M$, we denote by $K_{U,n}$ the number of componets into which $\mathcal{S}^{(n)}$ cuts U, and call it *the complexity of $\mathcal{S}^{(n)}$ on U.*

For $x \in M$ let us denote by $K_{x,n}$ the minimum number of components into which a sufficiently small open neighbourhood U of x is cut by $\mathcal{S}^{(n)}$. This $K_{x,n}$ will be called the *complexity of $\mathcal{S}^{(n)}$ at x.*

Finally, we define the complexity of $\mathcal{S}^{(n)}$ as $K_n := \sup_{x \in M} K_{x,n}$.

A very common assumption in the theory of singular hyperbolic dynamical systems is that the complexity K_n is a subexponential function of n, or at most $C(\lambda^n)$ where λ is strictly less than the smallest expansion Λ on the unstable cone. We will also have to assume this property (see further discussion later).

Theorem 2.3. *Let (M, T, μ) be a dispersing billiard map. Assume that the horizon is finite and that there are no corner points. Assume furthermore that the complexity K_n is a subexponential function of n.*

Then the system has exponential decay of correlations (EDC) for Hölder continuous observables. That is, for every $f, g : M \to \mathbb{R}$ which are Hölder continuous, there exist constants $C < \infty$ and $\gamma > 0$ such that

$$\left| \int_M f(x) g(T^n x) d\mu(x) - \int_M f(x) d\mu(x) \int_M g(x) d\mu(x) \right| \leq C e^{-\gamma n}.$$

Moreover, the central limit theorem (CLT) is satisfied for Hölder continuous observables. That is, for every $f : M \to \mathbb{R}$ which is Hölder continuous,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(T^i x)$$

converges in distribution to Gaussian.

2.2 Overview of the proof

The proof is based on the Young tower construction. However, we don't build the tower ourselves: formally the proof is an application of Chernov's 1999 theorem, which ensures EDC and CLT for uniformly hyperbolic systems, provided that a list of conditions is satisfied. Namely:

Theorem 2.4 (Chernov'99)**.** *Suppose that the dynamical system $(\tilde{M}, \tilde{T}, \tilde{\mu})$ features*

1. *Uniform hyperbolicity*
2. *Certain uniform regularity properties, namely*
   (a) *smoothness of unstable manifolds*
   (b) *distortion bounds along unstable manifolds*
   (c) *absolute continuity of the holonomy map between unstable manifolds*
3. *The "growth lemma" about the evolution of unstable manifolds*

Then the system has EDC and CLT for Hölder continuous observables.

Our proof consists of checking these conditions, which will be described later (to some extent).

Notice that for the theorem (both the conditions and the statement) to make sense, the phase space $\tilde{M}$ needs to be a **Riemannian manifold**. So to apply Chernov's theorem, we need to define

- a phase space $\tilde{M}$,
- a Riemannian structure g,

- a dynamics $\tilde{T}$
- and a measure $\tilde{\mu}$.

When we apply this theorem, the phase space and the measure will be those of the billiard: $\tilde{M} = M$, $\tilde{\mu} = \mu$. However, $\tilde{T}$ will be a sufficiently *high iterate* of the billiard map T, and the choice of Riemannian structure is also a delicate issue.

2.3 Overview of the talk

We will talk in some detail about

1. Uniform hyperbolicity and Riemannian structure (second lecture)
2. The growth lemma (third lecture)
3. Our assumption about subexponential complexity of the singularity structure (third lecture)

We will only talk very little about the regularity properties.

2.4 Uniform hyperbolicity and Riemannian structure

The phase space M has a natural Riemannian structure. However, we will now introduce a different one. The precise reason for doing that will only be clear during the next lecture, but we give some preliminary motivation now.

2.4.1 Motivation

- Goal: *one-step* uniform hyperbolicity: $|DTv| \geq \Lambda|v|$ for $v \in C_{\mathrm{u}}$.
- Difficulty: this fails with the usual metric: only true with an extra constant: $|DT^n v| \geq c\Lambda^n|v|$ with some $c > 0$.
- Difficulty 2: We *do not want to use higer iterates.*
  - That obviously would help to get the uniform hyperbolicity condition of Chernov,
  - but we *need one-step expansion* for something else: namely *in the proof of the growth lemma.*

2.4.2 Explanation: structure of the singularity set

The reason that we can not use a higher iterate of T in the proof of the growth lemma is the *substantial difference* between the structure of the singularities of T and T^n. Namely:

- The 1-step singulerity set $\mathcal{S}$ consists of (finitely many) *smooth* manifolds, with uniformly bounded curvature.
- However, the $n \geq 2$-step singularities in $\mathcal{S}^n$ are *not smooth*.

This is due to the vast anisotropy of the dynamics near the singularities: In dispersing billiards, near a singularity

- T not only fails to be continuous,
- but the derivative also blows up,
- *moreover*, in *high dimensions* this blow-up is strongly direction-dependent, even within unstable directions: while expansion is very strong in one dimension, it is moderate in the other(s).

Having noticed that, it is no surprise that the nice smooth components of $\mathcal{S}$ can get crumpled by the action of T^{-1}, to form non-smooth components of $\mathcal{S}^2$. See Figure 5. In [2] it was shown that *this is indeed always the case.*

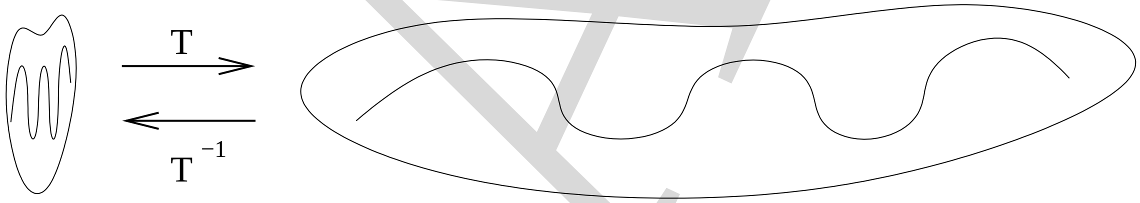


Figure 5: A smooth manifold crumpled by the highly anisotropic T^{-1}

For that reason, a local unstable manifold cut by an $n \geq 2$-step singularity should be pictured like Figure 6, and changing the scale does not help. Now we can understand why

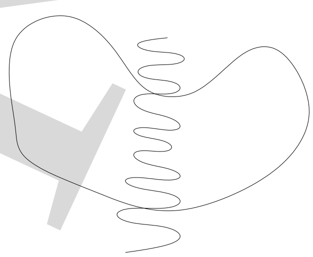


Figure 6: A small unstable manifold cut by a higher-order singularity.

we are unable to prove a growth lemma in this setting: the "new boundary" created by the cutting seems impossible to estimate with the old boundary.

2.4.3 Chernov-Dolgopyat metric

The difficulty is overcome by a smart new metric introduced by Chernov and Dolgopyat in [7]. To understand the essence, the two-dimensional picture is enough.

Consider a small divergent front of particles traveling from one scatterer to the other. Let $x = (r, \phi) \in M$ and $x' = x + \delta x = x + (\delta r, \delta\phi)$ be the state of two such particles at the first collision, and let $Tx = (r_1, \phi_1)$, $Tx' = Tx + DT\delta x = Tx + (\delta r_1, \delta\phi_1)$ be their images.

We want to compare the length of the tangent vectors $\delta x = (\delta r, \delta\phi)$ and $DT\delta x = (\delta r_1, \delta\phi_1)$.

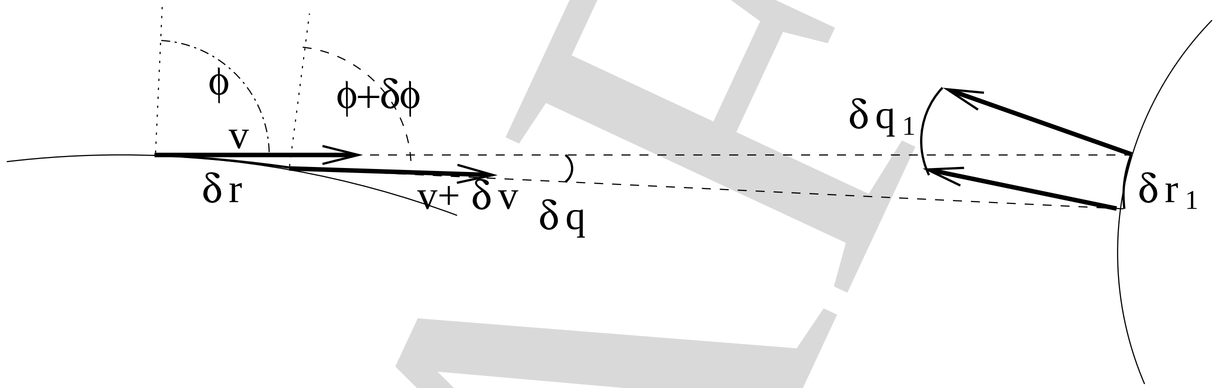


Figure 7: Expansion in the C-D metric and non-expansion in the Euclidean

The usual "length" of a tangent vector – which we will call the **Euclidean norm** is defined by

$$\|\delta x\|_e = \|(\delta r, \delta\phi)\|_e = \sqrt{|\delta r|^2 + |\delta\phi|^2}.$$

As we can see from Figure 7,

- this **can** very well **decrease**, even though δx corresponds to a divergent front.
- Indeed, if ϕ is close to $\pi/2$, and ϕ_1 is not, then δr is very long compared to δr_1.

Later this decrease will be compensated for, but that's too late for us: we really need one-step expansion.

The idea of the Chernov-Dolgopyat metric is to measure **distances on the front**. With the notation of Figure 7, this means

$$\|\delta x\|_{C-D} = \sqrt{|\delta q|^2 + |\delta v|^2}.$$

Here δq is the configurational distance of the two trajectories on the front, just after the collision. δv is the difference of the post-collision velocity vectors.

- This can be expressed in terms of δr, $\delta\phi$, ϕ and the curvature of the scatterer, so it is indeed a norm on $\mathcal{T}_x M$. (More precisely: almost a norm. See later.)
- For the problematic case considered, $\|\delta x\|_{C-D}$ is much smaller than $\|\delta x\|_e$.

- One-step expansion is easy: $|\delta q|$ always increases on divergent fronts, while $|\delta v|$ never decreases.

- As usual, defining the norm is equivalent to defining the metric tensor field g_{C-D}, through $\|v\|^2_{C-D} = g_{C-D}(v,v)$.

Good news:

- With this metric structure, T really features one-step expansion on the unstable cone, as well as one-step contraction on the stable one.

- The multidimensional generalization is not much harder.

Bad news: Unfortunately, this g_{C-D} is not a true Riemannian metric. Indeed, it is degenerate on the boundary of M, where $\phi = \pi/2$. At such phase points, there exists a nozero perturbation with length zero. (Namely, $\delta x = (\delta r, \delta\phi)$ for which $\delta v = 0$ (but $\delta q \neq 0$). Then $\|\delta x\|_{C-D} = 0$, since $\delta q = \cos\phi\delta r = 0$. Here the tensor is positive semi-definite, but not positive definite.

Solution: We introduce the **regularized Chernov-Dolgopyat metric** tensor field as

$$g = g_{C-D} + \varepsilon_g g_e$$

.

We formulate the results concerning this metric tensor field in the form of propositions:

Proposition 2.5. *For any $\varepsilon_g > 0$, the regularized Chernov-Dolgopyat metric tensor field is a C^1 Riemannian structure on M. If ε_g is small enough, then T is (one-step) uniformly hyperbolic w.r.t. g.*

The fact that one-step hyperbilicity is inherited from g_{C-D} is not completely obvious, since adding the regularizing term $\varepsilon_g g_e$ is not a small perturbation at points where g_{C-D} is nearly degenerate. So some work is needed here, but it can be done.

Proposition 2.6. *[2] The regularity properties required in Chernov's theorem are satisfied w.r.t. the Euclidean metric.*

Proposition 2.7. *Suppose that g_1 and g_2 are two arbitrary C^1 Riemannian structures on the phase space M. If the regularity properties are satisfied w.r.t. g_1, then they are also satisfied w.r.t. g_2, possibly with worse constants.*

Remark 2.8. *The need to regularize the Chernov-Dolgopyat metric arises not only to be able to use Chernov's theorem literally. For example, if we tried to use the non-regularized version, the bounded curvature property of the unstable manifolds would fail. Interestingly enough, in the two-dimensional situation of [7] these curvatures are bounded: a miracle that does not happen in higher dimensions.*

3 Lecture 3

Our aim is now to sketch the proof of the condition of Chernov about the growth of unstable manifolds. In the proof we will heavily use the following consequence of the regularity properties (namely, of the bounded curvature, distortion bounds, and smoothness of one-step singularities):

local flatness and linearity picture: On a sufficiently small scale, all unstable manifolds and all **one-step** singularities can be considered as planes, and the dynamics can be considered as linear. The error caused by using this picture can be made arbitrarily small by choosing the scale small enough.

Before reading on, please recall the statement of the growth condition from Section 1.4.2. We start the proof now.

Difficulties of the proof:

- We can not prove this property with $\tilde{T} = T$, because if W is cut by several singularities, then a lot of new boundary appears (see Figure 3), while on the other side we cannot guarantee the expansion to be large. For that, we will need to set $\tilde{T} = T^{n_0}$ with some $n_0 > 1$, and get $\tilde{\Lambda} = \Lambda^{n_0}$.
- On the other hand, we cannot directly work with T^{n_0} instead of T, for the reasons explained in the previous talk, namely the lack of smoothness of singularities of T^{n_0}

For these reasons we first state and prove a **one-step expansion lemma**.

Lemma 3.1. *Let $K_W = K_{W,1}$ be the complexity of $\mathcal{S} = \mathcal{S}^1$ on the small local unstable manifold W – that is, number of components into which the singularities of T cut W. Then*

$$m_W\left(\begin{Bmatrix}\textit{points who } \textbf{\textit{will be}}\ \varepsilon\textit{-close to}\\ \textit{the boundary after one step}\end{Bmatrix}\right) \le \gamma\, K_W\, m_W\left(\begin{Bmatrix}\textit{point who } \textbf{\textit{are}}\ \varepsilon/\Lambda\textit{-close}\\ \textit{to the boundary now}\end{Bmatrix}\right),$$

where γ is a constant greater than, but arbitrarily close to 1.

By saying "γ is arbitrarily close to 1" we mean that for any given $\gamma > 1$ we can choose the size limit for the local unstable manifolds W so small that the statement would hold with that γ.

To prove this, let V_1, V_2, ..., V_{K_W} be the components of W on which T is continuous.

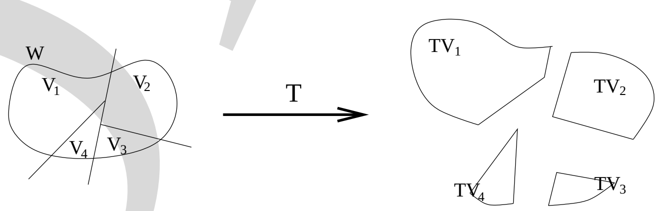


Figure 8: Notation for the one-step growth lemma

This means that

$$\left\{\begin{matrix}\text{points who } \textbf{will be } \varepsilon\text{-close to}\\ \text{the boundary after one step}\end{matrix}\right\} = \bigcup_{i=1}^{K_W} \left\{\begin{matrix}\text{points in } V_i \text{ who } \textbf{will be } \varepsilon\text{-close}\\ \text{to the boundary after one step}\end{matrix}\right\}.$$

Moreover,

$$\left\{\begin{matrix}\text{points in } V_i \text{ who } \textbf{will be } \varepsilon\text{-close}\\ \text{to the boundary after one step}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{points in } V_i \text{ who } \textbf{are } \varepsilon/\Lambda\text{-close}\\ \text{to the boundary of } V_i \text{ now}\end{matrix}\right\},$$

since the expansion is at least Λ, and the V_i are **not cut by singularities**. These two immediately imply

$$m_W\left(\left\{\begin{matrix}\text{points who } \textbf{will be } \varepsilon\text{-close to}\\ \text{the boundary after one step}\end{matrix}\right\}\right) \le \sum_{i=1}^{K_W} m_W\left(\left\{\begin{matrix}\text{points in } V_i \text{ who } \textbf{are } \varepsilon/\Lambda\text{-close}\\ \text{to the boundary of } V_i \text{ now}\end{matrix}\right\}\right).$$

To get the statement of the lemma, we only need to see that for every i

$$m_W\left(\left\{\begin{matrix}\text{points in } V_i \text{ who } \textbf{are } \varepsilon/\Lambda\text{-close}\\ \text{to the boundary of } V_i \text{ now}\end{matrix}\right\}\right) \le \gamma m_W\left(\left\{\begin{matrix}\text{points in } W \text{ who } \textbf{are } \varepsilon/\Lambda\text{-close}\\ \text{to the boundary of } W \text{ now}\end{matrix}\right\}\right).$$

This is the place where we use the locally flat picture, especially the flatness of the one-step singularity. For planar objects, the statement is obvious from Figure 9. The error mace by the application of this picture causes the γ to appear. Also note that **the one-step expansion of the dynamics was used.**

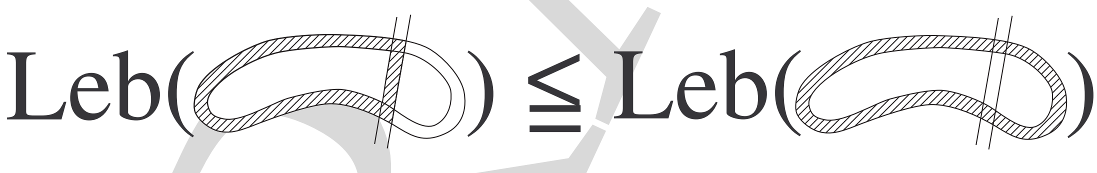


Figure 9: Application of the locally flat picture in the 1-step lemma.

□

From here we proceed to prove the "real" growth property. Namely, we will prove the **n-step growth lemma:**

Proposition 3.2. *Let $n \ge 1$ be arbitrary, and let $K_{W,n}$ be the complexity of $\mathcal{S}^{(n)}$ on the small local unstable manifold W – that is, the number of components into which the singularities of T^n cut W. Then*

$$m_W\left(\left\{\begin{matrix}\textit{points who } \textbf{\textit{will be}} \ \varepsilon\textit{-close to}\\ \textit{the boundary after } n \textit{ steps}\end{matrix}\right\}\right) \le \gamma\, K_{W,n}\, m_W\left(\left\{\begin{matrix}\textit{point who } \textbf{\textit{are}} \ \varepsilon/\Lambda^n\textit{-close}\\ \textit{to the boundary now}\end{matrix}\right\}\right), \tag{1}$$

where γ is a constant greater than, but arbitrarily close to 1.

For $n = 1$ this is exactly the 1-step lemma, so assume now inductively that the statement holds for $n-1$. As before, set $V_1, V_2, \ldots, V_{K_W}$ to be the components of W on which T is continuous. (*Warning: these are still the one-step smoothness components.*) Apply the inductive assumption on each TV_i, to get

$$m_{TV_i}\left(\left\{\begin{array}{l}\text{points in } TV_i \text{ who } \textbf{will be}\\ \varepsilon\text{-close to the boundary af-}\\ \text{ter another } n-1 \text{ steps}\end{array}\right\}\right) \le \gamma\, K_{TV_i,n-1}\, m_{TV_i}\left(\left\{\begin{array}{l}\text{point in } TV_i \text{ who } \textbf{are}\\ \varepsilon/\Lambda^{n-1}\text{-close to the}\\ \text{boundary of } TV_i\end{array}\right\}\right).$$

Now **apply the locally linear picture about the dynamics** to draw the measure estimate back from TV_i to V_i. The error of this picture is swallowed in γ, and we get

$$m_{V_i}\left(\left\{\begin{array}{l}\text{points in } V_i \text{ who } \textbf{will be}\\ \varepsilon\text{-close to the boundary}\\ \text{after } n \text{ steps}\end{array}\right\}\right) \le \gamma\, K_{TV_i,n-1}\, m_{V_i}\left(\left\{\begin{array}{l}\text{point in } V_i \text{ who } \textbf{will}\\ \textbf{be } \varepsilon/\Lambda^{n-1}\text{-close to the}\\ \text{boundary of } TV_i \text{ after}\\ \text{one step}\end{array}\right\}\right).$$

The measure of the set on the right hand side can be estimated using the 1-step growth lemma, with $K_{V_i} = 1$. The sum over i of the left hand side is just the left hand side of (1), so we get

$$m_W\left(\left\{\begin{array}{l}\text{points who } \textbf{will be } \varepsilon\text{-}\\ \text{close to the boundary}\\ \text{after } n \text{ steps}\end{array}\right\}\right) \le \gamma \sum_{i=1}^{K_W} K_{TV_i,n-1}\, m_{V_i}\left(\left\{\begin{array}{l}\text{point in } V_i \text{ who } \textbf{are}\\ \varepsilon/\Lambda^{n}\text{-close to the boun-}\\ \text{dary of } V_i \text{ now}\end{array}\right\}\right).$$

We apply again the locally flat picture to compare the ε/Λ^n-boundary of V_i (on the right hand side) to the ε/Λ^n-boundary of W, and get

$$m_W\left(\left\{\begin{array}{l}\text{points who } \textbf{will be } \varepsilon\text{-}\\ \text{close to the boundary}\\ \text{after } n \text{ steps}\end{array}\right\}\right) \le \gamma \left(\sum_{i=1}^{K_W} K_{TV_i,n-1}\right) m_{V_i}\left(\left\{\begin{array}{l}\text{points who } \textbf{are } \varepsilon/\Lambda^{n}\text{-}\\ \text{close to the boundary}\\ \text{of } W \text{ now}\end{array}\right\}\right).$$

Finally, the continuation property of the singularity set implies

$$\sum_{i=1}^{K_W} K_{TV_i,n-1} = K_{W,n},$$

which completes the proof.

$\square$

Now this n-step growth lemma, **together with the subexponential complexity condition** immediately implies the growth condition of Chernov for some iterate of T.

Indeed, let n_0 be so big that $K_{n_0} < \Lambda^{n_0}$, and

- Choose the size limit for the local unstable manifolds so small that $K_{W,n_0} < K_{n_0}$ for all of them,
- set $\tilde{T} = T^{n_0}$.

3.1 Discussion of the complexity assumption

This section is short, since we can make almost no positive statements. We begin with the
Bad news:

- We **cannot show** (or prove the existence of) **any example** of multi-dimensional dispersing billiard configuration, where the assumption is satisfied.
- Moreover, we are able to construct a (highly non-typical) example, where the condition is not satisfied.

In the **positive** direction, we only have a **conjecture** to formulate:

Conjecture 3.3. *The sequence of complexities K_n is bounded for typical billiard configurations, in any reasonable sense of typicality.*

This is based simply on counting dimensions. Heuristically, in a 3-dimensional system, if everything is typical,

- the Poincaré phase space is 4 dimensional,
- the singularity set has one dimension less, which is 3,
- double singularities have dimension 2,
- triple singularities are lines,
- four times singular points are isolated,
- and five times singular points don't exist.

If that picture were true, and (in a typical system) any trajectory could be at most 4 times singular, then K_n would be bounded by 2^4.

The notions of typicality that seem reasonable here:

1. Typicality **in an algebraic sense**. Say e.g., that the surfaces of the scatterers are desribed by algebraic equations, and the parameters of these equations can be adjusted. Then the set of "bad parameters" is clearly a countable union of closed submanifolds of the parameter space. (Indeed, in order for a phase point to have a trajectory which is tangent to five given scatterers, it has to satisfy five algebraic equations. Since there are only four variables, the set of solutions is expected to be empty, unless there's an unlucky coincidence of parameters.)
2. Typicality **in the C^3 topology**. The set of C^3 smooth scatterer configuratons for which no trajectory is tangent to some given five scatterers, is expected to be open and dense. So the set of "good" configurations is a countable intersection of open dense sets.

We are able to prove the two-dimensional version of the conjecture with the topological sense of typicality. However, that is of not much use, since in $2D$ the complexity is long known to be at most linear for every configuration.

Clearly, the **difficulty** in the proof of such conjectures is the phenomenon of **recollisions**: a trajectory may visit a scatterer (and even a collision point) many times, so when one perturbs the configuration, the effect on the trajectories is very difficult to know.

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