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On the asymptotic behaviour of geometric q-PushTASEP with particle creation model

TDK Thesis

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1 Introduction

The focus of this paper is to establish and offer a comprehensive proof for the scaling limit of the distribution of the geometric q-PushTASEP with particle creation model. Before delving into the intricacies, it's important to shed light on the Kardar-Parisi-Zhang (KZP) equation, which holds significance in the realm of random polymer models and interacting particle systems. These systems play a crucial role in understanding the behavior of various physical phenomena, aiding in the exploration of complex dynamics in diverse fields.

In the subsequent section, we introduce the q-PushTASEP, a form of stochastic interacting particle system, and its half-space variant, unraveling their connections to the random polymer models. Additionally, we delve into the Baik-Rains crossover distribution and present the central theorem. This theorem asserts that, under specific conditions, the scaling limit of the geometric q-PushTASEP with particle creation model converges to this distribution.

The fourth section of this paper entails a detailed exposition of the main theorem. Here, we employ the use of saddle point analysis techniques to achieve the convergence of Pfaffian formulas. This analysis serves as a crucial stepping stone in establishing the validity of the proposed scaling limit.

2 Kardar-Parisi-Zhang equation

The Kardar-Parisi-Zhang (KPZ) equation is a stochastic partial differential equation that has gained significant attention in the study of nonequilibrium statistical physics and the dynamics of fluctuating interfaces. It was introduced by Kardar, Parisi, and Zhang in 1986 as a model for the growth of interfaces in a random medium. It is given by

$$\partial_t H = \frac{1}{2}\partial_x^2 H + \frac{1}{2}(\partial_x H)^2 + \dot{W}, \tag{1}$$

where $H = H(x,t), x \in \mathbb{R}, t > 0$ is the height function and $\dot{W}$ is the space-time gaussian white noise with covariance $\mathbb{E}[\dot{W}(x,t)\dot{W}(y,s)] = \delta_{x-y}\delta_{t-s}$.

The KPZ equation is intimately related to the study of interacting particle systems and random polymers. In the context of interacting particle systems, the KPZ equation describes the scaling limit of various stochastic lattice models, such as the asymmetric simple exclusion process (ASEP) and the totally asymmetric simple exclusion process (TASEP). These models represent the dynamics of interacting particles on a lattice, where the KPZ equation emerges as the continuum limit of the corresponding height function.

Moreover, in the study of random polymers, the KPZ equation arises as a key tool for analyzing the statistical properties of polymers in disordered media. By considering the growth of a polymer chain or the movement of a directed random walk in a random environment, the KPZ equation provides insights into the fluctuations and roughening behavior of the polymer interface.

3 q-PushTASEP and its half-space variant

The q-PushTASEP, introduced in [MP16], is an extension of the totally asymmetric exclusion process (TASEP), incorporates an additional parameter q to capture more complex interactions between particles in a one-dimensional lattice. This model has gained prominence in nonequilibrium statistical physics, offering a more nuanced understanding of particle dynamics and collective phenomena.
The introduction of the q parameter allows for a deeper exploration of the interplay between particle interactions, making it a valuable tool in studying the scaling limits and universal properties of interacting particle systems. Additionally, it provides insights into the behavior of polymers in disordered environments, enriching the theoretical framework for analyzing polymer-related phenomena.

For the description of the models in this section we first need the definition of the *q-Geometric distribution*, which is defined via the limit of the *q-deformed beta binomial distribution.*

Definition 3.1. The q-deformed beta binomial distribution with parameters $q \in (0,1), \xi \in (0,1), \eta \in (0,\xi)$ is the probability distribution on $\{0, \cdots, m\}$ for $m \in \mathbb{N}$ given by the weight function

$$\phi_{q,\xi,\eta}(s|m) = \xi^s \frac{(\eta/\xi;q)_s(\xi;q)_{m-s}}{(\eta;q)_m} \frac{(q;q)_m}{(q;q)_s(q;q)_{m-s}}$$

where $(x;q)_s = \prod_{i=0}^{s}(1-xq^i)$ and $(x;q)_\infty = \prod_{i=0}^{\infty}(1-xq^i)$ are the q-Pochhammer symbols.

Definition 3.2. The q-Geometric distribution with parameters q, ξ is a probability distribution on $\mathbb{N}$ given by the weight function

$$\phi_{q,\xi}^{geo}(s) = \phi_{q,\xi,0}(s|\infty) = \xi^s \frac{(\xi;q)_\infty}{(q;q)_\infty}$$

which we denote by $q - Geo(\xi)$

Definition 3.3. Geometric q-PushTASEP with parameters N,q,$(a_i)_{i\in\{1,\cdots,N\}}$,$(b_i)_{i\in\mathbb{N}}$. In this system N particles are placed on the sites of the lattice $\mathbb{Z}$ at time 0, their position at any time $T \in \mathbb{N}$ is recorded in the array $Y(T) = (y_1(T) < \cdots < y_N(T))$. During the discrete time step from T to $T+1$, $Y(T)$ is updated sequentially from left to right by the following rules:

$$y_k(T+1) = y_k(T) + V_{k,T} + W_{k,T}, \qquad \text{for } k = 1, \ldots, N \tag{2}$$

where $V_{k,T}$ and $W_{k,T}$ are independent random variables defined as

$$\begin{aligned} V_{k,T} &\sim q - Geo(a_k b_{T+1}), \\ W_{k,T} &\sim \phi_{q^{-1},q^{gap_k(T)},0}(\cdot, y_{k-1}(T+1) - y_{k-1}(T)), \\ gap_k(T) &= y_k(T) - y_{k-1}(T) - 1, \end{aligned}$$

and by convention we set $y_0(T) = -\infty$.

In this model every particle's jump consists of a $q - Geometric$ independent jump $V_{k,T}$ and a random push from the previous particle's jump $W_{k,T}$. A jump occurs if and only if an empty space is available for the jump while preserving the original ordering of the particles.
This model is frequently viewed as a random particle system with infinitely many particles. Since the behaviour of a particle is independent of the behaviour of later particles, it is easy to extend the definition to incorporate countable many particles.

The Log Gamma Polymer model is a random polymer model on the planar lattice. On each site we place an inverse gamma random variable and for a site we consider the weighted sum of all possible upright paths from the origin. This defines the partition function of the model. The Log Gamma Polymer model arises as a scaled limit of the geometric q-PushTASEP.

The geometric q-PushTASEP with particle creation was introduced in [BBC20] as a new exactly solvable particle system. In scaled limit it corresponds to the half-space version of the Log Gamma Polymer model, where we restrict the planar lattice below the identity function.

Definition 3.4. The geometric q-PushTASEP with particle creation with parameters $\gamma > 0, (a_i)_{i\in\mathbb{N}}$ is a discrete-time Markov process on configurations of particles

$$0 = x_0(T) < x_1(T) < \cdots < x_T(T) < \infty,$$

At any time $T \in \mathbb{Z}_{\geq 0}$ there are $T+1$ particles in the system located on the nonnegative integers. The position of the i'th particle at time T are denoted by the above $x_i(T)$, $i \in \{0, \dots, T\}$.
Let $x(T) = (x_0(T), x_1(T), \cdots, x_T(T))$. In the discrete time step from T to $T+1$, the configuration gets updated from the particle with index 1 to the right by the q-PushTASEP rules:

$$x_k(T+1) = x_k(T) + V_{k,T} + W_{k,T}, \qquad \text{for } k = 1, \dots, T \tag{3}$$

where $V_{k,T}$ and $W_{k,T}$ are independent random variables defined as

$$\begin{aligned} V_{k,T} &\sim q - Geo(\gamma a_{T+1}), \\ W_{k,T} &\sim \phi_{q^{-1}, q^{gap_k(T)}, 0}(\cdot, x_{k-1}(T+1) - x_{k-1}(T)), \\ gap_k(T) &= x_k(T) - x_{k-1}(T) - 1, \end{aligned}$$

As in the q-PushTASEP a jump can occur if and only if an empty space is available for the jump while preserving the original ordering of the particles.
After completion of this sequence a new particle is added to the right of the already present particles following the rule $x_{T+1}(T+1) = x_T(T+1) + 1 + \tilde{V}_{T+1}$, where $\tilde{V}_{T+1} \sim q - Geo(\gamma a_{T+1})$ is an independent increment. Here we assume that $a_i a_j, \gamma a_i < 1$ for every $1 \leq i < j$.

In [IMS22] they state that in a special setting the geometric q-PushTASEP with particle creation converges to the Baik-Rains crossover distribution. This distribution appears as a continuous transition between the Tracy-Widom distributions corresponding to the symplectic and orthogonal gaussian ensembles (GSE, GOE) [TW96]. Our main result is the detailed proof of this theorem, which will be presented in section 4. In order to state the theorem, we first need to define the Pfaffian of a matrix kernel and the Baik-Rains crossover distribution.

Definition 3.5. Let $L(x, y)$ be 2×2 matrix kernel of the form

$$L(x,y) = \begin{bmatrix} k_{1,1}(x,y) & k_{1,2}(x,y) \\ -k_{1,2}(x,y) & k_{2,2}(x,y) \end{bmatrix}.$$

where $k_{i,j} \in L^(\Omega \times \Omega, \nu \otimes \nu)$ and $k_{i,i}(x, y) = -k_{i,i}(x, y)$ for all $x, y \in \Omega$ and $i = 1, 2.$. Then the Fredholm Pfaffian of L is defined as

$$Pf(J-L)_{L^2(\Omega)} = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int_{\Omega^l} Pf[L(x_i,x_j)]_{i,j=1}^{l} \nu(dx_1)\cdots\nu(dx_l) \qquad (4)$$

Definition 3.6. The Baik-Rains crossover distribution with parameter $\xi > 0$ is given by

$$F_{cross}(s;\xi) = Pf\left(J - K^{(\xi)}_{cross}\right)_{L^2(s,\infty)},\ s \in \mathbb{R} \qquad (5)$$

where $K^{(\xi)}_{cross}$ is the 2×2 matrix kernel

$$K^{(\xi)}_{cross}(u,v) = \begin{bmatrix} k^{(\xi)}(x,y) & -\partial_v k^{(\xi)}(x,y) \\ -\partial_u k^{(\xi)}(x,y) & \partial_u\partial_v k^{(\xi)}(x,y) \end{bmatrix}. \qquad (6)$$

$$k(x(u),y(v)) = \int_{C_\delta^{\pi/3}} \frac{d\alpha}{2\pi i} \int_{C_\delta^{\pi/3}} \frac{d\beta}{2\pi i} \frac{(\xi+\alpha)(\xi+\beta)}{(\xi-\alpha)(\xi-\beta)} \frac{\alpha-\beta}{4\alpha\beta(\alpha+\beta)} e^{\frac{\alpha^3}{3}-\alpha u} e^{\frac{\beta^3}{3}-\alpha v} \qquad (7)$$

where $C^\theta_p = \{p + |r|e^{sign(r)i\theta} : r \in \mathbb{R}\}$ is a V-shaped contour symmetric with respect to the real line, where the angle between the two semi-infinite rays extending away from the origin is 2θ, and its apex is located at point p. Here we choose $0 < \delta < \xi$.

Now we are able state the main theorem of the paper.

Theorem 3.7. *Let $x(T)$ be the geometric q-PushTASEP with particle creation and assume that the parameters are $a_1 = a_2 = \cdots a \in (0, 1)$ and $\gamma \in (0, 1]$. Then the following limit holds:*

rescaling $\gamma = 1 - \frac{\xi}{\sigma N^{1/3}}$ *we have:*

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_N(N) - (p+1)N}{\sigma N^{1/3}} \leq s'\right] = F_{cross}(s';\xi) \tag{8}$$

where $p = \frac{2}{\log q}\left[\log(1-q) + \psi_q(\log_q a)\right]$ *and* $\sigma = \frac{1}{\log q}\sqrt[3]{\psi_q^{(2)}(\log_q(a))}$.

4 Asymptotic behaviour of geometric q-PushTASEP with particle creation

As mentioned in the previous section, in [IMS22] they give the limiting distribution of the geometric q-PushTASEP with particle creation in a special setting and provide a sketch for the proof. In this section we prove this theorem in more detail following the argument outlined in the mentioned article with some modification and correction.

In [IMS22] they give and prove the following Pfaffian representation of the distribution function of $x_N(N)$ convoluted with specific independent random variables.

Proposition 4.1. *Let* $x(T)$ *be the geometric q-PushTASEP with particle creation with parameters* $\gamma, a_1, a_2, \dots \in (0,1)$ *and empty initial conditions. Let also* $\chi \sim q - Geo(q)$ *and* $S \sim Theta(\zeta^2, q^2)$ *be independent random variables. Then we have*

$$\mathbb{P}\left(x_N(N) - N + \chi + 2S < s\right) = Pf(J - L_{push})_{l^2(\mathbb{Z}'_{>s})}, \tag{9}$$

where L_{push} *is the* 2×2 *matrix kernel given by*

$$L_{push}(x,y) = \begin{bmatrix} k(x,y) & -\nabla_y k(x,y) \\ -\nabla_x k(x,y) & \nabla_x\nabla_y k(x,y) \end{bmatrix}. \tag{10}$$

In the next equations we define all necessary components of L_{push}:

$$\begin{aligned}
k(x,y) &= \frac{1}{(2\pi i)^2}\oint_{|z|=r}\frac{dz}{z^{x+3/2}}\oint_{|w|=r'}\frac{dw}{w^{y+5/2}}F(z)F(w)k^{hs}(z,w) \\
F(z) &= \frac{(\gamma/z;q)_\infty}{(\gamma z;q)_\infty}\prod_{i=1}^{N}\frac{(a_i/z;q)_\infty}{(a_i z;q)_\infty} \\
k^{hs}(z,w) &= \frac{(q,q,w/z,qz/w;q)_\infty}{(1/z^2,1/w^2,1/zw,qwz;q)_\infty}\frac{\vartheta_3(\zeta^2z^2w^2;q)}{\vartheta_3(\zeta^2;q^2)} \qquad (11)\\
\vartheta_3(\zeta;q) &= (q,-\sqrt{q}\zeta,-\sqrt{z}/\zeta;q)_\infty \\
(x_1,x_2,\dots,x_n;q)_\infty &= (x_1;q)_\infty(x_2;q)_\infty\cdots(x_n;q)_\infty \\
\nabla_x f(x) &= \frac{1}{2}[f(x+1)-f(x-1)]
\end{aligned}$$

Proof of Theorem 3.7. The starting point is Fredholm Pfaffian representation given in Proposition 4.1. We use the following scaling:

$$s=(p-1)N+\sigma N^{1/3}s',\ x=(p-1)N+\sigma N^{1/3}u,\ y=(p-1)N+\sigma N^{1/3}v, \qquad (12)$$

where we assume that $u,v>s'$.
Then $k(x,y)$ from 11 can be written as

$$k(x(u),y(v)) = \oint_{|z|=r}\frac{dz}{2\pi i z^{3/2}}\oint_{|w|=r'}\frac{dw}{2\pi i w^{5/2}}\frac{(\gamma/z,\gamma/w;q)_\infty}{(\gamma z,\gamma w;q)_\infty}\kappa(z,w)\times e^{Nh(z)-\sigma N^{1/3}(u)\log z}e^{Nh(w)-\sigma N^{1/3}(v)\log w} \qquad (13)$$

Where $u,v>s'$ and h in the exponents is defined as:

$$h(z)=\log(a/z;q)_\infty-\log(az;q)_\infty-p\log z \qquad (14)$$

The main contribution to the value of the integral comes from the term $e^{Nh(Z)+Nh(w)}$ when N is large.

Step 1: Saddle point analysis of h.

h can be rewritten as an infinite sum:

$$h(z)=-p\log z+\sum_{k=0}^{\infty}\log(1-\frac{a}{z}q^k)-\log(1-azq^k) \qquad (15)$$

Using this we can find its derivatives up to the third order and show that at $z=1$

it has a double critical point.

$$
\begin{aligned}
h(1) &= 0 \\
h'(z) &= \frac{\psi_q\left(\frac{\log\left(\frac{a}{z}\right)}{\log(q)}\right) + \psi_q\left(\frac{\log(az)}{\log(q)}\right) - p\log(q) + 2\log(1-q)}{z\log(q)} \\
h'(1) &= \frac{2\left(\psi_q\left(\frac{\log(a)}{\log(q)}\right) + \log(1-q)\right)}{\log(q)} - p = 0 \\
h''(z) &= \frac{1}{z^2\log^2(q)}\left(-\log(q)\left(\psi_q^{(0)}\left(\frac{\log\left(\frac{a}{z}\right)}{\log(q)}\right) + \psi_q^{(0)}\left(\frac{\log(az)}{\log(q)}\right)\right)\right. \\
&\quad \left. -\psi_q^{(1)}\left(\frac{\log\left(\frac{a}{z}\right)}{\log(q)}\right) + \psi_q^{(1)}\left(\frac{\log(az)}{\log(q)}\right) + \log(q)(p\log(q) - 2\log(1-q))\right) \\
h''(1) &= p - \frac{2\left(\psi_q^{(0)}\left(\frac{\log(a)}{\log(q)}\right) + \log(1-q)\right)}{\log(q)} = 0 \\
h'''(1) &= \frac{2\psi_q^{(2)}\left(\frac{\log(a)}{\log(q)}\right)}{\log^3(q)} = 2(\sigma)^3
\end{aligned} \tag{16}
$$

These calculations show that the function h has a double critical point at 1 and $h'''(1) = 2(\sigma)^3 > 0$

Now we will show that the contour $|z| = r$ is steep descent for the function h. We substitute $z = re^{\theta i}$ and compute the derivate with respect to θ.:

$$
\begin{aligned}
h(re^{\theta i}) &= -p\log(re^{\theta i}) + \sum_{k=0}^{\infty} \log(1 - \frac{a}{r}e^{-\theta i}q^k) - \log(1 - are^{\theta i}q^k) \\
\Re h(re^{\theta i}) &= -p\frac{1}{2}\log(r^2) + \sum_{k=0}^{\infty} \frac{1}{2}\log\left(\frac{a^2q^{2k} - 2aq^kr\cos(\theta) + r^2}{r^2}\right) \\
&\quad - \frac{1}{2}\log\left(a^2q^{2k}r^2 - 2aq^kr\cos(\theta) + 1\right) \\
\frac{\partial}{\partial\theta}\Re h(re^{\theta i}) &= \sum_{k=0}^{\infty} \frac{aq^kr\sin(\theta)\left(a^2q^{2k}r^2 - a^2q^{2k} - r^2 + 1\right)}{(a^2q^{2k} - 2aq^kr\cos(\theta) + r^2)(a^2q^{2k}r^2 - 2aq^kr\cos(\theta) + 1)}
\end{aligned} \tag{17}
$$

From this it is easy to see that for $\theta \in (0, \pi)$ $\frac{\partial}{\partial\theta}\Re h(re^{\theta i}) < 0$ and for $\theta \in (\pi, 2\pi)$ $\frac{\partial}{\partial\theta}\Re h(re^{\theta i}) > 0$, which proves that the integration contour $|z| = r$ steep descent for the function h. Figure 1 shows the real part of h and deformed integration contour which will be defined in the next step.

Step 2: Deformation of the integration contours.

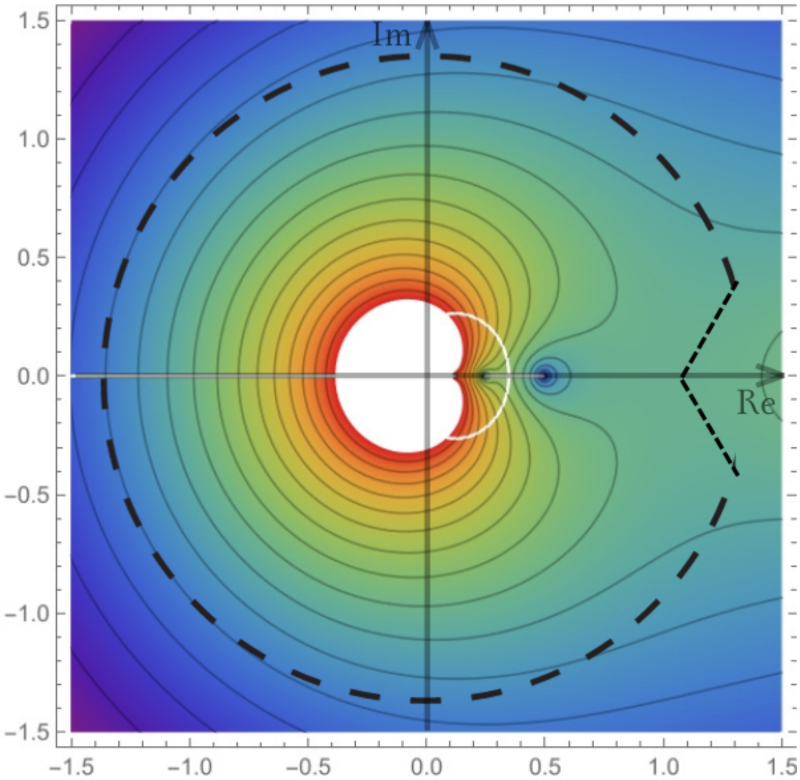

Figure 1: Real part of the function h.

By the Cauchy theorem we can deform the integration contours $|z| = r$, $|w| = r'$ to be

$$C^{\pi/3}\left(1 + \frac{\delta}{\sigma N^{1/3}}; r\right) \cup D(\bar{\theta}, r) \tag{18}$$

where

$$C^{\varphi}(a; r) = (z = a + \rho e^{sign(\rho) i\varphi} : |z| < r), \quad D(\varphi, r) = \{re^{i\varphi} : \theta \in (\varphi, 2\pi - \varphi)\} \tag{19}$$

and $\bar{\theta} > 0$ solves the equation $1 + \frac{\delta}{\sigma N^{1/3}} + \rho e^{i\frac{\pi}{3}} = re^{i\bar{\theta}}$ for some $\rho > 0$; see fig 2, which ensures that the two parts of the contour are connected. We choose δ so that the pole γ^{-1} remains outside of the contour. By simple derivation it can be also shown that for δ small enough this modified contour is also steep descent for the function h, where $\Re h(z)$ attains its maximum at $z = 1 + \frac{\delta}{\sigma N^{1/3}}$

When $N \to \infty$ such integrals are dominated by the contribution over the curve $C^{\pi/3}\left(1 + \frac{\delta}{\sigma N^{1/3}}; r\right) \cap \{z : |1 - z| < d\}$ for any $d > 0$ small enough.

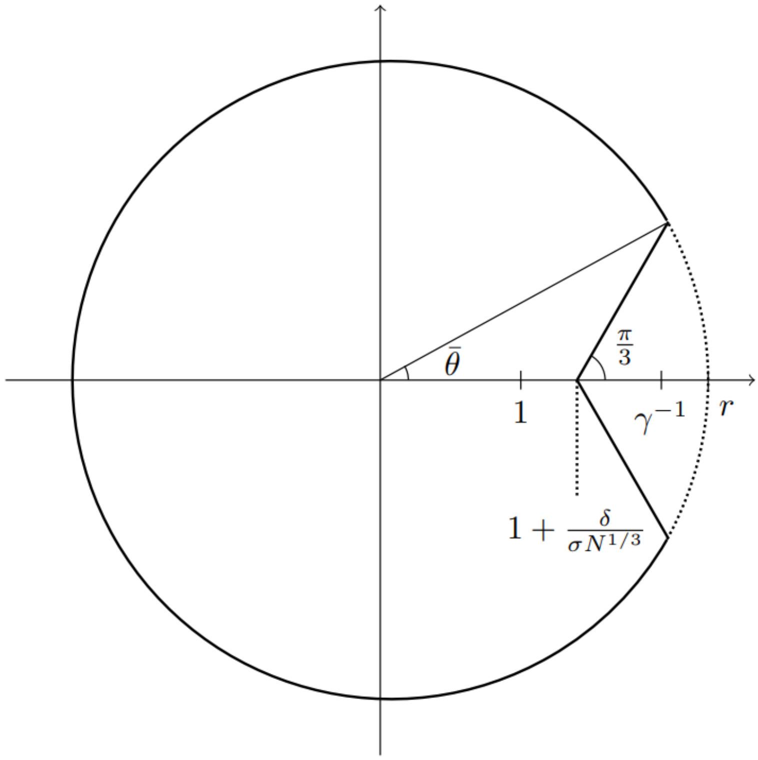

Figure 2: Deformation of the contour.

Step 3: Localization of the integral and change of variables.
Now we can localize our integral in both variables to the curve $C := C^{\pi/3}\left(1 + \frac{\delta}{\sigma N^{1/3}}; r\right) \cap \{z : |1-z| < d\}$ where we can approximate the functions h and log by their Taylor-expansion around the critical point $z = 1$:

$$\begin{aligned} h(z) &= (\sigma)^3 \frac{(z-1)^3}{3} + \mathcal{O}((z-1)^4) \\ \log(z) &= (z-1) + \mathcal{O}((z-1)^2) \end{aligned} \tag{20}$$

Now let $z = 1 + \frac{\alpha}{\sigma N^{1/3}}$ and $w = 1 + \frac{\beta}{\sigma N^{1/3}}$. Then in the exponent we get that

$$\begin{aligned} Nh(z) - \sigma N^{1/3} u \log z &= \frac{\alpha^3}{3} - \alpha u + \mathcal{O}\left(N^{-1/3}\left(\alpha^4 + \alpha^2\right)\right) \\ Nh(w) - \sigma N^{1/3} v \log z &= \frac{\beta^3}{3} - \alpha v + \mathcal{O}\left(N^{-1/3}\left(\beta^4 + \beta^2\right)\right) \end{aligned} \tag{21}$$

as $N \to \infty$.

We also need to calculate the limit and asymptotic behaviour of the remaining term of the integrand. First let us consider only $\frac{(\gamma/z;q)_\infty}{(\gamma z;q)_\infty}$:

$$\begin{aligned}\frac{(\gamma/z;q)_\infty}{(\gamma z;q)_\infty} &= \frac{1-\gamma/z}{1-\gamma z}\frac{\prod_{k=1}^\infty(1-\gamma/zq^k)}{\prod_{k=1}^\infty(1-\gamma zq^k)}\\ &= \frac{1-\frac{1-\frac{\xi}{\sqrt[3]{N}\sigma}}{\frac{\alpha}{\sqrt[3]{N}\sigma}+1}}{1-\left(\frac{\alpha}{\sqrt[3]{N}\sigma}+1\right)\left(1-\frac{\xi}{\sqrt[3]{N}\sigma}\right)}\frac{\prod_{k=1}^\infty(1-\frac{q^k\left(1-\frac{\xi}{\sqrt[3]{N}\sigma}\right)}{\frac{\alpha}{\sqrt[3]{N}\sigma}+1})}{\prod_{k=1}^\infty(1-q^k\left(\frac{\alpha}{\sqrt[3]{N}\sigma}+1\right)\left(1-\frac{\xi}{\sqrt[3]{N}\sigma}\right))}\end{aligned} \tag{22}$$

where the second fraction goes to 1 as $N\to\infty$ and for the first fraction we get the following limit and asymptotic

$$\begin{aligned}\frac{1-\frac{1-\frac{\xi}{\sqrt[3]{N}\sigma}}{\frac{\alpha}{\sqrt[3]{N}\sigma}+1}}{1-\left(\frac{\alpha}{\sqrt[3]{N}\sigma}+1\right)\left(1-\frac{\xi}{\sqrt[3]{N}\sigma}\right)} &= \frac{\xi+\alpha}{\xi-\alpha}+\frac{\alpha(\alpha+\xi)\left(\alpha\xi-\sqrt[3]{N}\sigma(\alpha-2\xi)\right)}{(\alpha-\xi)\left(\alpha+\sqrt[3]{N}\sigma\right)\left(\alpha\xi+\sqrt[3]{N}\sigma(\xi-\alpha)\right)} =\\ &= \frac{\xi+\alpha}{\xi-\alpha}+\frac{\Theta(N^{1/3})}{\Theta(N^{2/3})}\\ &= \frac{\xi+\alpha}{\xi-\alpha}+\mathcal{O}\left(\frac{1}{N^{1/3}}\right)\end{aligned} \tag{23}$$

Considering this for all the q-Pochhammer symbols in the remaining term of the integrand we get

$$\begin{aligned}\frac{1}{z^{3/2}w^{5/2}}\frac{(\gamma/z,\gamma/w;q)_\infty}{(\gamma z,\gamma w;q)_\infty}\kappa(z,w) &=\\ &= \frac{(\xi+\alpha)(\xi+\beta)}{(\xi-\alpha)(\xi-\beta)}\frac{\alpha-\beta}{4\alpha\beta(\alpha+\beta)}+\mathcal{O}\left(\frac{1}{N^{1/3}}\right)\\ &\text{as } N\to\infty.\end{aligned} \tag{24}$$

Putting everything together we get that:

$$\begin{aligned}k(x(u),y(v)) = \int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}\left(\frac{(\xi+\alpha)(\xi+\beta)}{(\xi-\alpha)(\xi-\beta)}\frac{\alpha-\beta}{4\alpha\beta(\alpha+\beta)}+\mathcal{O}\left(\frac{1}{N^{1/3}}\right)\right)\times\\ \times e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}+\mathcal{O}\left(e^{-\epsilon N}\right)\end{aligned} \tag{25}$$

Where the integration contour C_N is given by $C_N = C^{\pi/3}\left(1+\frac{\delta}{\sigma N^{1/3}};\sigma N^{1/3}r\right)\cap\{z:|1-z|<\sigma N^{1/3}d\}$.

Step 4: Limit calculations.

Now we can divide the above integral into two parts by expanding in the multiplication. Then the first term is the following:

$$\begin{aligned}
&\int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}f(\alpha,\beta,\xi)e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}=\\
&=\int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}f(\alpha,\beta,\xi)e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}+\\
&+\int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}f(\alpha,\beta,\xi)e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}\left(e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}-1\right)
\end{aligned}\tag{26}$$

where $f(\alpha,\beta,\xi)=\frac{(\xi+\alpha)(\xi+\beta)}{(\xi-\alpha)(\xi-\beta)}\frac{\alpha-\beta}{4\alpha\beta(\alpha+\beta)}$.

The second term in the sum is the error term which goes to 0 as $N\to\infty$, since we can upper bound its absolute value in the following way using $|e^x-1|\leq|x|e^{|x|}$:

$$\begin{aligned}
&\left|\int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}f(\alpha,\beta,\xi)e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}\left(e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}-1\right)\right|\leq\\
&\int_{C_N}\frac{d\alpha}{2\pi}\int_{C_N}\frac{d\beta}{2\pi}|f(\alpha,\beta,\xi)|\left|e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}\right|\left|\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)\right|\times\\
&\times\left|e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}\right|\leq\\
&\int_{C_N}\frac{d\alpha}{2\pi}\int_{C_N}\frac{d\beta}{2\pi}|f(\alpha,\beta,\xi)|e^{\Re\left(\frac{\alpha^3}{3}+d\frac{|\alpha^3|}{(\sigma)^3}-\alpha u(1-\frac{1}{\sigma})\right)}e^{\Re\left(\frac{\beta^3}{3}+d\frac{|\beta^3|}{(\sigma)^3}-\beta v(1-\frac{1}{\sigma})\right)}\times\\
&\times\left|\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)\right|
\end{aligned}\tag{27}$$

Where we used that $|\alpha|\leq d\sigma N^{1/3}$. d can be chosen to be small enough in order to $\Re\left(\frac{\alpha^3}{3}+d\frac{|\alpha^3|}{(\sigma)^3}\right)<0$, which ensures that the integral remains finite. By the dominated convergence theorem this integral converges to 0, which proves that the error term vanishes as $N\to\infty$.

The same argument can be applied to the other term as well:

$$\begin{aligned}
&\int_{C_N}\frac{d\alpha}{2\pi i}\int_{C_N}\frac{d\beta}{2\pi i}\mathcal{O}\left(\frac{1}{N^{1/3}}\right)\times\\
&\times e^{\frac{\alpha^3}{3}-\alpha u}e^{\frac{\beta^3}{3}-\alpha v}e^{\mathcal{O}\left(N^{-1/3}\left(\alpha^4+\beta^4+\alpha^2+\beta^2\right)\right)}\to 0,\text{ as }N\to\infty
\end{aligned}\tag{28}$$

These together show that

$$\lim_{N\to\infty} k(x(u),y(v)) = k^{(\xi)}(u,v) \qquad \text{pointwise.} \tag{29}$$

Step 5: Exponential decay of $k(x(u),y(v))$
For this, we will follow the arguments presented in Lemma 4.1. of [FV12].
Now we show an that the function $k(x(u),y(v))$ has exponential decay in both variables. This results in uniform convergence.
For this consider $u = \frac{N^{2/3}}{\sigma}U$ and $v = \frac{N^{2/3}}{\sigma}V$. With this scaling the term in the exponential in the integrand of $k(x(u),y(v))$ becomes the following

$$Nh(z) - \sigma N^{1/3} u \log z = Nh(z) - NU \log z = N\tilde{h}(z) \tag{30}$$

where $\tilde{h}(z) = \log(a/z;q)_\infty - \log(az;q)_\infty - (p+U)\log z$. This $\tilde{h}$ function has a critical point near $1 + \sqrt{\frac{U}{\sigma^3}}$, since

$$\begin{gathered}\tilde{h}'(z) = h'(z) - \frac{U}{z} \approx \sigma^3(z-1)^3 - \frac{U}{z} \\ \text{which equals zero if } z \approx 1 + \sqrt{\frac{U}{\sigma^3}}\end{gathered} \tag{31}$$

We can use the same saddle point analysis methods that were employed for h to calculate the limit of the integral $k(x(u),y(v))$.
Let $\epsilon > 0$ which will be determined later and let $\bar{\epsilon} = \min(\frac{U}{\sigma^3}, \frac{\epsilon}{\sigma^3})$. Consider the integral contour $\Gamma_{\bar{\epsilon}} = C^{\pi/3}(1+e;r) \cup D(\bar{\theta}, r)$ as in 18. We modify both integral contours in k to be Γ_e, but then as $N \to \infty$ the pole γ^{-1} will enter, so we need to subtract the corresponding residues to get equality.

$$\begin{aligned}k(x(u),y(v)) = &\oint_{\Gamma_{\bar{\epsilon}}} \frac{dz}{2\pi i z^{3/2}} \oint_{\Gamma_{\bar{\epsilon}}} \frac{dw}{2\pi i w^{5/2}} \frac{(\gamma/z, \gamma/w;q)_\infty}{(\gamma z, \gamma w;q)_\infty} \kappa(z,w) e^{N\tilde{h}(z)+N\tilde{h}(w)} - \\ &-A(u)B(v) + A(v)B(u)\end{aligned} \tag{32}$$

where

$$\begin{aligned}A(u) &= e^{N\tilde{h}(1/\gamma)} \\ B(v) &= \sqrt{\gamma} \oint_{|w|=r'} \frac{dw}{2\pi i w^{5/2}} \frac{(\gamma^2, \gamma/w;q)_\infty}{(\gamma w;q)_\infty (q;q)_\infty} k^{hs}(1/\gamma, w) e^{N\tilde{h}(w)}\end{aligned} \tag{33}$$

For the double integral term it can be shown that the integral Γ_e is steep descent for ϵ small enough, so after substituting $z = 1 + \frac{\alpha}{\sigma N^{1/3}}$ and $w = 1 + \frac{\beta}{\sigma N^{1/3}}$ and using that $\tilde{h}(z) = \frac{\sigma^3}{3}U^{3/2} - U\log(z) + \mathcal{O}\left((z-1)^2\right)$ we get that

$$\tilde{h}\left(1 + \sqrt{\frac{U}{\sigma^3}}\right) = -\frac{2}{3}\frac{U^{3/2}}{\sigma^{3/2}} + \mathcal{O}(U),$$

and for some positive constants M, m, m'

$$\left|\oint_{\Gamma_{\bar{\epsilon}}} \frac{dz}{2\pi i z^{3/2}} \oint_{\Gamma_{\bar{\epsilon}}} \frac{dw}{2\pi i w^{5/2}} \frac{(\gamma/z, \gamma/w; q)_\infty}{(\gamma z, \gamma w; q)_\infty} \kappa(z,w) e^{N\tilde{h}(z) + N\tilde{h}(w)}\right| \leq \tag{34}$$

$$\leq M e^{m(-\frac{2}{3}(\frac{U^{3/2}}{\sigma^{3/2}} + \frac{V^{3/2}}{\sigma^{3/2}}))} \leq$$

$$\leq M e^{-m'(u^{3/2} + v^{3/2})}$$

It is easy to see for the function A that it has an exponential decay in u, and for the exponential decay of B we can use the argument presented here, but only in one variable. These together show that $k(x(u), y(v))$ has exponential decay in both variables.

$$|k(x(u), y(v))| \leq K e^{-k'(u+v)}, \text{ for some } K, k' \text{positive constants.} \tag{35}$$

The same arguments can be applied for $-\nabla_y k, -\nabla_x k, \nabla_x \nabla_y k$ which proves by the dominated convergence theorem

$$Pf[J - L_{push}]_{l^2(\mathbb{Z}_{>s})} \longrightarrow Pf[J - K^{(\xi)_{cross}}]_{L^2(s', \infty)}. \tag{36}$$

This concludes the proof of the theorem.

□

5 Summary

In conclusion, this paper delved into the scaling limit of the distribution of the geometric q-PushTASEP with particle creation model. We began by exploring the Kardar-Parisi-Zhang equation and its connection to random polymer models and interacting particle systems.

We then introduced the q-PushTASEP and its half-space variant, discussing their connections to random polymer models and the Baik-Rains crossover distribution. The main theorem stated that the scaling limit of the geometric q-

PushTASEP with particle creation model aligns with this distribution.

In the third section, we used saddle point analysis methods to confirm the convergence of Pfaffian formulas, providing solid evidence for the main theorem. Overall, this paper contributes to our understanding of stochastic interacting particle systems and their role in elucidating complex physical phenomena.

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