The geometry of coalescing random walks, the Brownian web distance and KPZ universality

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Abstract

Coalescing simple random walks in the plane form an infinite tree. A natural directed distance on this tree is given by the number of jumps between branches when one is only allowed to move in one direction. The Brownian web distance is the scale-invariant limit of this directed metric. It is integer-valued and has scaling exponents 0 : 1 : 2 as compared to 1 : 2 : 3 in the KPZ world. However, we show that the shear limit of the Brownian web distance is still given by the Airy process. We conjecture that our limit theorem can be extended to the full directed landscape.

1 Introduction

1.1 The discrete web distance

Consider a system of coalescing random walks Y on the even points

$$\mathbb{Z}^2_{\mathrm{e}} = \{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\} \tag{1.1}$$

of the planar lattice so that from each $(i,n) \in \mathbb{Z}^2_{\mathrm{e}}$ there are two outgoing directed edges: to $(i+1,n-1)$ and to $(i+1,n+1)$. Assign independent random variables $(\xi_{(i,n)})$ with $\mathbf{P}(\xi_{(i,n)}=1)=\mathbf{P}(\xi_{(i,n)}=-1)=1/2$ to the vertices in $\mathbb{Z}^2_{\mathrm{e}}$. The *random walk web* Y is a family of coalescing random walks starting at each point of $\mathbb{Z}^2_{\mathrm{e}}$. For all $(i,n) \in \mathbb{Z}^2_{\mathrm{e}}$, set $Y_{(i,n)}(i)=n$ and let

$$Y_{(i,n)}(j+1) = Y_{(i,n)}(j) + \xi_{(j,Y_{(i,n)}(j))} \qquad \text{for all } j=i,i+1,\dots. \tag{1.2}$$

For any $(i,n;j,m) \in \mathbb{Z}^4$ we define $D^{\mathrm{RW}}(i,n;j,m)$ to be the smallest integer k such that (j,m) can be reached from (i,n) by following the directed random walk paths in

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the random walk web Y and by performing k jumps between different random walk paths along all possible directed paths from (i,n) to (j,m) in the graph $\mathbb{Z}^2_{\mathrm{e}}$. This defines a directed metric of positive sign in the sense introduced in [DV21] hence we call the function D^{RW} as *random walk web distance* or *discrete web distance*.

Definition 1.1. *For any $(i,n;j,m) \in \mathbb{Z}^4$ let $D^{\mathrm{RW}}(i,n;j,m)$ be the smallest non-negative integer k such that there are $(i_1,n_1),\ldots,(i_k,n_k) \in \mathbb{Z}^2_{\mathrm{e}}$ with the following property. There are random walk paths in Y from (i,n) to (i_1,n_1), from $(i_l+1, n_1-\xi_{(i_l,n_l)})$ to (i_{l+1},n_{l+1}) for $l=1,\ldots,k-1$ and from $(i_k+1,n_k-\xi_{(i_k,n_k)})$ to (j,m). We set $D^{\mathrm{RW}}(i,n;j,m)=\infty$ if there is no such k.*

For the function D^{RW} it holds that $D^{\mathrm{RW}}(i,n;i,n)=0$ for all $(i,n)\in\mathbb{Z}^2_{\mathrm{e}}$ and it satisfies the triangle inequality: for all $(i,n),(k,l),(j,m)\in\mathbb{Z}^2_{\mathrm{e}}$

$$D^{\mathrm{RW}}(i,n;j,m) \le D^{\mathrm{RW}}(i,n;k,l) + D^{\mathrm{RW}}(k,l;j,m) \tag{1.3}$$

holds. Hence D^{RW} is a directed metric of positive sign on $\mathbb{Z}^2_{\mathrm{e}}$ according to [DV21]. We mention that $D^{\mathrm{RW}}(i,n;j,m)$ can alternatively be defined as a first passage time between (i,n) and (j,m) where the passage time of an edge is 0 if it is used by some random walk in Y, and 1 otherwise.

The paths of coalescing random walks in Y form a subset of the geodesics of this first passage percolation model and those of the discrete web distance D^{RW}.

1.2 The Brownian web distance

As the system of coalescing random walks converge to their Brownian counterparts, the discrete web distance also has a limit. It turns out that the limiting object can be defined based on the Brownian web and its dual hence we call it the *Brownian web distance*.

The Brownian web consists of independent coalescing Brownian motions starting at each point of the space-time $\mathbb{R}^2$. Based on work of Arratia [Arr81], the Brownian web was first rigorously constructed by Tóth and Werner [TW98]. In their work, the Brownian web it describes the local time profile for the true self-repelling motion. The term Brownian web first appeared in [FINR04].

The Brownian web distance can be introduced as an integer valued function as follows.

Definition 1.2. *For any $(t,x;s,y)\in\mathbb{R}^4$ let $D^{\mathrm{Br}}(t,x;s,y)$ denote the infimum of those non-negative integers k for which there exist points $(t_1,x_1),\ldots,(t_k,x_k)\in\mathbb{R}^2$ such that $t<t_1<\cdots<t_k<s$ and there exists a continuous path $\pi:[t,s]\to\mathbb{R}$ so that π restricted to the intervals $[t,t_1],[t_1,t_2],\ldots,[t_{k-1},t_k],[t_k,s]$ coincides with trajectories in the Brownian web B. The infimum is equal to $+\infty$ if there is no such $k\in\mathbb{N}$.*

In Definition 1.2 we mean that a continuous path $\pi:[t,s]\to\mathbb{R}$ restricted to an interval $[t_0,s_0]\subset[t,x]$ coincides with a trajectory in the Brownian web B if π agrees with one of the outgoing paths starting at $(t_0,\pi(t_0))$ and that they are equal up to time s_0.

We explain some properties of the Brownian web distance which are investigated in this paper. The scale invariance with exponents $0:1:2$ is an immediate consequence of Brownian scaling. The value of the exponents is in contrast with the KPZ scaling exponents $1:2:3$, see for example section 2.1 in [Gan21].

Proposition 1.3. *For any $\alpha > 0$, we have the equality in distribution*

$$\left(D^{\mathrm{Br}}(\alpha^2 t, \alpha x; \alpha^2 s, \alpha y), (t, x; s, y) \in \mathbb{R}^4\right) \stackrel{\mathrm{d}}{=} \left(D^{\mathrm{Br}}(t, x; s, y), (t, x; s, y) \in \mathbb{R}^4\right). \tag{1.4}$$

A model related to the Brownian web distance is the Brownian castle which arises as the scaling limit of the infinite temperature version of the ballistic deposition model. It was introduced in [CH23] as a scale-invariant Markov process with scaling exponents $1:1:2$ which is different from the one in the Edwards–Wilkinson class as well as from the KPZ class. The Brownian castle is constructed based on the Brownian web and its dual in the following way. To each segment of path in the dual Brownian web a centered Gaussian random variable is associated with variance equal to the length of the corresponding time interval and independently for disjoint paths. Then the value of the process at a space-time point is the sum of the Gaussian variables along the dual Brownian web path started from the point.

As one may expect the value of the Brownian web distance $D^{\mathrm{Br}}(t, x; s, y)$ can change dramatically by a small perturbation of the starting point (t, x) or by that of the endpoint (s, y). As we shall see from the description given in Subsection 3.1 the Brownian web distance is more sensitive to the changes of the endpoint. Hence the Brownian web distance is not a continuous function of its variables. However the following continuity still holds.

Theorem 1.4. *On an event of probability one the mapping $(t, x, s, y) \mapsto D^{\mathrm{Br}}(t, x; s, y)$ is lower semicontinuous.*

As a consequence, the Brownian web distance D^{Br} is a random variable with values in the space of lower semicontinuous functions on $\mathbb{R}^4$. Lower semicontinuity of functions is equivalent with having a closed epigraph. The natural metric is a partially compactified version of the Hausdorff distance between the epigraphs of the functions where the range of the functions is mapped to a compact interval. Then the space of lower semicontinuous functions becomes a separable metric space.

For an integer n and $(t, x, s, y) \in \mathbb{R}^4$ we define the rescaled discrete web distance as

$$D_n^{\mathrm{RW}}(t, x; s, y) = \begin{cases} D^{\mathrm{RW}}(nt, n^{1/2}x; ns, n^{1/2}y), & \text{if } (nt, n^{1/2}x, ns, n^{1/2}y) \in \mathbb{Z}^4, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

We prove the convergence of the rescaled discrete web distance to the Brownian web distance in the epigraph sense which we define below. Let $f : \mathbb{R}^4 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. The epigraph of f is the set

$$\mathfrak{e} f = \{(x, y) \in \mathbb{R}^4 \times \overline{\mathbb{R}} : y \geq f(x)\} \tag{1.6}$$

which is closed by the lower semicontinuity of f. The epigraph of lower semicontinuous functions are elements of the space $\mathcal{E}_*$, the set of all closed subsets $\Gamma \subset \mathbb{R}^4 \times \overline{\mathbb{R}}$ such that $\Gamma \cap (\{x\} \times \overline{\mathbb{R}}) \neq \emptyset$ for all $x \in \mathbb{R}^4$. We equip $\mathcal{E}_*$ with the following version of the Hausdorff topology. Consider the map $E : \mathbb{R}^4 \times \overline{\mathbb{R}} \to \mathbb{R}^4 \times [-1, 1]$ given by

$$E(r, s) = \left(r, \frac{|u - v| e^{-|r|} s}{1 + |s|}\right) \tag{1.7}$$

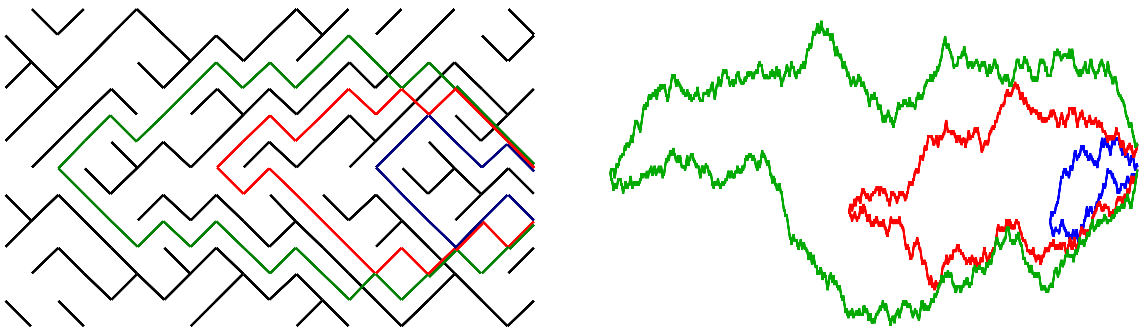

Figure 1: Boundaries of left disks with radii 0, 1 and 2 for random walk and Brownian web distance. The centers of nontrivial left Brownian web disks are atypical points.

for any $(r, s) \in \mathbb{R}^4 \times \overline{\mathbb{R}}$ with $r = (u, v) \in \mathbb{R}^4$ with the convention $E(r, \pm\infty) = (r, \pm|u - v|e^{-|r|})$. For two elements $e_1, e_2 \in \mathcal{E}_*$ we define their distance $d_*(e_1, e_2)$ to be the Hausdorff distance of the images $E(e_1)$ and $E(e_2)$. The space $(\mathcal{E}_*, d)$ is compact by Lemma 7.1 of [DV21].

Theorem 1.5. *There is a coupling of the Brownian web B and the sequence Y_n of random walk webs such that the epigraphs of the lower semicontinuous functions D_n^{RW} on $\mathbb{R}^4$ defined in terms of Y_n converge to the epigraph of D^{Br}, that is, $\mathfrak{e}D_n^{\mathrm{RW}} \to \mathfrak{e}D^{\mathrm{Br}}$ in $\mathcal{E}_*$ almost surely as $n \to \infty$.*

The proof Theorem 1.5 is based on understanding the boundaries of regions that are distance at most k from a spatial half line. These are closely related to boundaries of disks, see Figure 1.

1.3 KPZ universality

Even though the Brownian web distance is scale-invariant, in certain directions it still exhibits KPZ universality. This can be described in terms of the directed landscape. First, consider independent two-sided standard Brownian motions W_i for $i \in \mathbb{Z}$, and the directed metric on $\mathbb{R} \times \mathbb{Z}$ given by Brownian last passage percolation

$$L(s, m; t, n) = \max_{s = t_{m-1} \le t_m \le \cdots \le t_n = t} \sum_{i=m}^{n} (W_i(t_i) - W_i(t_{i-1})) \tag{1.8}$$

whenever $s \le t, m \le n$, and $L = -\infty$ elsewhere. It was shown in [DOV23] that Brownian last passage percolation has a distributional scaling limit, which can be taken to be the definition of the directed landscape

$$n^{1/6} \left(L(s + 2xn^{1/3}, \lfloor sn \rfloor; t + 2yn^{1/3}, \lfloor tn \rfloor) - 2(t-s)n^{1/2} - 2(y-x)n^{1/6} \right) \to \mathcal{L}(x, s; y, t) \tag{1.9}$$

as a function of $(x, s, y, t) \in \mathbb{R}^4$, in the topology generated by uniform convergence on compact subsets with $s < t$. The directed landscape has been shown to be the scaling limit of several last passage percolation models, as well as the KPZ equation, see [DV21, QS23, Vir20, Wu23].

The function $x \mapsto \mathcal{L}(0,0;x,1) + x^2$ is called the stationary Airy process, first introduced in [PS02] as the scaling limit of the interface in the polynuclear growth model. For $s < t$ the real random variable $\mathcal{L}(x,s;y,t) + (x-y)^2/(t-s)$ has GUE Tracy–Widom law scaled by $(t-s)^{1/3}$.

We prove that the shear limit of the Brownian web distance to a half-line in the first pair of variables is given in terms of the directed landscape.

Theorem 1.6. *As $m \to \infty$, the Brownian web distance after a shear mapping satisfies*

$$\frac{tm^2 + 2zm^{4/3} - D^{\mathrm{Br}}(-t, 2tm + 2zm^{1/3}; 0, \mathbb{R}_-)}{m^{2/3}} \to \mathcal{L}(0,0;z,t) \tag{1.10}$$

in law with respect to the topology of uniform convergence on compact sets for $(z,t) \in \mathbb{R} \times (0,\infty)$.

Note that by the $0:1:2$ scale invariance, as a process in z, t,

$$D^{\mathrm{Br}}(-t, 2tm + 2zm^{1/3}; 0, \mathbb{R}_-) \overset{\mathrm{d}}{=} D^{\mathrm{Br}}(-tm^2, 2tm^2 + 2zm^{4/3}; 0, \mathbb{R}_-). \tag{1.11}$$

Thus Theorem 1.6 also gives the more customary scaling limit in the direction $(-1,2)$ with the usual scaling exponents. Note also that all directions $(-1,\eta)$ when $\eta \neq 0$ are equivalent by Brownian scaling.

In the $t = 1$ case the limit in (1.10) is the parabolic Airy process, $\mathfrak{A}(y) = \mathcal{L}(0,0;1,y)$. We expect that the Airy scaling limit holds in the second space variable as well and the joint limit should be the Airy sheet, $\mathcal{S}(x,y) = \mathcal{L}(0,x;1,y)$.

Conjecture 1.7. *The rescaled Brownian web distance has Airy fluctuations in the other space variable as well, that is,*

$$\frac{m^2 - 2ym^{4/3} - D^{\mathrm{Br}}(-1, 2m; 0, (-\infty, 2ym^{1/3}])}{m^{2/3}} \to \mathfrak{A}(y) \tag{1.12}$$

as $m \to \infty$.

Conjecture 1.8. *The fluctuations of the rescaled Brownian web distance are given by the Airy sheet $\mathcal{S}$, that is*

$$\frac{m^2 + 2(z-y)m^{4/3} - D^{\mathrm{Br}}(-1, 2m + 2zm^{1/3}; 0, (-\infty, 2ym^{1/3}])}{m^{2/3}} \to \mathcal{S}(y,z) \tag{1.13}$$

as $m \to \infty$.

We expect that if we replace the half-line $(-\infty, 2ym^{1/3}]$ by its endpoint $2ym^{1/3}$ in (1.13), the conjecture holds in the sense of hypograph convergence. In our scaling, the half-line corresponds to an increasingly steeper half-wedge, and so it converges to a narrow wedge initial condition.

The next results give information about the limiting fluctuations of the random walk web distance. In any direction different from horizontal, the random walk web distance to a vertical half-line has Airy fluctuations.

Theorem 1.9. *For any $\eta \in (0,1)$,*

$$\frac{(1-\eta^2)^{1/6}}{(\eta/2)^{2/3}} n^{-1/3} \left(\frac{1-\sqrt{1-\eta^2}}{2} n - \frac{\eta^{1/3}(1-\eta^2)^{1/6}}{2^{1/3}} z n^{2/3} \right.$$
$$\left. - D^{\mathrm{RW}}(0,0;n,(-\infty,-\eta n - 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3})) \right) \to \mathfrak{A}(z) \quad (1.14)$$

as $n \to \infty$ uniformly in t on compact intervals.

The main term $(1-\sqrt{1-\eta^2})n/2$ in Theorem 1.9 gives a description of the asymptotic shape of disks in D^{RW}, see Figure 2. There is not conceptual difficulty in extending Theorem 1.9 to two-parameter convergence as in Theorem 1.6. However, in [DV21] only hypograph convergence was shown for the Seppäläinien–Johansson model. The required uniform convergence extension is straightforward but too technical for the present paper.

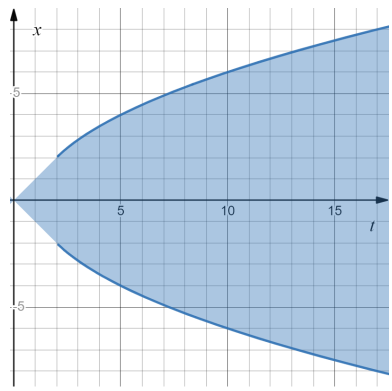

Figure 2: The asymptotic disk $t - \sqrt{t^2 - x^2} \le 2$ for the shift term in Theorem 1.9

The case of horizontal direction is covered by Theorem 1.5. The random walk web distance of two points along the same horizontal line is surprisingly completely different. We prove logarithmic upper and lower bounds on the horizontal distance and we expect it to satisfy a central limit theorem with normal fluctuations.

Theorem 1.10. *For some $c > 1$,*

$$\lim_{n\to\infty} \mathbf{P}\left(1/c \le \frac{D^{\mathrm{RW}}(0,0;n,0)}{\log n} \le c\right) = 1. \quad (1.15)$$

The rest of the paper is organized as follows. We define the Brownian web in the appropriate metric space with the corresponding topology in Section 2 and we describe some properties of the Brownian web in particular the convergence of the discrete web to the Brownian web in Theorem 2.2. Section 3 contains the most important properties we prove about the Brownian web distance which are used for the proof of our main results. We prove Theorem 1.5 about the convergence of discrete web distance to Brownian web distance in Section 4. Theorem 1.6 about the shear limit of Brownian distance and Theorem 1.9 on the Tracy–Widom fluctuations of the random walk web distance are proved in Section 5, Theorem 1.10 about the fluctuations in the horizontal direction is shown in Section 6. The proofs of continuity properties of Brownian web distance are postponed to Section 7.

2 The Brownian web and its dual

The Brownian web distance is constructed based on the Brownian web and its dual, which we introduce next. The most natural construction of the Brownian web is based on Theorem 2.1 below which can be found as Theorem 2.1 in [FINR04] explicitly and it follows from the proof of Theorem 2.1 in [TW98].

We first introduce the metric spaces which are used in the definition of the Brownian web according to [FINR04]. By writing $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, let $\Phi : \overline{\mathbb{R}}^2 \to [-1, 1]^2$ be defined as

$$\Phi(t, x) = \left(\tanh(t), \frac{\tanh(x)}{1 + |t|}\right) \tag{2.1}$$

and let ρ be a metric which naturally compactifies $\overline{\mathbb{R}}^2$ given by

$$\rho((t_1, x_1), (t_2, x_2)) = \|\Phi(t_1, x_1) - \Phi(t_2, x_2)\|_1. \tag{2.2}$$

Then $(\overline{\mathbb{R}}^2, \rho)$ is a compact metric space.

For any $t_0 \in \overline{\mathbb{R}}$, let $C[t_0]$ be the set of functions $f : [t_0, \infty] \to \overline{\mathbb{R}}$ for which $\Phi_2(t, f(t))$ is continuous. We let

$$\Pi = \bigcup_{t_0 \in \overline{\mathbb{R}}} \{t_0\} \times C[t_0] \tag{2.3}$$

denote the set of paths along with their starting points. For a $(t_0, f) \in \Pi$ let $\widehat{f}$ be the extension of f to $\overline{\mathbb{R}}$ by letting it equal to the constant $f(t_0)$ on $[-\infty, t_0]$. We define the distance d on Π by

$$d((t_1, f_1), (t_2, f_2)) = |\Phi_1(t_1, f_1(t_1)) - \Phi_1(t_2, f_2(t_2))| \vee \sup_{t \in \overline{\mathbb{R}}} \left|\Phi_2(t, \widehat{f_1}(t)) - \Phi_2(t, \widehat{f_2}(t))\right|. \tag{2.4}$$

With this metric, (Π, d) is a complete separable metric space.

Further let (H, d_H) denote the metric space which consists of compact collections of paths in (Π, d) with the Hausdorff metric

$$d_H(K_1, K_2) = \sup_{k_1 \in K_1} \inf_{k_2 \in K_2} d(k_1, k_2) \vee \sup_{k_2 \in K_2} \inf_{k_1 \in K_1} d(k_1, k_2). \tag{2.5}$$

The space (H, d_H) is also a complete separable metric space. Let $\mathcal{F}_H$ be the Borel σ-algebra generated by the metric d_H.

The Brownian web is defined as a random variable taking values in $(H, \mathcal{F}_H)$.

Theorem 2.1. *The Brownian web B is an $(H, \mathcal{F}_H)$-valued random variable whose distribution is uniquely determined by the properties below.*

1. *For any deterministic point $(t, x) \in \mathbb{R}^2$, there is almost surely a unique path $B_{(t,x)}$ starting at (t, x).*

2. *For any deterministic n and $(t_1, x_1), \ldots, (t_n, x_n) \in \mathbb{R}^2$, the joint distribution of $B_{(t_1,x_1)}, \ldots, B_{(t_n,x_n)}$ is the same as that of coalescing Brownian motions starting at $(t_1, x_1), \ldots, (t_n, x_n)$.*

3. *For any deterministic countable dense set D in $\mathbb{R}^2$, the closure of $\{B_{(t,x)}, (t,x) \in D\}$ in (H, d_H) is equal to B almost surely.*

Then the Brownian web is constructed as follows. We fix a countable dense subset D of $\mathbb{R}^2$ as in the third property in Theorem 2.1 and we enumerate the points of D as $z_i = (t_i, x_i)$ for $i = 1, 2, \ldots$. We sample Brownian motions B_{z_i} starting from the points z_i for each i inductively so that they are independent until they hit one of the trajectories sampled so far and they merge with the trajectory they first hit. More precisely with an i.i.d. sequence of Brownian motions $(\widetilde{B}_i(t), t \geq 0)_{i=1}^{\infty}$ and for all $i = 1, 2, \ldots$ we let $B_{z_i}(t) = x_i + \widetilde{B}_i(t - t_i)$ for $t \in [t_i, \tau_i)$ where

$$\tau_i = \inf\{t \geq t_i : \exists j \in \{1, 2, \ldots, i-1\} : x_i + \widetilde{B}_i(t - t_i) = B_{z_j}(t)\}. \tag{2.6}$$

If $\iota(i) = j \in \{1, 2, \ldots, i-1\}$ as above in (2.6), that is, for which $x_i + \widetilde{B}_i(t - t_i) = B_{z_j}(t)$, then we define $B_{z_i}(t) = B_{z_{\iota(i)}}(t)$ for $t \geq \tau_i$. Note that $\tau_1 = \infty$ but τ_i is finite almost surely for $i \geq 2$. The union of the paths $(B_{z_i})_i$ for $z_i \in D$ excluding the starting points z_i is the skeleton of the Brownian web. Once the skeleton is sampled, the Brownian web B is determined uniquely by Theorem 2.1. Equivalently the Brownian web B is the closure of its skeleton in the metric space (H, d_H). For any point $(t, x) \in \mathbb{R}^2$, let $B_{(t,x)}$ denote the unique Brownian path started from (t, x) in the Brownian web.

The dual of the Brownian web consists of coalescing Brownian paths running backwards in time and it is called the backward (dual) Brownian web. It can be constructed based on the same countable dense subset D of $\mathbb{R}^2$ as follows. The skeleton of the backward Brownian web from each point of D is the almost surely unique continuous curve going backwards in time which does not cross the forward lines. The backward Brownian web $\widehat{B}$ is the closure of this set of paths in the metric space of backward paths $(\widehat{H}, d_H)$ with the same d_H as in (2.5). By Theorem 2.3 in [TW98], the backward Brownian web has the same distribution as the time-reversed trajectories of the forward web.

Due to Theorem 2.1 for any deterministic $(t, x) \in \mathbb{R}^2$, there is almost surely a unique forward path $B_{(t,x)}$ in B starting at (t, x) and a unique backward path $\widehat{B}_{(t,x)}$ in $\widehat{B}$. The Brownian web and its dual however has random exceptional points where more than one path are passing through or starting from. They are characterized by their types as follows.

Any two paths $b, b' \in B$ are said to be equivalent paths entering the point (t, x) if $b|_{[t-\varepsilon,t]} = b'|_{[t-\varepsilon,t]}$ for some $\varepsilon > 0$. The number of equivalence classes defines $m_{\text{in}}(t, x)$. Also $m_{\text{out}}(t, x)$ can be defined similarly as the number of equivalence classes of outgoing paths. Then the pair $(m_{\text{in}}(t, x), m_{\text{out}}(t, x))$ is the type of the point (t, x). By Proposition 2.4 in [TW98], almost surely all points of $\mathbb{R}^2$ have one of the following six types: $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 1)$. For topological reasons if a point has type $(m_{\text{in}}, m_{\text{out}})$ in the forward Brownian web B, then its type in the backward web $\widehat{B}$ is $(\widehat{m}_{\text{in}}, \widehat{m}_{\text{out}}) = (m_{\text{out}} - 1, m_{\text{in}} + 1)$.

The $(1, 2)$ points have a special importance in the construction of the Brownian web distance. These points can be characterized as follows. A point is of type $(1, 2)$ if and only if both a forward and a backward path pass through it. The unique incoming path in a $(1, 2)$ point (t, x) continues along exactly one of the two outgoing paths which is

$B_{(t,x)}$. Let us introduce the notation $B'_{(t,x)}$ for the other outgoing path starting at the $(1,2)$ point (t,x). It is also referred to as the newly born path at (t,x) in the literature, see e.g. [NRS10]. Let $B_{(t,x)+}$ and $B_{(t,x)-}$ denote the highest and lowest outgoing paths starting at (t,x) which coincide with $B_{(t,x)}$ for almost all points (t,x). The backward paths $\widehat{B}_{(t,x)\pm}$ are defined similarly.

Donsker's invariance principle implies the convergence of the random walk web to the Brownian web for any finite collection of paths. The convergence extends to the full web and its dual as follows. First we introduce the dual backward system of coalescing random walks $\widehat{Y}$ on the dual lattice

$$\mathbb{Z}^2_{\mathrm{o}} = \{(i,n) \in \mathbb{Z}^2 : i+n \text{ is odd}\}. \tag{2.7}$$

For any $(i,n) \in \mathbb{Z}^2_{\mathrm{o}}$ a path starts at $\widehat{Y}_{(i,n)}(i) = n$ and the paths evolve as

$$\widehat{Y}_{(i,n)}(j-1) = \widehat{Y}_{(i,n)}(j) - \xi_{(j-1,\widehat{Y}_{(i,n)}(j))} \tag{2.8}$$

for all $j = i, i-1, \dots$ using the same random variables $\xi_{(i,n)}$ as in (1.2). Let $Y^{(n)}$ denote the rescaled random walk web which is defined as follows. For all t,x with $(nt, n^{1/2}x) \in \mathbb{Z}^2_{\mathrm{e}}$ we let $Y^{(n)}_{(t,x)}(s) = n^{-1/2} Y_{(nt,n^{1/2}x)}(ns)$ if $ns \in \mathbb{Z}$ and we define $Y^{(n)}_{(t,x)}(s)$ by linear interpolation between these values which yields a continuous function for all $s \geq t$. Let $\widehat{Y}^{(n)}$ be the rescaled backward random walk web defined similarly.

Theorem 2.2. *The pair $(B,\widehat{B})$ of forward and backward Brownian webs together with the rescaled forward and backward random walk webs $(Y^{(n)}, \widehat{Y}^{(n)})$ can be realized on the same probability space in a way that*

$$(Y^{(n)}, \widehat{Y}^{(n)}) \to (B, \widehat{B}) \tag{2.9}$$

almost surely in $(H, d_H) \times (\widehat{H}, d_H)$.

The convergence in distribution in (2.9) is the $b=0$ special case of Theorem 5.4 in [SS08]. Since (H, d_H) and $(\widehat{H}, d_H)$ are a separable metric spaces Skorokhod's representation theorem implies the existence of the coupling and the almost sure convergence in Theorem 2.2.

3 Properties of Brownian web distance

3.1 Left neighborhoods of an interval

Proposition 3.3 below shows that $D^{\mathrm{Br}}(t,x;s,y) = \infty$ for any $(t,x;s,y) \in \mathbb{R}^4$ for which (s,y) is not hit by a Brownian path in the Brownian web B, that is, for almost all (s,y). Hence we define distances to intervals as follows. Let $I \subset \mathbb{R}$ be a possibly infinite interval. For $s \in \mathbb{R}$ let

$$D^{\mathrm{Br}}(t,x;s,I) = \inf_{y\in I} D^{\mathrm{Br}}(t,x;s,y), \qquad Q_k(s,I) = \{(t,x) : D^{\mathrm{Br}}(t,x;s,I) \leq k\}. \tag{3.1}$$

These regions are connected subsets of $\mathbb{R}^2$ satisfying $Q_k \subseteq Q_{k+1}$. Proposition 3.2 below describes the boundaries of the regions Q_k.

We define below the backward paths $\rho_k^{\pm}$ which according to Proposition 3.2 turn out to be the boundary curves for the regions Q_k. Let $\rho_0^+(t) = \widehat{B}_{(s,v)}(t)$ and $\rho_0^-(t) = \widehat{B}_{(s,u)}(t)$ for $t \le s$, that is, the backward Brownian web trajectories starting at time s at the two endpoints of the interval I. Let $\tau_0 = \sup\{t \le s : \rho_0^+(t) = \rho_0^-(t)\}$ denote the time when the two backward trajectories meet. Given the paths $\rho_k^{\pm}(t)$ for $t \le s$ and their time of collision τ_k, we define

$$\rho_{k+1}^+(t) = \sup_{r\in[\max(t,\tau_k),s]} \widehat{B}_{(r,\rho_k^+(r))+}, \qquad \rho_{k+1}^-(t) = \inf_{r\in[\max(t,\tau_k),s]} \widehat{B}_{(r,\rho_k^-(r))-} \tag{3.2}$$

for all $t \le s$ and we let $\tau_{k+1} = \sup\{t \le s : \rho_{k+1}^+(t) = \rho_{k+1}^-(t)\}$ be their time of collision.

The next results show that the curves $\rho_k^{\pm}$ are the boundaries for Q_k and that they arise as Brownian paths reflected off one another in the Skorokhod sense. The proof of these results are postponed to Subsection 7.1.

Proposition 3.1. *Let $I = [u, v] \subset \mathbb{R}$ and $s \in \mathbb{R}$ be fixed. Assume that for all $j = 0, 1, \ldots, k$ the curves $\rho_j^{\pm}(t)$ are given on $t \in [\tau_j, s]$. Conditionally given these curves the distributions of $\rho_{k+1}^{\pm}(t)$ are reflected backward Brownian paths off $\rho_k^{\pm}(t)$ in the Skorokhod sense on $t \in [\tau_k, s]$ and independent Brownian motions until collision at τ_{k+1} on $t \in [\tau_{k+1}, \tau_k]$. In particular, $\rho_{k+1}^{\pm}(t)$ are continuous.*

Proposition 3.2. *Let $I = [u, v] \subset \mathbb{R}$ and $s \in \mathbb{R}$ be fixed. The union of the interval $s \times I$ and the curves $\rho_k^{\pm}(t)$ on $t \in [\tau_k, s]$ is the boundary of the region Q_k given in (3.1).*

3.2 Continuity properties of the Brownian web distance

We start this subsection with the characterization that the Brownian web distance between two space-time points is finite if and only if the target is on the skeleton of the Brownian web. The statement is proved in Subsection 7.4

Proposition 3.3. *Let $(t, x) \in \mathbb{R}^2$ and let (s, y) be an interior point of the Brownian web path starting at (t, x), that is, $B_{(t,x)}(s) = y$ with $t < s$. Then $D^{\mathrm{Br}}(u, z; s, y)$ is finite for any $u < s$ and $z \in \mathbb{R}$. If (s, y) is such that it is not the interior point of any Brownian web path, then $D^{\mathrm{Br}}(u, z; s, y)$ is infinite for all (u, z).*

Proposition 3.3 enables us to give another natural definition of the Brownian web distance D^{Br} in the spirit of Definition 5.3 in [DV21] as an induced directed metric with an extra continuity property. By Proposition 3.5 below which is proved in Subsection 7.5 the new definition gives the same directed metric.

Definition 3.4. *Let d denote the restriction of D^{Br} to the skeleton of the Brownian web B, that is, if (t, x) and (s, y) are both on the skeleton, then let $d(t, x; s, y) = D^{\mathrm{Br}}(t, x; s, y)$.*

The function d on the skeleton extends naturally to the full $\mathbb{R}^4$ as an induced directed metric $\widetilde{D}^{\mathrm{Br}}$ as follows. Let $\widetilde{D}^{\mathrm{Br}}$ be the supremum of all directed metrics on $\mathbb{R}^4$ which are lower semicontinuous in all of their variables and whose values restricted to the skeleton of the Brownian web are upper bounded by d.

Proposition 3.5. *The induced directed metric $\widetilde{D}^{\mathrm{Br}}$ exists and it agrees with D^{Br} on $\mathbb{R}^4$.*

Finally the Brownian web distance is on $\mathbb{R}^4$ is determined by its values on a countably infinite subset. We postpone its proof to Subsection 7.5.

Proposition 3.6. *The Brownian web distance $D^{\mathrm{Br}}(t,x;s,y)$ for all $(t,x;s,y) \in \mathbb{R}^4$ are determined by the values $D^{\mathrm{Br}}(t,x;s,I)$ where $I=[u,v]$ and t,x,s,u,v are rational.*

3.3 Left neighbourhoods in the discrete web

Similarly to (3.1) we introduce

$$D^{\mathrm{RW}}(i,n;j,I) = \inf_{m\in I} D^{\mathrm{RW}}(i,n;j,m), \qquad R_k = \{(i,n) : D^{\mathrm{RW}}(i,n;j,I) \le k\} \tag{3.3}$$

for any interval I for the discrete web distance where R_k is the discrete analogue of the region Q_k in (3.1) which depends on the choice of j and I. The regions are connected subsets of $\mathbb{Z}_e^2$ for which $R_k \subseteq R_{k+1}$ clearly hold. Next we define the backward paths $r_k^\pm$ which according to Proposition 3.8 turn out to serve as boundary curves for the regions R_k.

Let $j\in\mathbb{Z}$ and $I=[u,v]\subset\mathbb{Z}$ be fixed such that $(j,u),(j,v)\in\mathbb{Z}_{\mathrm{e}}^2$. We let $r_0^+(i) = \widehat{Y}_{(j,v+1)}(i)$ and $r_0^-(i) = \widehat{Y}_{(j,u-1)}(i)$ for $i=j,j-1,\dots$ to be equal to the backward discrete Brownian web trajectories starting from two points of the dual lattice $\mathbb{Z}_{\mathrm{o}}^2$ at the endpoints of $j\times I$. Then we let $T_0 = \max\{i\le j : r_0^+(i) = r_0^-(i)\}$ be the time when the two backward trajectories meet. Given the path $r_k^\pm(i)$ for $i\in[T_k,j]$ we define

$$r_{k+1}^+(i) = \max_{l\in\{i,\dots,j\}} \widehat{Y}_{(l,r_k^+(l+1)+1)}(i), \qquad r_{k+1}^-(i) = \min_{l\in\{i,\dots,j\}} \widehat{Y}_{(l,r_k^-(l+1)-1)}(i) \tag{3.4}$$

for all $i\le j$ and we let $T_{k+1} = \max\{i\le T_k : r_{k+1}^+(i) = r_{k+1}^-(i)\}$.

The next results are the discrete analogues of Propositions 3.1 and 3.2. The statements are proved in Subsection 7.2.

Proposition 3.7. *Conditionally given the curves $r_j^\pm$ for all $j=0,1,\dots,k$, the path r_{k+1}^+ evolves as the discrete Skorokhod reflection of a backward random walk off r_k^+ until time T_k. More precisely there exists a backward random walk s_{k+1} such that*

$$r_{k+1}^+(i) = s_{k+1}(i) - \min_{l\in\{i,\dots,j\}} \left(s_{k+1}(l) - r_k^+(l+1) - 1\right) \tag{3.5}$$

holds for all $i\in[T_k,j]$ with the convention $r_k^+(j+1)=v$. Furthermore, there are backward random walks $s_0,s_1,\dots,s_k$ such that r_k^+ can be represented as a discrete last passage percolation as

$$r_k^+(i) = \max_{i=l_k\le l_{k-1}\le\dots\le l_{-1}=j-k} \sum_{m=0}^{k} \left(s_m(l_m) - s_m(l_{m-1})\right) + k + v + 1. \tag{3.6}$$

Proposition 3.8. *The paths $r_k^\pm(i)$ for $i=T_k,T_k+1,\dots,j$ together with the interval $j\times I$ serve as the boundary of the region R_k for all $k=0,1,2,\dots$.*

4 Convergence of discrete web distance to Brownian web distance

This section is devoted to the proof of Theorem 1.5, the main result about the epigraph convergence of D_n^{RW} to D^{Br}. The convergence is proved to hold in $(\mathcal{E}_*, d_*)$ which is a compact metric space by Lemma 7.1 of [DV21]. We first state the lower semicontinuous analogue of Lemma 7.3 of [DV21] which provides a sufficient condition for the convergence in $(\mathcal{E}_*, d_*)$ to hold. We postpone its proof to the end of the section.

Lemma 4.1. *Let $f, f_n : \mathbb{R}^4 \to \overline{\mathbb{R}}$ be lower semicontinuous functions. Assume that for any convergent sequence $x_n \to x \in \mathbb{R}^4$ it holds that*

$$\liminf_{n\to\infty} f_n(x_n) \geq f(x). \tag{4.1}$$

Further assume that for any $x \in \mathbb{R}^4$ we can find a convergent sequence $x_n \to x$ such that

$$\limsup_{n\to\infty} f_n(x_n) \leq f(x). \tag{4.2}$$

holds. Then the epigraphs converge, that is, $\mathfrak{e}f_n \to \mathfrak{e}f$ in $\mathcal{E}_*$ as $n \to \infty$.

The strategy of the proof of Theorem 1.5 is to understand the level curves of the distance functions D_n^{RW} and D^{Br}. One could hope that the convergence of these curves is a deterministic consequence of the random walk web converging to the Brownian web. We do not have a proof of this. The main difficulty is that the random walk distance uses microscopic information (two curves being a single edge away from each other) that does not behave well in the scaling limit (which only sees two curves being less-than-scaling away from each other). The remedy we use is Skorokhod reflection.

We fix an $s \in \mathbb{R}$ and an interval $I = [u, v]$ and for all n define the rescaled curves corresponding to the target time ns and and target interval $[n^{1/2}u, n^{1/2}v]$ as

$$r_{k,n}^{\pm}(t) = n^{-1/2} r_k^{\pm}(nt) \tag{4.3}$$

for all $t \leq s$ and with linear interpolation between integer values. Then by Donsker's invariance principle and by the continuity of Skorokhod reflection it follows that under the coupling given in Theorem 2.2 the rescaled curves $r_{k,n}^{\pm}$ converge uniformly to $\rho_k^{\pm}$ on compact intervals almost surely as $n \to \infty$. The convergence of single curves can be improved by the following result.

Proposition 4.2. *Consider a countable collection of s_i, I_i where $s_i \in \mathbb{R}$ and $I_i \subset \mathbb{R}$ is a closed interval for all i. Then almost surely for all i the rescaled boundaries $r_{k,n}^{\pm}$ of the regions $R_k(s_i, I_i)$ defined in (4.3) converge to the boundaries $\rho_k^{\pm}$ of $Q_k(s_i, I_i)$ uniformly on compact sets and for finitely many k.*

Proof of Proposition 4.2. We fix an $s \in \mathbb{R}$ and an $I = [u, v] \subset \mathbb{R}$ first and we consider Q_k defined in (3.1). By Proposition 3.2, $\rho_k^{\pm}(r)$ for $r \leq s$ are boundary curves of Q_k. We study the convergence to these curves. The definition (3.4) of r^+_{k+1} can be written in terms of the rescaled boundary as

$$r^+_{k+1,n}(t) = \sup_{r\in[t,s]} \widehat{Y}^{(n)}_{(r, r^+_{k,n}(r+1/n)+1/\sqrt{n})}(t). \tag{4.4}$$

for $t \le s$.

We fix the natural coupling of the random walk web and the Brownian web given in Theorem 2.2. In particular the rescaled backward random walk web $\widehat{Y}^{(n)}$ converges to $\widehat{B}$ as compact collection of paths in the Hausdorff topology almost surely. Let Γ_n be the closure of the set of all paths in $\widehat{Y}^{(n)}$ started from the points $(r, r^+_{k,n}(r+1)+1/\sqrt{n})$ for $r \in [t,s]$. Since Γ_n is a closed subset of the compact set $\widehat{Y}^{(n)}$ and by the convergence $\widehat{Y}^{(n)} \to \widehat{B}$ in (H, d_H), the sets Γ_n have a limit Γ that is a subset of $\widehat{B}$.

We claim that Γ is a subset of Γ', the closure of the set of all paths in $\widehat{B}$ started from $(r, \rho^+_k(r))$. Indeed, let $\gamma_n \in \Gamma_n$ be a convergent sequence. Then the set of starting points converges, and all starting points in the limit lie on ρ^+_k, since $r^+_{k,n} \to \rho^+_k$.

This implies that

$$\bigcup_{\gamma\in\Gamma_n} \operatorname{graph}(\gamma) \to \bigcup_{\gamma\in\Gamma} \operatorname{graph}(\gamma) \subset \bigcup_{\gamma\in\Gamma'} \operatorname{graph}(\gamma) \tag{4.5}$$

in the Hausdorff topology, and by considering the upper boundaries of these sets, we get

$$\limsup_{n\to\infty} r^+_{k,n}(r) \le \rho^+_k(r) \tag{4.6}$$

for all r almost surely. But note that $r^+_{k,n} \to \rho^+_k$ in law with respect to uniform convergence on compact sets (by the Skorokhod reflection representation and Donsker's theorem). These two statements imply $r^+_{k,n} \to \rho^+_k$ almost surely uniformly on compact sets. The argument works simultaneously for countably many pairs (s, I) and for finitely many k hence the proof is complete. □

Note that the proof above works only for countable collections of pairs (s, I) and not for all (s, I) because of the convergence in law part.

Proof of Theorem 1.5. We check the two conditions of Lemma 4.1. To see (4.1) for any sequence $x_n \to x$ in $\mathbb{R}^4$, we use that by Theorem 2.2 the rescaled discrete web $Y^{(n)}$ and the Brownian web B as (H, F_H)-valued random variables can be coupled so that $Y^{(n)}$ converges to B almost surely in (H, d_H) as $n \to \infty$.

If the left-hand side of (4.1) is equal to k for D^{RW}_n, then there is a sequence of convergent starting points $(t^{(n)}, x^{(n)}) \to (t, x)$ and convergent endpoints $(s^{(n)}, y^{(n)}) \to (s, y)$ such that $D^{\mathrm{RW}}_n(t^{(n)}, t^{(n)}; s^{(n)}, y^{(n)}) = k$. This means that for all n there are $k+1$ paths $\pi^{(n)}_0, \dots, \pi^{(n)}_k$ in $Y^{(n)}$ along the geodesic between $(t^{(n)}, x^{(n)})$ and $(s^{(n)}, y^{(n)})$ with k jumps. The convergence of $Y^{(n)}$ to B in (H, d_H) implies that for all $j = 0, 1, \dots, k$ and for all n there are paths $\widetilde{\pi}^{(n)}_j$ in B so that $d(\pi^{(n)}_j, \widetilde{\pi}^{(n)}_j) \to 0$ for all $j = 0, \dots, k$ by (2.5). Since B as an element of H is a compact collection of paths in (Π, d), the sequence $\widetilde{\pi}^{(n)}_j$ must have a subsequential limit $\widetilde{\pi}^{(\infty)}_j$ for all $j = 0, \dots, k$ between (t, x) and (s, y). This shows that $D^{\mathrm{Br}}(t, x; s, y) \le k$ proving (4.1).

Next, we assume that the claim of Proposition 4.2 holds for rationals s and intervals I with rational endpoints. To prove the second condition (4.2) in Lemma 4.1, assume that $k := D^{\mathrm{Br}}(t, x; s, y) < \infty$. Then we can find intervals $I_j = [u_j, v_j]$ with rational endpoints and rationals s_j so that

- the line segment $\{s_j\} \times I_j$ intersects the geodesic from (t, x) to (s, y),
- $|s_j - s|, |u_j - y|, |v_j - y| \leq 1/j$.

Then $D^{\mathrm{Br}}(t, x; s_j, I_j) \leq k$. To prove (4.2), we construct a sequence of indices n_j such that there are $t^{(n_j)}, x^{(n_j)}, s^{(n_j)}, y^{(n_j)} \in \mathbb{R}$ with $(t^{(n_j)}, x^{(n_j)}; s^{(n_j)}, y^{(n_j)}) \to (t, x; s, y)$ so that

$$\limsup_{j\to\infty} D^{\mathrm{RW}}_{n_j}(t^{(n_j)}, x^{(n_j)}; s^{(n_j)}, y^{(n_j)}) \leq k. \tag{4.7}$$

We define n_j inductively. We set $n_0 = 1$, and given n_{j-1} we let $n_j \geq n_{j-1}$ so that the kth rescaled boundaries $r^{\pm}_{k,n_j} = n_j^{-1/2} r^{\pm}_k(n_j\cdot)$ corresponding to s_j, I_j are at most $1/j$ away from their limits $\rho^{\pm}_k$ uniformly on the interval $[t-1, s_j]$. This can be achieved by Proposition 4.2.

Next, we choose a starting point $(t^{(n_j)}, x^{(n_j)})$ which is at most $1/j$ away from (t, x) so that $D^{\mathrm{RW}}_{n_j}(t^{(n_j)}, x^{(n_j)}; s_j, y^{(n_j)}) \leq k$ for some $y^{(n_j)} \in [u_j, v_j]$. Then $(s^{(n_j)}, y^{(n_j)}) = (s_j, y^{(n_j)})$ is also at most $2/j$ far from (s, y) showing the existence of the subsequence n_j with the required properties. This proves (4.2) and the theorem. $\square$

Proof of Lemma 4.1. The space $\mathcal{E}_*$ consists of closed subsets of $\mathbb{R}^4 \times \overline{\mathbb{R}}$. By the lower semicontinuity of f_n the epigraphs $\mathfrak{e}f_n$ are closed. Let Γ denote a subsequential limit of $\mathfrak{e}f_n$ in $\mathcal{E}_*$. This exists by the compactness of $\mathcal{E}_*$. Define the function $g : \mathbb{R}^4 \to \overline{\mathbb{R}}$ by

$$g(x) = \inf\{z \in \overline{\mathbb{R}} : (x, z) \in \Gamma\}. \tag{4.8}$$

Since Γ is a closed subset of $\mathbb{R}^4 \times \overline{\mathbb{R}}$ as an element of $\mathcal{E}_*$, the function g is lower semicontinuous and $\mathfrak{e}g = \Gamma$. Next we show that $g = f$.

On one hand Γ is a subsequential limit of $\mathfrak{e}f_n$ in $\mathcal{E}_*$. Hence for any $x \in \mathbb{R}^4$, the point $(x, g(x)) \in \Gamma$ can be approximated by points in $\mathrm{e}f_n$, that is, there is a sequence $x_n \to x$ such that

$$\liminf_{n\to\infty} f_n(x_n) = g(x). \tag{4.9}$$

Comparing this with (4.1) which holds for any convergent sequence x_n yields that $g \geq f$.

On the other hand for any $x \in \mathbb{R}^4$ there is a convergent sequence x_n for which $\lim_{n\to\infty} f_n(x_n) = f(x)$ holds by (4.2) and (4.1). Hence the convergence

$$E(x_n, f_n(x_n)) \to E(x, f(x)) \tag{4.10}$$

is also satisfied. The convergence of a subsequence of the epigraphs $\mathfrak{e}f_n$ to Γ in $\mathcal{E}_*$ means that a subsequence of $E(\mathfrak{e}f_n)$ converges to $E(\Gamma)$ in the Hausdorff distance. This fact together with the convergence (4.10) implies for the limit on the right-hand side of (4.10) that $E(x, f(x)) \in (E(\Gamma))_{+\varepsilon}$ for all $\varepsilon > 0$. Since $E(\Gamma)$ is a closed set this means that $E(x, f(x)) \in E(\Gamma)$ and $(x, f(x)) \in \Gamma$ which yields $f \geq g$. $\square$

5 The KPZ limit

This section contains the proofs of Theorem 1.6 and Theorem 1.9. Both proofs are based on Propositions 3.2 and 3.7 describing the boundary curves of the regions with different distances. These curves are Brownian motions and random walks reflected off each other in the Skorokhod sense. The distribution of the reflected paths can be represented as last passage values.

In the case of Theorem 1.6 about the Brownian web distance we specialize Proposition 3.2 to the semiinfinite interval $I = \mathbb{R}_- = (-\infty, 0]$. According to the proposition the upper boundary of the region with distance at most k

$$\rho_k^+(t) = \sup\{x \in \mathbb{R} : D^{\mathrm{Br}}(t, x; 0, \mathbb{R}_-) \leq k\} \tag{5.1}$$

for $t \leq 0$ is a backward Brownian motion reflected off ρ_{k-1}^+ in the Skorokhod sense. The initial ρ_0^+ is a backward Brownian motion. By Lemma 5.1 below the processes $\rho_k^+(-t)$ for $k = 0, 1, 2, \dots$ have the distribution of Brownian last passage percolation.

We introduce the Brownian last passage percolation with general boundary condition below, see also (1.8). Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function and for a non-negative integer n and for $t \geq 0$ let the Brownian last passage percolation with boundary condition f be given by

$$L^f(t, n) = \sup_{0 \leq t_1} (f(t_1) + L(t_1, 2; t, n)) = \sup_{0 = t_0 \leq t_1 \leq \dots \leq t_n = t} \left(f(t_1) + \sum_{i=2}^{n} (W_i(t_i) - W_i(t_{i-1})) \right) \tag{5.2}$$

where $W_2(t), W_3(t), \dots, W_n(t)$ are independent standard Brownian motions.

Lemma 5.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a fixed continuous function. For the Brownian last passage percolation with boundary condition f it holds that $L^f(t, 1) = f(t)$ and for $n = 2, 3, \dots$, $L^f(t, n)$ can be represented as a Skorokhod reflection as*

$$L^f(t, n) = W_n(t) - \inf_{s \in [0,t]} (W_n(s) - L^f(s, n-1)). \tag{5.3}$$

Proof. For $n = 1$, (5.3) holds by definition. By taking out the supremum over $t_n \in [0, t]$ in (5.2) one gets

$$L^f(t, n) = \sup_{t_n \in [0,t]} \left(L^f(t, n-1) + W_n(t) - W_n(t_n) \right). \tag{5.4}$$

This is the recursion (5.3). □

The second building block in the proof of Theorem 1.6 is the following convergence result of Brownian last passage percolation to the directed landscape from [DOV23,DV21], see also [FN11,For10,Sod14] for single-parameter versions.

Proposition 5.2. *The convergence of the rescaled processes*

$$n^{1/6} \left(\rho_{tn+2zn^{2/3}}^+(-t) - 2t\sqrt{n} - 2zn^{1/6} \right) \to \mathcal{L}(0, 0; z, t) \tag{5.5}$$

holds as $n \to \infty$ in distribution uniformly in z and $t > 0$ in any compact region.

Proof of Proposition 5.2. With the notations of Theorem 1.7 and Remark 1.10 in [DV21], we have $n = \sigma^3$ and $v = (1, -1), e = (0, -1)$. On the other hand since $\|(x, -y)\|_d = 2\sqrt{xy}$ for the Brownian last passage percolation, the Taylor expansion

$$\|v + xe\|_d = \|(1, -1 - x)\|_d = 2\sqrt{1 + x} = 2 + x - \frac{x^2}{4} + \mathcal{O}(x^3) \tag{5.6}$$

holds as $x \to 0$. Remark 1.9 in [DV21] implies that $\alpha = 2, \beta = 1, \chi/\tau^2 = 1/4$ and $\chi = 1$ for the Brownian last passage percolation. Then $\tau = 2$ holds as a consequence and $v = (0, -2)$, so we can choose $s = 0, x = 0, y = z$ and leave $t > 0$ a parameter. By (1.6) in Remark 1.10 of [DV21] with $\widetilde{\alpha} = 2, \widetilde{\eta} = -1$ we have $\widetilde{\gamma} = 2, \widetilde{\chi} = \widetilde{\omega} = 1$. Note that Brownian last passage percolation in [DV21] is obtained by optimizing over down-right paths whereas our definition (1.8) is the supremum over up-right paths. Hence the Brownian last passage time between $(0, 0)$ and $(tn, -tn - 2zn^{2/3})$ in [DV21] translates into

$$n^{-1/3}\left(L(0, 0; tn, tn + 2zn^{2/3}) - 2tn - 2zn^{2/3}\right) \to \mathcal{L}(0, 0; z, t). \tag{5.7}$$

The distributional identity $L(0, 0; ns, k) \stackrel{\mathrm{d}}{=} n^{1/2}L(0, 0; s, k)$ for the Brownian last passage percolation follows from Brownian scaling. Using it in (5.7) and writing the convergence in terms of the boundaries ρ_k^+ gives (5.5). □

Proof of Theorem 1.6. First we fix a compact interval K for z. The convergence in distribution in (5.5) happens in the space of continuous functions. We consider the supremum distance of continuous functions on K which generates the topology of uniform convergence on K. By Skorokhod's representation theorem the convergence in distribution in (5.5) on K can be realized by a coupling of the sequence ρ_k^+ for $k = 0, 1, 2, \dots$ and the limit as an almost sure convergence which is uniform on K.

The almost sure convergence which is uniform on K in (5.5) means that for almost all realizations of the randomness it holds that for every $\varepsilon > 0$ there exists a random k_0 such that for all $k \geq k_0$

$$\begin{aligned} 2t\sqrt{n} + 2zn^{1/6} + (\mathcal{L}(0, 0; z, t) - \varepsilon)n^{-1/6} &\leq \rho^+_{tn+2zn^{2/3}}(-t) \\ &\leq 2t\sqrt{n} + 2zn^{1/6} + (\mathcal{L}(0, 0; z, t) + \varepsilon)n^{-1/6} \end{aligned} \tag{5.8}$$

holds. The upper and lower bounds on the boundary between the regions with D^{Br} distance $tn + 2zn^{2/3}$ and $tn + 2zn^{2/3} + 1$ imply that

$$\begin{aligned} D^{\mathrm{Br}}(-t, 2t\sqrt{n} + 2zn^{1/6} + (\mathcal{L}(0, 0; z, t) + \varepsilon)n^{-1/6}; 0, \mathbb{R}_-) &\geq tn + 2zn^{2/3} + 1, \\ D^{\mathrm{Br}}(-t, 2t\sqrt{n} + 2zn^{1/6} + (\mathcal{L}(0, 0; z, t) - \varepsilon)n^{-1/6}; 0, \mathbb{R}_-) &\leq tn + 2zn^{2/3}. \end{aligned} \tag{5.9}$$

Now let $m(n) = \sqrt{n} + \frac{L}{2t}n^{-1/6}$ where $L \in \mathbb{R}$ is a parameter. Taylor expansion yields

$$\begin{aligned} 2tm(n) + 2z(m(n))^{1/3} &= 2t\sqrt{n} + 2zn^{1/6} + Ln^{-1/6} + \mathcal{O}(n^{-1/2}) \\ t(m(n))^2 + 2z(m(n))^{4/3} - L(m(n))^{2/3} &= tn + 2zn^{2/3} + \mathcal{O}(1) \end{aligned} \tag{5.10}$$

as $n \to \infty$. Applying (5.10) in (5.9) with $L = \mathcal{L}(0, 0; z, t) \pm \varepsilon$ and using the fact that the inequalities in (5.9) hold for any $\varepsilon > 0$ if n is large enough implies that

$$D^{\mathrm{Br}}(-t, 2tm + 2zm^{1/3}; 0, \mathbb{R}_-) = tm^2 + 2zm^{4/3} - \mathcal{L}(0, 0; z, t)m^{2/3} + \mathcal{O}(1) \tag{5.11}$$

as $m \to \infty$ which proves (1.10). □

The strategy of the proof of Theorem 1.9 is to rewrite the boundary curve r_k^+ in terms of the last passage value in the Seppäläinen–Johansson model, Proposition 3.7. Then we use the known fluctuation results of the Seppäläinen–Johansson model to conclude Airy fluctuations of the discrete web distance.

Let $T(m,n)$ denote the last passage time from $(0,0)$ to (m,n) in the Seppäläinen–Johansson model with parameter $1/2$ where all vertical edges of $\mathbb{Z}^2$ have 0 passage time and the horizontal edges have weight 0 or 1 with probability $1/2$ each and all edge weights are independent random variables. Our notation differs from [Joh01] but the first passage time from $(0,0)$ to (m,n) is equal in distribution to $m-T(m,n)$ where $T(m,n)$ is the last passage time. By [Joh01] for a single z and by Corollary 6.6 of [DV21] for all z uniformly on compact intervals it holds that if $x>y$ then

$$\begin{aligned}&T\left(xn+2x\frac{(x-y)^{1/3}}{(xy)^{1/3}}zn^{2/3},yn\right)\\&=\left(x-\frac{(\sqrt{x}-\sqrt{y})^2}{2}\right)n+(x+\sqrt{xy})\frac{(x-y)^{1/3}}{(xy)^{1/3}}zn^{2/3}+\frac{(x-y)^{2/3}}{2(xy)^{1/6}}\mathfrak{A}(z)n^{1/3}+o(n^{1/3})\end{aligned}\tag{5.12}$$

as $n\to\infty$. By the same idea including Skorokhod's representation as in the proof of Theorem 1.6 we have that for any $\varepsilon>0$ the left-hand side of (5.12) can be almost surely upper and lower bounded by the right-hand side of (5.12) with $\mathfrak{A}(z)$ replaced by $\mathfrak{A}(z)\pm\varepsilon$ uniformly in z on a compact interval.

Proof of Theorem 1.9. Proposition 3.7 gives a representation of the boundary curve r_k^+ as a last passage time which can be written in terms of $T(m,n)$ as

$$\begin{aligned}r_k^+(i)&=\max_{i=l_k\le l_{k-1}\le\cdots\le l_{-1}=j-k}\sum_{m=0}^{k}(s_m(l_m)-s_m(l_{m-1}))+k+v+1\\&\overset{\mathrm{d}}{=}2T(j-k-i,k)+v+2k+i-j+1\end{aligned}\tag{5.13}$$

where the second identity above maps the ±1 weights of horizontal edges into 0 or 1 weights in the last passage percolation.

We specify the target interval to be $\{0\}\times(-\infty,0]$, hence we choose $j=0$ and $v=0$. Further we set $i=-n$ and $k=\kappa n+czn^{2/3}+\beta n^{1/3}=\kappa\widetilde{n}$ with κ, c and β to be determined and with $\widetilde{n}=n+\frac{cz}{\kappa}n^{2/3}+\frac{\beta}{\kappa}n^{1/3}$. We get from (5.12) and (5.13) that

$$\begin{aligned}r_k^+(-n)&=2T\left((1-\kappa)n-czn^{2/3}-\beta n^{1/3},\kappa n+czn^{2/3}+\beta n^{1/3}\right)\\&\qquad-n+2(\kappa n+czn^{2/3}+\beta n^{1/3})+1\\&=2T\left((1-\kappa)\widetilde{n}-\frac{c}{\kappa}\widetilde{z}\widetilde{n},\kappa\widetilde{n}\right)-n+2\kappa\widetilde{n}\end{aligned}\tag{5.14}$$

where $\widetilde{z}=z+(-\frac{2c}{3\kappa}z^2+\frac{\beta}{c})n^{-1/3}+o(n^{-1/3})$. Then the right-hand side of (5.14) is of the form found in (5.12) with $x=1-\kappa$, $y=\kappa$, $n=\widetilde{n}$, $z=\widetilde{z}$ and $c=-2\kappa x(x-y)^{1/3}/(xy)^{1/3}$. Then (5.12) yields that

$$\begin{aligned}r_k^+(-n)&=2\sqrt{\kappa(1-\kappa)}n+2(1-2\kappa)^{4/3}(\kappa(1-\kappa))^{1/6}zn^{2/3}\\&\qquad+\left(\frac{1-2\kappa}{\sqrt{\kappa(1-\kappa)}}\beta+\frac{(1-2\kappa)^{2/3}}{(\kappa(1-\kappa))^{1/6}}\mathfrak{A}(z)+o(1)\right)n^{1/3}\end{aligned}\tag{5.15}$$

where the $o(1)$ term above is almost surely uniform as z varies in a compact interval. We choose $\kappa = (1 - \sqrt{1-\eta^2})/2$ so that (5.15) simplifies to

$$r_k^+(-n) = \eta n + 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3} + \frac{2\sqrt{1-\eta^2}}{\eta}\left(\beta + \frac{\eta^{2/3}}{2^{2/3}(1-\eta^2)^{1/6}}\mathfrak{A}(z) + o(1)\right) n^{1/3}. \tag{5.16}$$

This means that with

$$\beta = -\frac{\eta^{2/3}}{2^{2/3}(1-\eta^2)^{1/6}}\mathfrak{A}(z) - \varepsilon \tag{5.17}$$

for any $\varepsilon > 0$ fixed the right-hand side of (5.16) is at most $\eta n + 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3}$ if n is large enough. This implies that

$$\begin{aligned} &D^{\mathrm{RW}}\left(-n, \eta n + 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3}; 0, \mathbb{R}_-\right) \\ &\quad \geq \frac{1-\sqrt{1-\eta^2}}{2} n - \frac{\eta^{1/3}(1-\eta^2)^{1/6}}{2^{1/3}} z n^{2/3} - \frac{\eta^{2/3}}{2^{2/3}(1-\eta^2)^{1/6}}\mathfrak{A}(z) - \varepsilon \end{aligned} \tag{5.18}$$

for any $\varepsilon > 0$ if n is large enough. By changing the sign of ε to positive in (5.17) we see that the right-hand side of (5.16) is at least $\eta n + 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3}$ which yields the corresponding upper bound on the random walk web distance in (5.18) with $-\varepsilon$ replaced by $+\varepsilon$. This completes the proof since $D^{\mathrm{RW}}(-n, \eta n + 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3}; 0, \mathbb{R}_-)$ and $D^{\mathrm{RW}}(0, 0; n, (-\infty, -\eta n - 2^{2/3}\eta^{1/3}(1-\eta^2)^{2/3} z n^{2/3}))$ have the same distribution by the shift invariance. $\square$

6 Discrete web distance in the horizontal direction

This section is devoted to the proof of Theorem 1.10 about the horizontal scaling of the random walk web distance.

Proof of Theorem 1.10. First we prove the lower bound in (1.15). Since $D^{\mathrm{RW}}(0, 0; n, 0)$ and $D^{\mathrm{RW}}(-n, 0; 0, 0)$ have the same distribution we can consider the upper and lower boundary curves $r_k^\pm$ of the regions R_k corresponding to the single point $(0, 0)$ in the place of the target interval $\{j\} \times I$. The upper and lower curves $r_k^\pm$ meet at time T_k for each k. We will find a dominating sequence $E_k \geq |T_k|$ so that the increments of $\log E_k$ have a uniform exponential tail decay.

The sequence E_k is chosen so that the region R_k is included in the rectangle $[-E_k, 0] \times [-\sqrt{E_k}, \sqrt{E_k}]$. This is done by induction on k as follows. Assume that E_k along with R_k is given. Then we sample two backward random walks $e_{k+1}^\pm$ starting at $e_{k+1}^\pm(0) = \pm\sqrt{E_k}$ which are reflected off the constant levels $\pm\sqrt{E_k}$ upwards respectively downwards on the time interval $[-E_k, 0]$ and the two random walks run beyond $-E_k$ until they collide. We let the paths $e_{k+1}^\pm$ follow the steps of the backward random walk web $\widehat{Y}$ when they are away from the barriers at $\pm\sqrt{E_k}$.

Let E_{k+1} be the maximum of the absolute value of the collision time and the square of the largest absolute value that these two random walks ever had until collision. In other words, E_{k+1} is the smallest number for which the full trajectories of the two random walks $e_{k+1}^\pm$ are included in the rectangle $[-E_{k+1}, 0] \times [-\sqrt{E_{k+1}}, \sqrt{E_{k+1}}]$.

Let $\mathcal{F}_k = \sigma(E_1, \dots E_k)$. We claim that the tail of the conditional law of $\log E_{k+1} - \log E_k$ given $\mathcal{F}_k$ decays exponentially uniformly in k. Thus the increments are dominated by an i.i.d. sequence with finite mean, which then implies the lower bound of the theorem.

First we study the time $-S$ the two processes $e^{\pm}_{k+1}$ meet. Until time $-E_k$ these are random walks which are reflected from the boundaries of the rectangle. Afterwards, their difference performs a lazy random walk. For lazy walks started at $s > 0$, the time τ they hit zero satisfies $\mathbf{P}(\tau > r) \le 3s/\sqrt{r}$, see Corollary 2.28 in [LPW06]. So for our walks

$$\mathbf{P}\big(S > (e^x + 1)E_k \,|\, \mathcal{F}_k, e^{\pm}_{k+1}(-E_k)\big) \le \frac{e^{+}_{k+1}(-E_k) - e^{-}_{k+1}(-E_k)}{e^{x/2}\sqrt{E_k}} \tag{6.1}$$

if x is large. Taking conditional expectations with respect to $\mathcal{F}_k$ we get that

$$\mathbf{P}\big(S > (e^x + 1)E_k \,|\, \mathcal{F}_k\big) \le \frac{c\sqrt{E_k}}{e^{x/2}\sqrt{E_k}} = ce^{-x/2}. \tag{6.2}$$

The walks e^{+}_{k+1} and $-e^{-}_{k+1}$ are dominated by random walks reflected off the level $\sqrt{E_k}$ upwards all the way until time $-S$ which have the same distribution as the absolute value of random walks shifted by $\sqrt{E_k}$. This helps us to bound the tail of $M = \max_{-S\le j\le 0} \max(e^{+}_{k+1}(j), -e^{-}_{k+1}(j))$. Let X_j be a simple random walk on $\mathbb{Z}$ independent of the rest. Then

$$\begin{aligned}\mathbf{P}\big(M > (e^x + 1)\sqrt{E_k},\, S \le (e^x + 1)E_k \,|\, \mathcal{F}_k\big) &\le 2\mathbf{P}\left(\max_{j\le(e^x+1)E_k} |X_j| \ge e^x\sqrt{E_k} \,|\, \mathcal{F}_k\right)\\ &\le 8\mathbf{P}\left(X_{(e^x+1)E_k} \ge e^x\sqrt{E_k} \,|\, \mathcal{F}_k\right)\\ &\le ce^{-x/2}\end{aligned} \tag{6.3}$$

for large x where Chebyshev's inequality is used in the last line. So we get

$$\begin{aligned}&\mathbf{P}\left(E_{k+1} > e^{2x}E_k \,|\, \mathcal{F}_k\right)\\ &\quad\le \mathbf{P}\big(S > (e^x + 1)E_k \,|\, \mathcal{F}_k\big) + \mathbf{P}\left(M > (e^x + 1)\sqrt{E_k}, S \le (e^x + 1)E_k \,|\, \mathcal{F}_k\right) \le ce^{-x/2}.\end{aligned} \tag{6.4}$$

This implies the lower bound, since

$$\mathbf{P}\left(D^{\mathrm{RW}}(-n, 0; 0, 0) < a \log n\right) \le \mathbf{P}\left(|T_{a\log n}| \ge n\right) \le \mathbf{P}\left(\log E_{a\log n} \ge \log n\right) \to 0 \tag{6.5}$$

by the law of large numbers, as long as $a > 0$ is small enough.

For the upper bound it suffices to show that there is an $\varepsilon > 0$ so that $|T_{k+1}|/|T_k| > 1 + \varepsilon$ with a uniformly positive probability as k varies. Since, $r^{\pm}_{k+1}$ are reflected walks, there are independent simple random walks $s^{\pm}_{k+1}(n)$ on $n = 0, 1, \dots, |T_k|$ so that the $r^{+}_{k+1}(-n)$ stochastically dominates $s^{+}_{k+1}(n)$ and $s^{-}_{k+1}(n)$ stochastically dominates $r^{-}_{k+1}(-n)$ for all $n = 0, 1, \dots, |T_k|$. The rescaled distance $(r^{+}_{k+1}(T_k) - r^{-}_{k+1}(T_k))/\sqrt{|T_k|}$ therefore stochastically dominates $(s^{+}_{k+1}(|T_k|)-s^{-}_{k+1}(|T_k|))/\sqrt{|T_k|}$. The latter converges to a normal random variable as $|T_k| \to \infty$. This holds since $|T_k| \ge 2k$.

This implies that there is a $\varepsilon > 0$ for which it takes at least $\varepsilon|T_k|$ steps beyond T_k for $r^{\pm}_{k+1}$ to collide with probability $p > 0$ for all k. So $\log |T_k|$ dominates $\log(1 + \varepsilon)$ times a

Bernoulli random walk with success probability p. For any $0 < \nu < p\log(1+\varepsilon)$ by the law of large numbers,

$$\mathbf{P}\left(|T_{(\log n)/\nu}| \leq n\right) \to 0 \tag{6.6}$$

as $n \to \infty$. Assume now that $|T_{(\log n)/\nu}| \geq n$ and the point $(-n, 0)$ is not in the set $R_{(\log n)/\nu}$ but $R_{(\log n)/\nu}$ intersects the vertical line $\{-n\} \times \mathbb{Z}$ above 0. As a consequence of Proposition 4.2 the sequence $D^{\mathrm{RW}}(-n, 0; 0, \mathbb{Z}_+)$ converges in distribution to $D^{\mathrm{Br}}(-1, 0; 0, \mathbb{R}_+)$ and in particular $D^{\mathrm{RW}}(-n, 0; 0, \mathbb{Z}_+)$ is tight. Hence there is a path with a tight number of steps in the forward random walk web from $(-n, 0)$ to $\{0\} \times \mathbb{Z}_+$ which must cross $R_{(\log n)/\nu}$. It proves that $D^{\mathrm{RW}}(-n, 0; 0, 0)$ is at most a constant times $\log n$ with probability tending to 1 as required for the upper bound in (1.15). $\square$

7 Proofs of continuity properties

7.1 Properties of continuous region boundaries

In this subsection we prove Propositions 3.1 and 3.2. First we give in alternative characterization of the boundary curves $\rho_k^\pm$ in Proposition 7.2 below which is used in the proofs of Propositions 3.1 and 3.2. Proposition 3.1 is shown using two different couplings of the same sequence of processes that approximate the Brownian motion reflected off a function in the Skorokhod sense. The proof of Proposition 3.2 uses the continuity of the paths $\rho_k^\pm$ which follows from Proposition 3.1.

Next we give an alternative description of the boundary curves $\rho_k^\pm$ which we introduce at $\widetilde{\rho}_k^\pm$ which are shown to be the same in Proposition 7.2. Let $\widetilde{\rho}_0^\pm(t)$ and $\widetilde{\tau}_0$ to be the same as $\rho_0^\pm(t)$ and τ_0, that is, the backward Brownian web paths $\widehat{B}_{(s,v)}(t)$ and $\widehat{B}_{(s,u)}(t)$ and $\widetilde{\tau}_0$ their collision time. Then we proceed by induction on k. We assume that the paths $\widetilde{\rho}_k^\pm(t)$ and their collision time $\widetilde{\tau}_k$ are given and we introduce $\widetilde{\rho}_{k+1}^\pm(t)$ as follows. We first define a decreasing sequence of trajectories $\widetilde{\rho}_{k+1}^{+,n}(t)$ which converge to $\widetilde{\rho}_{k+1}^+(t)$. We start the trajectory $\widetilde{\rho}_{k+1}^{+,n}(t)$ at $(s, v + 2^{-n+1})$ and follow the backward Brownian web path started from there until it reaches $\widetilde{\rho}_k^+(t) + 2^{-n}$ when we reset it to $\widetilde{\rho}_k^+(t) + 2^{-n+1}$. That is, we set $\kappa_0^n = s$ and we let

$$\kappa_{j+1}^n = \sup\{t \in [\tau_k, \kappa_j^n) : \widehat{B}_{(\kappa_j^n, \widetilde{\rho}_k^+(\kappa_j^n) + 2^{-n+1})}(t) \leq \widetilde{\rho}_k^+(t) + 2^{-n}\} \tag{7.1}$$

where the supremum is meant to be $-\infty$ if there is no such t. Then we let

$$\widetilde{\rho}_{k+1}^{+,n}(t) = \widehat{B}_{(\kappa_j^n, \widetilde{\rho}_k^+(\kappa_j^n) + 2^{-n+1})}(t) \tag{7.2}$$

on the interval $t \in (\kappa_{j+1}^n, \kappa_j^n]$ for $j = 0, 1, 2, \ldots$ until $\kappa_{j+1}^n = -\infty$.

Lemma 7.1. *We have that*

$$\widetilde{\rho}_{k+1}^{+,n+1}(t) \leq \widetilde{\rho}_{k+1}^{+,n}(t) \tag{7.3}$$

for all $t \leq s$.

Proof of Lemma 7.1. First observe that

$$\widetilde{\rho}_k^+(t) + 2^{-n} \leq \widetilde{\rho}_{k+1}^{+,n}(t) \tag{7.4}$$

holds for any $t \le s$ by the definition of the stopping times κ_j^n and that of $\widetilde{\rho}_{k+1}^{+,n}$ in (7.1)–(7.2). Then (7.3) can be shown as follows. It clearly holds for $t = s$, then the inequality remains valid until κ_1^{n+1} by the fact that both sides follow the evolution of the backward Brownian web. At time $t = \kappa_j^{n+1}$ by definition $\widetilde{\rho}_{k+1}^{+,n+1}(\kappa_j^{n+1}) = \widetilde{\rho}_k^+(\kappa_j^{n+1}) + 2^{-n}$ which is still a lower bound for $\widetilde{\rho}_{k+1}^{+,n}(\kappa_j^{n+1})$ by (7.4) for any $j = 1, 2, \dots$. This completes the proof because at κ_j^n the left-hand side of (7.3) changes continuously while the right-hand side has a jump of size 2^{-n} and at any other time the evolution of the backward Brownian web does not allow the two paths to cross each other. □

Hence the sequence of paths $\widetilde{\rho}_{k+1}^{+,n}(t)$ for $t \le s$ is non-increasing in n. This allows us to define

$$\widetilde{\rho}_{k+1}^{+}(t) = \lim_{n\to\infty} \widetilde{\rho}_{k+1}^{+,n}(t). \tag{7.5}$$

Very similarly given $\widetilde{\rho}_k^-(t)$ for $t \le s$ one defines $\widetilde{\rho}_{k+1}^-(t)$ as the limit of a non-decreasing sequence of paths. By the obvious inequality $\widetilde{\rho}_{k+1}^-(t) \le \widetilde{\rho}_{k+1}^+(t)$ both curves are well-defined and the limits in (7.5) and in its analogue for $\widetilde{\rho}_{k+1}^-$ are finite. Then we let

$$\widetilde{\tau}_{k+1} = \sup\{t \le s : \widetilde{\rho}_{k+1}^+(t) = \widetilde{\rho}_{k+1}^-(t)\}. \tag{7.6}$$

Proposition 7.2. *For all $k = 0, 1, 2, \dots$ and for all $t \le s$ almost surely $\rho_k^\pm(t) = \widetilde{\rho}_k^\pm(t)$ holds.*

Proof of Proposition 7.2. All points of $\widetilde{\rho}_k^+$ for all $k = 0, 1, 2, \dots$ have an incoming backward Brownian web path. The statement is true for $k = 0$ by definition, it holds by construction for $\widetilde{\rho}_{k+1}^+$ if it is away from $\widetilde{\rho}_k^+$ and at the points where they meet the statement follows by induction. Hence all point of $\widetilde{\rho}_k^+$ are of type $(1, 1)$ or $(1, 2)$ in the backward Brownian web. Note that all $(1, 1)$ points on $\widetilde{\rho}_k^+$ with a single outgoing backward Brownian web path b, we have $b \le \widetilde{\rho}_k^+$ at all times that they are defined. These statements imply that the supremum (3.2) equals the supremum of all backward Brownian web paths starting at all $(1, 2)$ points along the trajectory of $\widetilde{\rho}_k^+(r)$ for $r \in [t, s]$.

Assume now that the $\rho_k^+(t) = \widetilde{\rho}_k^+(t)$ holds for k and we prove it for $k + 1$. We first show that

$$\rho_{k+1}^+(t) \le \widetilde{\rho}_{k+1}^+(t) \tag{7.7}$$

holds for all $t \in [\tau_k, s]$. This follows from the inequalities $\rho_{k+1}^+(t) \le \widetilde{\rho}_{k+1}^{+,n}(t)$ which hold for all n and for all $t \in [\tau_k, s]$. They are indeed true because $\rho_{k+1}^+(t) > \widetilde{\rho}_{k+1}^{+,n}(t)$ for some $t \le s$ would mean the existence of an $r \in [t, s]$ such that $\widehat{B}_{(r,\rho_k^+(r))+}(t) > \widetilde{\rho}_{k+1}^{+,n}(t)$, that is, the backward Brownian web path $\widehat{B}_{(r,\rho_k^+(r))+}$ would cross $\widetilde{\rho}_{k+1}^{+,n}$ where the latter consists of backward Brownian web path parts with possible upward jumps between the parts. This is impossible, hence (7.7).

To prove the reverse inequality we assume that there is a $t \in [\tau_k, s]$ such that $\widetilde{\rho}_{k+1}^+(t) > \widetilde{\rho}_k^+(t)$ because otherwise the equality in (7.7) holds automatically. In case of $\widetilde{\rho}_{k+1}^+(t) > \widetilde{\rho}_k^+(t)$, the path $\widetilde{\rho}_{k+1}^+$ coincides with a backward Brownian web trajectory around t, hence there are at least two different outgoing paths from the point $(t, \widetilde{\rho}_{k+1}^+(t))$ in the forward Brownian web. We consider the lowest forward path $B_{(t,\widetilde{\rho}_{k+1}^+(t))-}$ starting at $(t, \widetilde{\rho}_{k+1}^+(t))$ which is below $\widetilde{\rho}_{k+1}^+$ locally. By definition, the trajectory $\widetilde{\rho}_{k+1}^+$ must always follow a backward Brownian web path if it is away from $\widetilde{\rho}_k^+$. For this reason $B_{(t,\widetilde{\rho}_{k+1}^+(t))-}$ cannot cross $\widetilde{\rho}_{k+1}^+$ as long as $\widetilde{\rho}_{k+1}^+$ is away from $\widetilde{\rho}_k^+$. Take $\tau = \inf\{r \in [t,s] : \widetilde{\rho}_{k+1}^+(r) = \widetilde{\rho}_k^+(r)\}$ which has to exist since $\widetilde{\rho}_{k+1}^+(s) = \widetilde{\rho}_k^+(s)$.

Since $\widetilde{\rho}_{k+1}^+(r) > \widetilde{\rho}_k^+(r)$ for $r \in [t,\tau)$ by the definition of τ, the path $\widetilde{\rho}_{k+1}^+$ is equal to a backward Brownian web path on $[t,\tau)$. Hence $\widetilde{\rho}_{k+1}^+$ itself on $[t,\tau)$ is one of the outgoing backward paths starting from $(\tau, \widetilde{\rho}_k^+(\tau))$ which implies

$$\widetilde{\rho}_{k+1}^+(r) \le \widehat{B}_{(\tau,\rho_k^+(\tau))+}(r) \le \rho_{k+1}^+(r) \tag{7.8}$$

for all $r \in [t,\tau)$ and it completes the proof. □

The proof of Proposition 3.1 is based on Proposition 7.3 below. To state it, we introduce the following notation. Let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous functions with $f(0) = 0$. The **reflection of g off f in the Skorokhod sense** is

$$g_{f\uparrow}(t) = g(t) - \inf_{s\in[0,t]} (g(s) - f(s)) . \tag{7.9}$$

We will apply this concept with $g = W$, standard Brownian motion.

Next, we keep f general, and will present a sequence of approximations to $W_{f\uparrow}$ that is compatible with the Brownian web. We will use these approximations to understand the law of the paths $\rho_k^{\pm} = \widetilde{\rho}_k^{\pm}$. In these approximations, we will mimic the definition of $\widetilde{\rho}_{k+1}^+$ given $\widetilde{\rho}_k^+$. That is in the nth step we run $W(t)$ until it approaches $f(t)$ by 2^{-n} and then we reset it to $f(t) + 2^{-n+1}$.

First, we fix f and a positive integer n, and describe the marginal law of the approximation $W_{f,n}$. We define $W_{f,n}(t)$ together with the sequence of stopping times ι_j^n for $j = 0, 1, 2, \ldots$ as follows. We let $\iota_0^n = 0$ and define $W_{f,n}$ to evolve as a Brownian motion started at $W_{f,n}(0) = 2^{-n+1}$ until time ι_1^n defined by

$$\iota_{j+1}^n = \inf\left\{t > \iota_j^n : W_{f,n}(t) \le f(t) + 2^{-n}\right\} . \tag{7.10}$$

For any $j = 1, 2, \ldots$, we set $W_{f,n}(\iota_j^n) = f(\iota_j^n) + 2^{-n+1}$ and we let $W_{f,n}$ evolve further as a Brownian motion independent of the past of $W_{f,n}$. In this way $W_{f,n}(t)$ is defined on the intervals $t \in [\iota_j^n, \iota_{j+1}^n)$ to run as Brownian motion and with jumps at ι_j^n for $j = 1, 2, \ldots$ from $f(\iota_j^n) + 2^{-n}$ to $f(\iota_j^n) + 2^{-n+1}$.

Next, we describe a specific coupling of W and the $W_{f,n}$, $n = 1, 2, \ldots$ which is compatible with the Brownian web. We refer to this as the **Brownian web coupling**. Under this coupling, Brownian motion paths follow a coalescing rule. Let us sample W first, which gives us $W_{f\uparrow}$. Then we add the paths $W_{f,n}$ one by one as follows. The path $W_{f,1}$ is a Brownian motion started at 1 and if it reaches $f + 1/2$ then it jumps to $f + 1$. We set $W_{f,1}$ to be a Brownian motion started at 1 which is independent of $W_{f\uparrow}$ until the time when $W_{f,1}$ hits $W_{f\uparrow}$. Then $W_{f,1}$ follows $W_{f\uparrow}$ until the next jump of $W_{f,1}$. After its jump, $W_{f,1}$ evolves independently of $W_{f\uparrow}$ again until they collide.

Suppose that $W_{f\uparrow}$ and $W_{f,1}, \ldots, W_{f,n}$ are already sampled. The trajectory $W_{f,n+1}$ starts above $W_{f\uparrow}$ and below $W_{f,1}, \ldots, W_{f,n}$ and, as it will be clear from the construction, this relation is kept for all time under the Brownian web coupling. The path $W_{f,n+1}$ runs independently of those sampled so far ($W_{f\uparrow}$ and $W_{f,1}, \ldots, W_{f,n}$) as long as it is away from them. Note that there are two possible collisions of $W_{f,n+1}$ with previously sampled paths. First, if $W_{f,n+1}$ hits $W_{f,n}$ from below, then $W_{f,n+1}$ coalesces with $W_{f,n}$ and they run together until the next jump of $W_{f,n}$ at ι_j^n for some j when $W_{f,n}$ jumps up and $W_{f,n+1}$ runs further as an independent Brownian motion. Second, if $W_{f,n+1}$ hits $W_{f\uparrow}$ from above, then $W_{f,n+1}$ coalesces with $W_{f\uparrow}$ until the next jump of $W_{f,n+1}$ when $W_{f,n+1}$ jumps up and it evolves further as an independent Brownian motion again.

By construction, we have

$$W_{f,n}(t) \geq W_{f,n+1}(t) \geq W_{f\uparrow}(t), \qquad \text{for all } t \geq 0. \tag{7.11}$$

By monotonicity, the limit of $W_{f,n}$ exists and it satisfies

$$W_{f*}(t) = \lim_{n\to\infty} W_{f,n}(t) \geq W_{f\uparrow}(t) \qquad \text{for all } t \geq 0. \tag{7.12}$$

The inequality in (7.12) is an equality according to the next result.

Proposition 7.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $f(0) = 0$. Then*

$$W_{f*}(t) = W_{f\uparrow}(t) \tag{7.13}$$

holds almost surely for all t.

Proof of Lemma 7.3. We define another coupling of the sequence of processes $W_{f,n}(t)$ where each of them are driven by the same Brownian motion $W(t)$. We denote these processes by $\widetilde{W}_{f,n}(t)$ under this coupling which we define by

$$\widetilde{W}_{f,n}(t) = f(\iota_j^n) + 2^{-n+1} + W(t) - W(\iota_j^n) \tag{7.14}$$

for $t \in [\iota_j^n, \iota_{j+1}^n)$ where ι_j^n is given by (7.10). Note that for every n, $\widetilde{W}_{f,n}$ defined in (7.14) has the same law as $W_{f,n}$.

For a given integer n and given $t \geq 0$, let j be such that $\iota_j^n \leq t < \iota_{j+1}^n$, that is ι_j^n is the last time in $[0, t]$ where $W_{f,n}$ jumps or time 0 if there is no jump in $[0, t]$. Next we use the property of Skorokhod reflection that for any continuous $f, g : \mathbb{R}_+ \to \mathbb{R}$ and $0 \leq s \leq r$ the inequality $g(r) - g(s) \leq g_{f\uparrow}(r) - g_{f\uparrow}(s)$ holds. For this reason, the increment of W on the right-hand side of (7.14) can be upper bounded almost surely as

$$\widetilde{W}_{f,n}(t) \leq f(\iota_j^n) + 2^{-n+1} + W_{f\uparrow}(t) - W_{f\uparrow}(\iota_j^n). \tag{7.15}$$

Another property of Skorokhod reflection is that $W_{f\uparrow}(\iota_j^n) \geq f(\iota_j^n)$, hence (7.15) can be further upper bounded as

$$\widetilde{W}_{f,n}(t) \leq 2^{-n+1} + W_{f\uparrow}(t) \tag{7.16}$$

which holds almost surely for all t uniformly. Also, by construction $W_{f\uparrow} \leq \widetilde{W}_{f,n}$. This implies that $\sup_t |\widetilde{W}_{f,n}(t) - W_{f\uparrow}(t)| \to 0$, and hence the sequence $\widetilde{W}_{f,n}$ is tight in the sup-norm topology. So $W_{f,n}$ is also tight since it has the same distribution. In particular, its pointwise limit W_{f*} is continuous. So for every $t \geq 0$, almost surely

$$W_{f\uparrow}(t) \stackrel{\mathrm{d}}{=} W_{f*}(t) \geq W_{f\uparrow}(t), \tag{7.17}$$

and so $W_{f\uparrow}(t) = W_{f*}(t)$. The continuity of W_{f*} and $W_{f\uparrow}$ now implies the claim. $\square$

Proof of Proposition 3.1. By definition (3.2) the curve ρ_{k+1}^{+} is independent of the randomness above ρ_{k+1}^{+} which implies that the sequence $\rho_k^{\pm}(t)$ on $[\tau_k, s]$ is Markovian in k. Hence the distribution of ρ_{k+1}^{+} only depends on ρ_k^{+}. It remains to show that ρ_{k+1}^{+} is a Skorokhod reflected Brownian motion off ρ_k^{+} on $[\tau_k, s]$, the evolution on $[\tau_{k+1}, \tau_k]$ is clear.

By Proposition 7.2 it holds that $\rho_{k+1}^{+} = \widetilde{\rho}_{k+1}^{+}$. We show next that $\widetilde{\rho}_{k+1}^{+}$ is a Brownian motion reflected off $\widetilde{\rho}_k^{+}$ in the Skorokhod sense. Instead of specifying the backward Brownian motion which is reflected off $\widetilde{\rho}_k^{+}(t)$ to construct $\widetilde{\rho}_{k+1}^{+}(t)$ we use Proposition 7.3 backward in time with the continuous curve $f(t) = \widetilde{\rho}_k^{+}(s-t)$ for $t \geq 0$. Then $W_{f,n}(t) = \widetilde{\rho}_{k+1}^{+,n}(s-t)$ holds for all integer n and $t \geq 0$ under the Brownian web coupling of $W_{f,n}$. Proposition 7.3 yields that the almost sure decreasing limit $\widetilde{\rho}_{k+1}^{+}$ is a Brownian path reflected off $\widetilde{\rho}_k^{+}$ in the Skorokhod sense which completes the proof. □

Proof of Proposition 3.2. By Proposition 7.2, we have $\rho_k^{\pm} = \widetilde{\rho}_k^{\pm}$ for all k, hence it is enough to show that the curves $\widetilde{\rho}_k^{\pm}$ serve as the boundary of the region Q_k. We proceed by induction on k. The statement clearly holds for $k = 0$ because the points $\{(t,x) : D^{\mathrm{Br}}(t,x;s,I) = 0\}$ are exactly those for which $B_{(t,x)}(s) \in I$ and these points are the ones between $\widetilde{\rho}_0^{-}(t) = \widehat{B}_{(s,u)}(t)$ and $\widetilde{\rho}_0^{+}(t) = \widehat{B}_{(s,v)}(t)$ for $t \in [\tau_0, s]$.

Next we assume that

$$Q_k = \left\{(t,x) : t \in [\tau_k, s], x \in [\widetilde{\rho}_k^{-}(t), \widetilde{\rho}_k^{+}(t)]\right\} \tag{7.18}$$

holds for some k and we prove it for $k+1$ below. We choose an arbitrary $(t,x) \notin Q_k$ and we prove that if the distance $D^{\mathrm{Br}}(t,x;s,I)$ is $k+1$, then the point (t,x) belongs to the right-hand side of (7.18) and if the distance is larger than $k+1$, then (t,x) is not in the right-hand side of (7.18).

If the forward Brownian web path $B_{(t,x)}$ hits the boundary of the set Q_k, then by definition $D^{\mathrm{Br}}(t,x;s,I) = k+1$ and $(t,x) \in Q_{k+1}$. Without loss of generality we can assume that $B_{(t,x)}$ hits the curve $\widetilde{\rho}_k^{+}$ at some $(t^*, \widetilde{\rho}_k^{+}(t^*))$. We show that the point (t,x) is below the curve $\widetilde{\rho}_{k+1}^{+}$, that is, $x \leq \widetilde{\rho}_{k+1}^{+}(t)$ which by definition (7.5) means that $x \leq \widetilde{\rho}_{k+1}^{+,n}(t)$ for all n. This is however true because $B_{(t,x)}(r)$ is a continuous forward trajectory for $r \in [t, t^*]$ starting at $B_{(t,x)}(t) = x$ and ending at $B_{(t,x)}(t^*) = \widetilde{\rho}_k^{+}(t^*)$. The backward trajectory $\widetilde{\rho}_{k+1}^{+,n}(r)$ for $r \in [t,t^*]$ starts at $\widetilde{\rho}_{k+1}^{+,n}(t^*) \geq \widetilde{\rho}_k^{+}(t^*) + 2^{-n}$, that is, above the endpoint of $B_{(t,x)}$. Furthermore $\widetilde{\rho}_{k+1}^{+,n}(r)$ is defined to follow a backward Brownian web trajectory on the intervals $(\kappa_{j+1}^n, \kappa_j^n]$ and to jump up by 2^{-n} at every κ_j^n. For this reason $\widetilde{\rho}_{k+1}^{+,n}(r)$ can never cross $B_{(t,x)}(r)$ and it remains above $B_{(t,x)}(r)$ for $r \in [t, t^*]$. In particular $x = B_{(t,x)}(t) \leq \widetilde{\rho}_{k+1}^{+,n}(t)$ for all n which proves that $x \leq \widetilde{\rho}_{k+1}^{+}(t)$.

If the path $B_{(t,x)}$ avoids the set Q_k, then $D^{\mathrm{Br}}(t,x;s,I) > k+1$ and $(t,x) \notin Q_{k+1}$. In this case we may assume that $B_{(t,x)}$ passes above the continuous upper boundary $\widetilde{\rho}_k^{+}$ (continuity follows from Proposition 3.1). Then $B_{(t,x)}$ has a positive distance from $\widetilde{\rho}_k^{+}$, that is, there is an $\varepsilon > 0$ such that $B_{(t,x)}(r) \geq \widetilde{\rho}_k^{+}(r) + \varepsilon$ for all $r \in [t,s]$. Let $n \geq 1 - \log_2 \varepsilon$ which means that $2^{-n+1} \leq \varepsilon$. We show that the point (t,x) is above $\widetilde{\rho}_{k+1}^{+,n}$, that is, $x \geq \widetilde{\rho}_{k+1}^{+,n}(t) \geq \widetilde{\rho}_{k+1}^{+}(t)$ where the second inequality holds due to the fact that $\widetilde{\rho}_{k+1}^{+}$ is the decreasing limit of $\widetilde{\rho}_{k+1}^{+,n}$. The inequality $x \geq \widetilde{\rho}_{k+1}^{+,n}(t)$ is true because $B_{(t,x)}(r)$ is a continuous forward trajectory for $r \in [t,s]$ starting at $B_{(t,x)}(t) = x$ and ending at $B_{(t,x)}(s) \geq v + \varepsilon$. The backward trajectory $\widetilde{\rho}_{k+1}^{+,n}(r)$ for $r \in [t,s]$ starts at $v + 2^{-n+1}$

which is below the endpoint of $B_{(t,x)}$. As long as $\widetilde{\rho}^{+,n}_{k+1}$ follows a backward Brownian web path it cannot cross $B_{(t,x)}$. In the points κ^n_j the path $\widetilde{\rho}^{+,n}_{k+1}$ jumps from $\widetilde{\rho}^+_k + 2^{-n}$ to $\widetilde{\rho}^+_k + 2^{-n+1}$ which is still below $\widetilde{\rho}^+_k + \varepsilon \le B_{(t,x)}$. Therefore $\widetilde{\rho}^{+,n}_{k+1}$ remains below $B_{(t,x)}$, hence $x \ge \widetilde{\rho}^{+,n}_{k+1}(t) \ge \widetilde{\rho}^+_{k+1}(t)$. □

7.2 Properties of discrete region boundaries

This subsection contains the proofs of Propositions 3.7 and 3.8. about the boundaries of regions with different discrete web distances. First we prove the following discrete analogue of Proposition 7.2 which describes the evolution of the discrete region boundaries. Heuristically r^+_{k+1} follows the evolution of a backward discrete Brownian web trajectory of $\widehat{Y}$ as long as it is away from r^+_k, cf. (2.8), and r^+_{k+1} is forced to jump up if it is equal to r^+_k until time T_k. Beyond T_k, r^+_{k+1} follows a backward discrete Brownian web trajectory. The evolution of r^-_{k+1} given r^-_k can be given similarly.

Proposition 7.4. *For any $k = 0, 1, 2, \dots$, conditionally given the trajectory of r^+_k and the time T_k when $r^\pm_k$ meet, the evolution of r^+_{k+1} is given by*

$$r^+_{k+1}(i-1) = \begin{cases} r^+_{k+1}(i) - \xi_{(i-1, r^+_{k+1}(i))} & \text{if} \quad r^+_{k+1}(i) > r^+_k(i) \\ r^+_{k+1}(i) + 1 & \text{if} \quad r^+_{k+1}(i) = r^+_k(i) \end{cases} \tag{7.19}$$

for all $i = j, j-1, \dots, T_k + 1$. For times $i = T_k, T_k - 1, \dots$ we have that

$$r^+_{k+1}(i-1) = r^+_{k+1}(i) - \xi_{(i-1, r^+_{k+1}(i))}. \tag{7.20}$$

Proof of Proposition 7.4. Given the path r^+_k up to time T_k let $\widetilde{r}^+_{k+1}$ denote the trajectory given by the evolution rules (7.19)–(7.20). We prove that for any $k = 0, 1, 2, \dots$ given r^+_k it holds that $r^+_{k+1}(i) = \widetilde{r}^+_{k+1}(i)$ for all $i \le j$ by induction backwards on i. The case $i = j$ is true by definition. We assume that r^+_{k+1} and $\widetilde{r}^+_{k+1}$ agree for $i+1, i+2, \dots, j$. To see the equality $r^+_{k+1}(i) = \widetilde{r}^+_{k+1}(i)$ we rewrite (3.4) by separating the $l = i$ term as

$$\begin{aligned} r^+_{k+1}(i) &= \max\left(\widehat{Y}_{(i, r^+_k(i+1)+1)}(i), \max_{l \in \{i+1, \dots, j\}} \widehat{Y}_{(l, r^+_k(l+1)+1)}(i)\right) \\ &= \max\left(r^+_k(i+1) + 1, r^+_{k+1}(i+1) - \xi_{(i, r^+_{k+1}(i+1))}\right). \end{aligned} \tag{7.21}$$

Since $r^+_{k+1} \ge r^+_k$ always holds there are two possibilities: either $r^+_{k+1}(i+1) = r^+_k(i+1)$ or $r^+_{k+1}(i+1) > r^+_k(i+1)$. If $r^+_{k+1}(i+1) = r^+_k(i+1)$, then $r^+_{k+1}(i) = r^+_k(i+1) + 1$ since the first term in the maximum on the right-hand side of (7.21) cannot be smaller than the other one. Also $\widetilde{r}^+_{k+1}(i) = r^+_{k+1}(i+1) + 1 = r^+_k(i+1) + 1$ by (7.19) hence $r^+_{k+1}(i) = \widetilde{r}^+_{k+1}(i)$. If $r^+_{k+1}(i+1) > r^+_k(i+1)$, then $r^+_{k+1}(i+1) \ge r^+_k(i+1) + 2$, hence $r^+_{k+1}(i) = r^+_{k+1}(i+1) - \xi_{(i, r^+_{k+1}(i+1))}$ because the second term in the maximum on the right-hand side of (7.21) cannot be smaller than the first one. On the other hand by (7.19) we have $\widetilde{r}^+_{k+1}(i) = r^+_{k+1}(i+1) - \xi_{(i, r^+_{k+1}(i+1))}$ which means that $r^+_{k+1}(i) = \widetilde{r}^+_{k+1}(i)$ as required for $i \ge T_k$. The evolution beyond T_k clearly follows that of the backward Brownian web given in (7.20). □

Proof of Proposition 3.7. To see (3.5) we proceed by induction on k. Given r_k^+, the path r_{k+1}^+ clearly starts at $(j, v+1)$ as well as the right-hand side of (3.5). Next we check that the right-hand side of (3.5) satisfies the evolution rule (7.19) by induction on i backwards. For this we rewrite the right-hand side of (3.5) by separating the $l = i$ case in the minimum to get that

$$\begin{aligned} & s_{k+1}(i) - \min_{l \in \{i,\dots,j\}} \left(s_{k+1}(l) - r_k^+(l+1) - 1 \right) \\ &= s_{k+1}(i) - \min \left(s_{k+1}(i) - r_k^+(i+1) - 1, \min_{l \in \{i+1,\dots,j\}} \left(s_{k+1}(l) - r_k^+(l+1) - 1 \right) \right) \\ &= s_{k+1}(i) - \min \left(s_{k+1}(i) - r_k^+(i+1) - 1, s_{k+1}(i+1) - r_{k+1}^+(i+1) \right) \\ &= \max \left(r_k^+(i+1) + 1, s_{k+1}(i) - s_{k+1}(i+1) + r_{k+1}^+(i+1) \right) \end{aligned} \tag{7.22}$$

where the second equality above follows by the induction hypothesis (3.5) with i replaced by $i+1$. The right-hand side of (7.22) means an independent random walk step compared to $r_{k+1}^+(i+1)$ except for the case when $r_{k+1}^+(i+1) = r_k^+(i+1)$ in which case it is a forced to jump up by one. This corresponds to the evolution of $r_{k+1}^+(i)$ described in (7.19) hence it proves (3.5).

The proof of (3.6) follows an induction on k. It clearly holds for $k = 0$ and the induction step is based on (3.5) which one can write as

$$\begin{aligned} r_k^+(i) &= \max_{l \in \{i,\dots,j\}} \left(s_k(i) - s_k(l) + r_{k-1}^+(l+1) + 1 \right) \\ &= \max_{l \in \{i,\dots,j\}} \Bigg(s_k(i) - s_k(l) \\ &\qquad + \max_{l+1 \le l_{k-1} \le \dots \le l_{-1} = j-k+1} \sum_{m=0}^{k-1} \left(s_m(l_m) - s_m(l_{m-1}) \right) + k + v + 1 \Bigg) \end{aligned} \tag{7.23}$$

where we used the induction hypothesis (3.6) for $r_{k-1}^+(l+1)$ in the last equality above. Then the right-hand side above is equal in distribution to that of (3.6) which can be seen by shifting all indices $l_{k-1}, \dots, l_{-1}$ by 1. $\square$

Proof of Proposition 3.8. We proceed by induction on k. First we prove the statement for $k = 0$. Take any $i \in [T_0, j]$ and $m \in \mathbb{Z}$ so that $(i, m) \in \mathbb{Z}_\mathrm{e}^2$. The forward random walk web path $Y_{(i,m)}$ cannot cross either of the backward paths $r_0^+ = \widehat{Y}_{(j,v+1)}$ and $r_0^- = \widehat{Y}_{(j,u-1)}$ on the interval $[i, j]$. Hence if $r_0^-(i) < m < r_0^+(i)$, then $Y_{(i,m)}(j) \in I$ and $D^{\mathrm{RW}}(i, m; j, I) = 0$. If $m < r_0^-(i)$ or $m > r_0^+(i)$, then $Y_{(i,m)}(j) \notin I$ and $D^{\mathrm{RW}}(i, m; j, I) > 0$. Also if $i < T_0$, then the forward random walk web path $Y_{(i,m)}$ avoids $j \times I$ and $D^{\mathrm{RW}}(i, m; j, I) > 0$.

Assume that the statement is true for k, that is, $D^{\mathrm{RW}}(i, m; j, I) \le k$ for points $(i, m) \in \mathbb{Z}_\mathrm{e}^2$ between r_k^+ and r_k^- on $(T_k, j]$ and $D^{\mathrm{RW}}(i, m; j, I) > k$ for all other points $(i, m) \in \mathbb{Z}_\mathrm{e}^2$. Now we prove it for $k + 1$ by another induction on i backwards in time as follows. The statement of the proposition clearly holds for $k + 1$ and $i = j$. Indeed we have that $D^{\mathrm{RW}}(j, m; j, I) = 0 \le k + 1$ exactly if $r_{k+1}^-(j) = u - 1 < m < r_{k+1}^+(j) = v + 1$, otherwise $D^{\mathrm{RW}}(j, m; j, I) = \infty > k + 1$.

Next we assume that the statement holds for $k + 1$ and $i \in [T_k, j]$ and we prove it for $k + 1$ and $i - 1$ in what follows. By assumption around the upper boundary the points

$(i,m) \in \mathbb{Z}^2_{\mathrm{e}}$ with $r_k^+(i) < m < r_{k+1}^+(i)$ have $D^{\mathrm{RW}}(i,m;j,I) = k+1$. We have two cases depending on weather there are such ms or not, that is $r_k^+(i) < r_{k+1}^+(i)$ or $r_k^+(i) = r_{k+1}^+(i)$.

We deal with the case $r_k^+(i) < r_{k+1}^+(i)$ first. Then $D^{\mathrm{RW}}(i-1,m;j,I) \le k+1$ certainly holds for all points $(i-1,m) \in \mathbb{Z}^2_{\mathrm{e}}$ with $r_k^+(i-1) < m \le r_{k+1}^+(i) - 2$ because these points $(i-1,m)$ are connected to either of $(i,m\pm 1)$ by an edge in the random walk web Y and $D^{\mathrm{RW}}(i,m\pm 1;j,I) \le k+1$ by assumption. If $\xi_{(i-1,r_{k+1}^+(i))} = -1$, then there is an edge in Y from $(i-1,r_{k+1}^+(i))$ to $(i,r_{k+1}^+(i)-1)$, hence $D^{\mathrm{RW}}(i-1,r_{k+1}^+(i);j,I) = D^{\mathrm{RW}}(i,r_{k+1}^+(i)-1) = k+1$ and $(i-1,r_{k+1}^+(i)) \in R_{k+1}$. If $\xi_{(i-1,r_{k+1}^+(i))} = 1$, then there is an edge in Y from $(i-1,r_{k+1}^+(i))$ to $(i,r_{k+1}^+(i)+1)$, hence $D^{\mathrm{RW}}(i-1,r_{k+1}^+(i);j,I) = D^{\mathrm{RW}}(i,r_{k+1}^+(i)-1)+1 = k+2$ and $(i-1,r_{k+1}^+(i)) \notin R_{k+1}$. This means that the boundary of R_{k+1} changes by $-\xi_{(i-1,r_{k+1}^+(i))}$ in the step $i \to i-1$ which is the same as the evolution of r_{k+1}^+ in the first case in (7.19).

In the case $r_k^+(i) = r_{k+1}^+(i)$ regardless of the value of $\xi_{(i-1,r_k^+(i))}$ and whether the edge from $(i-1,r_k^+(i))$ to $(i,r_k^+(i)-1)$ is in Y or not, we can bound $D^{\mathrm{RW}}(i-1,r_k^+(i);j,I) \le D^{\mathrm{RW}}(i,r_k^+(i)-1;j,I)+1 \le k+1$, hence $(i-1,r_k^+(i)) \notin R_{k+1}$. This means that the boundary of R_{k+1} increases by 1 in this step and it corresponds to the second case in (7.19) where $r_{k+1}^+(i-1) = r_{k+1}^+(i)+1$.

For $i \le T_k$, we know by assumption that $D^{\mathrm{RW}}(i,m;j,I) = k+1$ if $(i,m) \in \mathbb{Z}^2_{\mathrm{e}}$ with $r_{k+1}^+(i) < m < r_{k+1}^-(i)$ and $D^{\mathrm{RW}}(i,m;j,I) > k+1$ for all other values of m. Hence the boundary of R_{k+1} changes by $-\xi_{(i-1,r_{k+1}^\pm(i))}$ which means that the boundary follows the same backward random walk web trajectory as $r_{k+1}^\pm$, cf. (7.20). □

7.3 Lower semicontinuity

The aim of this subsection is to prove the lower semicontinuity of the Brownian web distance D^{Br} stated in Theorem 1.4. The lower semicontinuity relies on the following property of the Brownian web.

Proposition 7.5. *Fix a compact subset $K \subset \mathbb{R}^2$ and $\varepsilon > 0$. Almost surely there are only finitely many disjoint paths of the Brownian web B with length at least ε which are contained in K.*

Proof of Proposition 7.5. Assume that there is a K and $\varepsilon > 0$ such that there are infinitely many disjoint paths of B with length ε in K with positive probability. By dividing the time axis into subintervals of length $\varepsilon/2$ we see that there are also infinitely many disjoint paths of B with length $\varepsilon/2$ in K which all start at the same time which we denote by s. This means that there are infinitely many starting points on the compact set $(\{s\}\times\mathbb{R}) \cap K$ which have disjoint outgoing paths of length $\varepsilon/2$.

Next we show a basic bounds on the Brownian motion $W(t)$ which starts form 0 and has variance $\sigma^2 t$ at time t. For $\lambda > 0$,

$$\mathbf{P}\left(\sup_{t\in[0,T]} W(t) \le \lambda\right) = 1 - \mathbf{P}\left(\sup_{t\in[0,T]} W(t) \ge \lambda\right) = 1 - 2\mathbf{P}(W(T) \ge \lambda) \le \frac{2\lambda}{\sqrt{2\pi\sigma^2 T}} \tag{7.24}$$

holds where the second equality follows from the reflection principle and the last inequality can be seen by upper bounding the density of $W(T)$ by its value $1/\sqrt{2\pi\sigma^2 T}$ at 0.

We assume that two paths in the Brownian web B start from (s, w) and from $(s, w+\delta)$ for some $\delta > 0$. The probability that the two trajectories $B_{(s,w)}$ and $B_{(s,w+\delta)}$ do not meet until time $\varepsilon/2$ can be written as

$$\begin{aligned} \mathbf{P}\left(\inf_{t\in[0,\varepsilon/2]}\left(B_{(s,w+\delta)}(t) - B_{(s,w)}(t)\right) > 0\right) &= \mathbf{P}\left(\sup_{t\in[0,\varepsilon/2]}\left(\delta - B_{(s,w+\delta)}(t) + B_{(s,w)}(t)\right) < \delta\right) \\ &\le \sqrt{\frac{2}{\pi\varepsilon}}\delta \end{aligned} \tag{7.25}$$

where the last inequality follows from (7.24) with $\sigma^2 = 2$.

Now assume that there are infinitely many paths starting in $(\{s\} \times \mathbb{R}) \cap K$ which remain disjoint for time at least $\varepsilon/2$. Then the starting points have an accumulation point which yields that for any $\delta > 0$ there are two starting points closer to each other than δ. The two paths emanating from these two starting points also have to be disjoint which has a probability at most $\sqrt{2/(\pi\varepsilon)}\delta$ by (7.25). This implies that the probability of infinitely many disjoint paths of length $\varepsilon/2$ in B starting in $(\{s\} \times \mathbb{R}) \cap K$ can be upper bounded by $\sqrt{2/(\pi\varepsilon)}\delta$ for any $\delta > 0$, that is, this probability is 0 as required. $\square$

Lemma 7.6. *On an event of probability one the function $(s, y) \mapsto D^{\mathrm{Br}}(t, x; s, y)$ is lower semicontinuous.*

Proof. Assume that lower semicontinuity fails to hold for some $(t, x; s, y) \in \mathbb{R}^4$ for which $D^{\mathrm{Br}}(t, x; s, y) = k$ where $k = \infty$ is also allowed. This means that there is a sequence $(s^{(n)}, y^{(n)})$ converging to (s, y) in $\mathbb{R}^2$ such that $D^{\mathrm{Br}}(t, x; s^{(n)}, y^{(n)}) = l < k$. We may assume that l is minimal. Let $(s_l^{(n)}, y_l^{(n)})$ denote the point of the lth jump along the geodesic path from (t, x) to $(s^{(n)}, y^{(n)})$ which verifies that their Brownian web distance is l. No subsequence of $(s_l^{(n)}, y_l^{(n)})$ can converge to (s, y) in $\mathbb{R}^2$ because then the paths from (t, x) to $(s_l^{(n)}, y_l^{(n)})$ would have a subsequence with Brownian web distance $l - 1$ which would contradict the minimality of l. Hence there is a $\delta > 0$ such that the points $(s_l^{(n)}, y_l^{(n)})$ are at least δ apart from (s, y) in $\mathbb{R}^2$.

Let ε_n denote the Euclidean distance between the point (s, y) and the closest point along the Brownian web path $B_{(s_l^{(n)}, y_l^{(n)})}$ starting at $(s_l^{(n)}, y_l^{(n)})$ for all n. Note that the paths $B_{(s_l^{(n)}, y_l^{(n)})}$ do not terminate at $(s^{(n)}, y^{(n)})$ but we consider their continuation. We have $\varepsilon_n \to 0$ as $n \to \infty$ because the point $(s^{(n)}, y^{(n)})$ is on the path $B_{(s_l^{(n)}, y_l^{(n)})}$ for each n and $(s^{(n)}, y^{(n)}) \to (s, y)$. On the other hand $\varepsilon_n = 0$ is impossible for any n because it would mean that $D^{\mathrm{Br}}(t, x; s, y) = l$. By the properties of ε_n there is a subsequence n_j along which ε_{n_j} is strictly decreasing. The corresponding paths starting at $(s_l^{(n_j)}, y_l^{(n_j)})$ have disjoint portions from the boundary of the ball of radius δ around (s, y) to distance ε_{n_j} from (s, y). For j large enough the length of these paths are at least $\delta/2$. The existence of infinitely many disjoint paths of length $\delta/2$ contradicts Proposition 7.5. $\square$

Lemma 7.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a lower semicontinuous function. We extend it to intervals $I \subset \mathbb{R}$ as $f(I) = \inf_{y \in I} f(y)$. Then it holds for any $y \in \mathbb{R}$ that*

$$f(y) = \sup_{I \subset \mathbb{R}: y \in I} f(I). \tag{7.26}$$

Proof. Fix $y \in \mathbb{R}$. Then $f(y) \geq \sup_{I \subset \mathbb{R}: y \in I} f(I)$ clearly holds because $f(y) \geq f(I)$ if $y \in I$. The strict inequality $f(y) > \sup_{I \subset \mathbb{R}: y \in I} f(I)$ would mean that there is an $\varepsilon > 0$ such that $f(I) \leq f(y) - \varepsilon$ for all I with $y \in I$. By letting y_n be an approximate minimizer of f in the interval I_n so that $y \in I_n$ and $|I_n| \to 0$, one would get a sequence for which $\liminf_{y_n \to y} f(y_n) \leq f(y) - \varepsilon/2 < f(y)$ contradicting the lower semicontinuity of f. $\square$

Proof of Theorem 1.4. We have seen in Lemma 7.6 that $D^{\mathrm{Br}}(t, x; s, y)$ is lower semicontinuous in the second component (s, y). Now we prove that lower semicontinuity also holds in the first component. For any $(t, x; s, y)$, one has by Lemma 7.7 that

$$D^{\mathrm{Br}}(t, x; s, y) = \sup_{I \subset \mathbb{R}: y \in I} D^{\mathrm{Br}}(t, x; s, I) \tag{7.27}$$

holds. It is clear from the description given in Propositions 3.2 and 3.1 that for any $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ the mapping $(t, x) \mapsto D^{\mathrm{Br}}(t, x; s, I)$ is lower semicontinuous for any (t, x). Since the supremum of lower semicontinuous functions is lower semicontinuous (cf. Definition 2.8 (c) in [Rud87]), $(t, x) \mapsto D^{\mathrm{Br}}(t, x; s, y)$ is also lower semicontinuous by (7.27).

Finally we show that $(t, x, s, y) \mapsto D^{\mathrm{Br}}(t, x; s, y)$ is jointly lower semicontinuous almost surely. For this we assume that $D^{\mathrm{Br}}(t, x; s, y) = k$ with $k \in \mathbb{N} \cup \{+\infty\}$ and that there are $(t^{(n)}, x^{(n)}) \to (t, x)$ and $(s^{(n)}, y^{(n)}) \to (s, y)$ in $\mathbb{R}^2$ such that $D^{\mathrm{Br}}(t^{(n)}, x^{(n)}; s^{(n)}, y^{(n)}) = l < k$ and l is minimal. By the minimality of l, the sequence of the first points of jump $(t_1^{(n)}, x_1^{(n)})$ along the paths between $(t^{(n)}, x^{(n)})$ and $(s^{(n)}, y^{(n)})$ remain at least ε far from (t, x) for some $\varepsilon > 0$.

For any $\delta > 0$, the Brownian web paths starting at $(t^{(n)}, x^{(n)})$ can exit the ball of radius δ around (t, x) only in finitely many points by Proposition 7.5. Since the Brownian web paths cannot cross each other and the starting points converge to (t, x), all Brownian web paths starting at $(t^{(n)}, x^{(n)})$ exit the ball of radius δ around (t, x) in the same point $z_\delta \in \mathbb{R}^2$ for all $n \geq n_0$ for some $n_0 = n_0(\delta)$ depending on δ.

Using the above fact with $\delta = \varepsilon/2^j$ for $j = 0, 1, 2, \ldots$, one gets the existence of a sequence of Brownian web paths between $z_{\varepsilon/2^j}$ and z_ε such that the path with larger j extends the previous paths. This implies that there is a Brownian web path $B'_{(t,x)}$ from (t, x) which passes through all the points $z_{\varepsilon/2^j}$. This may also coincide with $B_{(t,x)}$. For all $n \geq n_0(\varepsilon)$, the path $B_{(t^{(n)}, x^{(n)})}$ passes through z_ε. Hence for $n \geq n_0(\varepsilon)$, the paths $B_{(t^{(n)}, x^{(n)})}$ and $B'_{(t,x)}$ collide within the ball of radius ε around (t, x), in particular $B'_{(t,x)}$ passes through $(t_1^{(n)}, x_1^{(n)})$. Using the Brownian web path $B'_{(t,x)}$ between (t, x) and $(t_1^{(n)}, x_1^{(n)})$ one gets a sequence of paths between (t, x) and $(s^{(n)}, y^{(n)})$ with D^{Br} distance l. This contradict the lower semicontinuity of D^{Br} in the second component given in Lemma 7.6. $\square$

7.4 Finiteness of the Brownian web distance

We prove Proposition 3.3 about the finiteness of the Brownian web distance in this subsection. In order to characterize the case when the Brownian distance D^{Br} is infinite in terms of the second statement in Proposition 3.3, we recall the Brownian last passage percolation with general boundary condition f defined in (5.2). By Lemma 5.1, the Brownian last passage percolation with boundary f can be obtained as an iterated Skorokhod reflection of Brownian paths started from f. Next we show that the Brownian last passage percolation diverges everywhere.

Lemma 7.8. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a fixed continuous function. For the Brownian last passage percolation with boundary condition f it holds that for any $t > 0$ fixed*

$$\mathbf{P}\left(\lim_{n\to\infty} L^f(t,n) = +\infty\right) = 1. \tag{7.28}$$

Proof. The right-hand side in the definition (5.2) is lower bounded by

$$\sup_{0=t_0\le t_1\le\cdots\le t_n=t}\left(f(t_1)+\sum_{i=2}^n (W_i(t_i)-W_i(t_{i-1}))\right) \ge \max_{i=2,\dots,n}(W_i(t)) \ge \max_{i=2,\dots,n}(W_i(t)\wedge M) \tag{7.29}$$

for any $M \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary and note that

$$\mathbf{P}(\max_{i=2,\dots,n}(W_i(t)\wedge M) < M-\varepsilon) = \mathbf{P}(W_2(t) < M-\varepsilon)^n \to 0 \tag{7.30}$$

as $n \to \infty$, that is $\max_{i=2,\dots,n}(W_i(t)\wedge M)$ converges to M in distribution and in probability because the limit is a constant. Since the sequence $\max_{i=2,\dots,n}(W_i(t)\wedge M)$ is non-decreasing in n, there is an almost sure convergence. This implies by (5.2) that $\liminf_{n\to\infty} L^f(t,n) \ge M$ almost surely for any $M \in \mathbb{R}$ which completes the proof. □

Proof of Proposition 3.3. Assume that (s,y) is an interior point of the path $B_{(t,x)}$. Then the point (s,y) can be of type $(1,1)$ or $(2,1)$ in the forward Brownian web B and of type $(0,2)$ or $(0,3)$ in the dual $\widehat{B}$, that is, there are two or three dual paths starting at (s,y).

We consider the regions $\widetilde{Q}_k = \{(u,z) : D^{\mathrm{Br}}(t,x;s,y) \le k\}$ for $k = 0,1,2,\dots$ which are the analogues of Q_k defined in (3.1) but with the interval $s \times I$ replaced by the single point (s,y). Let $\widetilde{\rho}_k^{\pm}$ denote the upper and lower boundaries of the regions $\widetilde{Q}_k$ which can be obtained similarly to $\rho_k^{\pm}$ that are the boundaries for Q_k, see Proposition 3.2. The boundaries $\widetilde{\rho}_0^{\pm}$ on the interval $[t,s]$ are $\widehat{B}_{(s,y)\pm}$, that is, the highest and lowest dual Brownian web paths starting at (s,y). The boundaries $\widetilde{\rho}_k^{\pm}$ for $k = 1,2,\dots$ on $[t,s]$ can be given inductively and they are Brownian motions reflected off in the Skorokhod sense upwards and downwards from $\widetilde{\rho}_{k-1}^{\pm}$.

By letting $f(v) = \widetilde{\rho}_0^{+}(s-v)$ for $v \in [0,s-t]$, Lemma 5.1 implies that the distribution of $(\widetilde{\rho}_k^{+}(s-v), v \in [0,s-t])$ is equal to that of Brownian last passage percolation $(L^f(v,k), v \in [0,s-t])$ with boundary condition f. By (7.28) in Lemma 7.8, for any $u \in [t,s)$, $\widetilde{\rho}_k^{+}(u) \to \infty$ as $k \to \infty$ almost surely and very similarly $\widetilde{\rho}_k^{-}(u) \to -\infty$. Hence almost surely for any (u,z) with $u \in [t,s)$ and $z \in \mathbb{R}$, we have $D^{\mathrm{Br}}(u,z;s,y) < \infty$. If $u < t$, one simply follows the Brownian web path $B_{(u,z)}$ until time t to reach a point with finite Brownian distance to (s,y).

In the case when (s,y) is not the interior point of any Brownian web path, that is, it is not on the skeleton of the Brownian web B, then there is no incoming trajectory of B in this point, hence by Definition 1.2, $D^{\mathrm{Br}}(t,x;s,y)$ cannot be finite for any $(t,x) \in \mathbb{R}^2$. $\square$

7.5 Brownian web distance as an induced metric and countable generating set

Finally we prove Propositions 3.5 and 3.6 in this subsection.

Proof of Proposition 3.5. The function D^{Br} satisfies the triangle inequality and the distance of any point from itself is 0, hence D^{Br} is a directed metric. Since D^{Br} agrees with d on the skeleton of the Brownian web B, the induced directed metric $\widetilde{D}^{\mathrm{Br}}$ exists and the inequality

$$D^{\mathrm{Br}}(t,x;s,y) \leq \widetilde{D}^{\mathrm{Br}}(t,x;s,y) \tag{7.31}$$

holds for any $(t,x;s,y) \in \mathbb{R}^4$ by Definition 3.4. Furthermore, if (t,x) and (s,y) are both on the skeleton of the Brownian web B, then we also have by definition that

$$\widetilde{D}^{\mathrm{Br}}(t,x;s,y) \leq d(t,x;s,y). \tag{7.32}$$

In order to show that (7.31) holds with an equality, we choose $(t,x;s,y) \in \mathbb{R}^4$ such that $D^{\mathrm{Br}}(t,x;s,y) = k$ for some finite integer k. By Definition 1.2, there are points $(t_1,x_1),\dots,(t_k,x_k)$ at which the optimal path from (t,x) to (s,y) switches between different Brownian web trajectories. In particular, the path starting at (t,x) passes through (t_1,x_1). The point (t_1,x_1) is a $(1,2)$ point for B and for its dual $\widehat{B}$. Between the two backward Brownian web path starting at (t_1,x_1), one can find a sequence of points $(t^{(n)},x^{(n)})$ on the skeleton of the Brownian web B such that $(t^{(n)},x^{(n)}) \to (t,x)$ as $n \to \infty$ and that $B_{(t^{(n)},x^{(n)})}(t_1) = x_1$ for all n. This means that $d(t^{(n)},x^{(n)};s,y) \leq k$ for all n, hence also $\widetilde{D}^{\mathrm{Br}}(t^{(n)},x^{(n)};s,y) \leq k$ holds for all n by (7.32). Since $\widetilde{D}^{\mathrm{Br}}$ is a lower semicontinuous directed metric as being the supremum of such directed metrics, it follows that $\widetilde{D}^{\mathrm{Br}}(t,x;s,y) \leq k$ proving equality in (7.31). $\square$

Proof of Proposition 3.6. We assume that the values of $D^{\mathrm{Br}}(t,x;s,I)$ are given for all rational t,x,s,u,v where $I = [u,v]$. We show in several steps that it uniquely extends to $\mathbb{R}^4$.

First we prove that the values of $D^{\mathrm{Br}}(t,x;s,I)$ are determined for all $t,x \in \mathbb{R}$ and for all rational s,u,v. We fix a rational s and an interval $I = [u,v]$ with rational endpoints. By the characterization of regions with distance k from a fixed s and I given in Proposition 3.2, it holds that for any $t \leq s$ the set

$$\{x \in \mathbb{R} : D^{\mathrm{Br}}(t,x;s,I) \leq k\} = [\rho_k^-(t), \rho_k^+(t)] \tag{7.33}$$

is an interval. This implies that knowing the values of $D^{\mathrm{Br}}(t,x;s,I)$ for fixed s and I and for (t,x) from a dense subset of $\mathbb{R}^2$ settles the continuous curves $\rho_k^\pm$ for all $k = 0,1,\dots$ by (7.33). Hence the values of $D^{\mathrm{Br}}(t,x;s,I)$ are also determined for all $(t,x) \in \mathbb{R}^2$.

Second we show that the values of $D^{\mathrm{Br}}(t,x;s,y)$ are determined for all $t,x,y \in \mathbb{R}$ and for all rational s. By Theorem 1.4, the Brownian distance is lower semicontinuous in all its variables. In particular, $y \mapsto D^{\mathrm{Br}}(t,x;s,y)$ is lower semicontinuous for any t,x,s and by applying Lemma 7.7 the equality (7.27) holds. The supremum on the right-hand side of (7.27) does not change if we replace it with the supremum over all intervals I with rational endpoints which contain y. This proves that $D^{\mathrm{Br}}(t,x;s,y)$ for $t,x,y \in \mathbb{R}$ and s rational are determined.

Finally we prove that $D^{\mathrm{Br}}(t,x;s,y)$ are determined for all $(t,x;s,y) \in \mathbb{R}^4$. We claim that for any $(t,x;s,y) \in \mathbb{R}^4$, $D^{\mathrm{Br}}(t,x;s,y) \leq k$ holds if and only if there is an increasing sequence of rationals s_n converging to s and a sequence $y_n \to y$ such that $D^{\mathrm{Br}}(t,x;s_n,y_n) \leq k$. The claim is seen as follows.

If $D^{\mathrm{Br}}(t,x;s,y) \leq k$, then there is a last jump between different Brownian web paths along the geodesic from (t,x) to (s,y) which we call (t_k,x_k) for simplicity. We choose any increasing sequence of rationals s_n from the interval (t_k,s) so that $s_n \to s$ and we let $y_n = B_{(t_k,x_k)}(s_n)$ to be the value of the Brownian web path along the geodesic at s_n. Then $y_n \to y$ and $D^{\mathrm{Br}}(t,x;s_n,y_n) \leq k$ holds. Conversely, if the sequence $(s_n,y_n) \to (s,y)$ exists with the desired properties, then the lower semicontinuity of the Brownian distance (see Theorem 1.4) implies that

$$D^{\mathrm{Br}}(t,x;s,y) \leq \liminf_{n\to\infty} D^{\mathrm{Br}}(t,x;s_n,y_n) \leq k \tag{7.34}$$

which proves the claim. The claim implies that all values of $D^{\mathrm{Br}}(t,x;s,y)$ for $(t,x;s,y) \in \mathbb{R}^4$ can be determined by those for $t,x,y \in \mathbb{R}$ and s rational. This completes the proof. $\square$

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