Practice Problems ♯10

1. Evaluate the integral
$$\iint_R (x^2y - 2xy)\, dx\, dy$$
over the rectangle $R = [-2, 0] \times [0, 3]$.

2. Evaluate the integral
$$\iint_R y \cos xy\, dx\, dy$$
over the rectangle $R = [0, \pi] \times [0, 1]$.

3. Evaluate the integral
$$\iint_D (x^2y - 2xy)\, dx\, dy$$
over the domain
$$D = \{(x, y) : 1 \le x \le 2, \quad x \le y \le 2x\}.$$

4. Find volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top is bounded by the plane $z = x + 4$.

5. Evaluate the integral by reversing the order of integration:
$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx.$$

6. *(The Euler method for the approximate solution of differential equations)*
The Euler method gives the following approximate solution of an initial value problem
$$y' = f(x, y), \qquad y(x_0) = y_0.$$
Choose an $h > 0$, and then let
$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h\, f(x_n, y_n) \qquad (n = 0, 1, 2, \ldots).$$
Apply the Euler method for the initial value problem $y' = y$, $y(0) = 1$, with $h = 1/5$. Compare the numerical solution of $y(1)$ and its exact value.

7. *(The Runge–Kutta method for the approximate solution of differential equations)*
The Runge–Kutta method gives the following approximate solution of the initial value problem discussed in the previous problem. Choose an $h > 0$, and then let
$$\begin{aligned} k_1 &= h\, f(x_n, y_n), \qquad k_2 = h\, f(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2}), \\ k_3 &= h\, f(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2}), \qquad k_4 = h\, f(x_n + h,\ y_n + k_3) \\ x_{n+1} &= x_n + h, \qquad y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (n = 0, 1, 2, \ldots). \end{aligned}$$

Apply the Runge–Kutta method for the initial value problem $y' = y$, $y(0) = 1$, with $h = 1/5$. Compare the numerical solution of $y(1)$ and its exact value.

8. *(Explosion in a non-linear differential equations)*
In contrast to linear differential equations, a non-linear one can "explode" in finite time. It means that its solution becomes infinite at a finite value of the independent variable.

Show that the solution of the initial value problem $y' = x^2 + y^2$, $y(0) = 1$ increases faster on the interval $0 \leq x < 1$ than does the solution of the initial value problem $y' = y^2$, $y(0) = 1$. Solve the latter problem by separation of variables and thus show that the solution of the original problem becomes infinite at a value of x not greater than 1. (The value is about 0.96981.)