Homework Exercises $\sharp 11$

1. Show that the function $f(x,y) = \ln\sqrt{x^2+y^2}$ satisfies a Laplace equation
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$
in the domain $D = \mathbb{R}^2 \backslash \{(0,0)\}$. (Such a function f is called a *harmonic function* in D.)

2. Show that the function $w = \sin(x+ct)$ is solution of the wave equation. *Hint:* If we stand in water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. If physics, this beautiful symmetry is expressed by the one-dimensional wave equation:
$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$
where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.

3. Express dw/dt in terms of t. Then evaluate the derivative at the given value of t.
$$w = \ln(x^2+y^2+z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}; \quad t = 3$$

4. Draw a tree diagram and write a chain rule formula for $\partial w/\partial x$ and $\partial w/\partial y$ if $w = g(u,v), \quad u = h(x,y), \quad v = k(x,y)$.

5. *Changing voltage in a circuit.* The voltage in a circuit that satisfies Ohm's law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance is increasing as the resistor heats up. Use chain rule to find how the current is changing at the instant when $R = 600\,\Omega$, $I = 0.04$ Amp, $dR/dt = 0.5\,\Omega$/sec, and $dV/dt = -0.01$ volts/sec.

6. Let $f(0,0) = 0$ and
$$f(x,y) = \frac{2xy}{x^2+y^2}, \qquad \text{if} \qquad (x,y) \neq (0,0).$$
Show that at every point of $\mathbb{R}^2$, $\partial f/\partial x$ and $\partial f/\partial y$ exist, but f is not continuous at $(0,0)$.

7. Find ∇f at the given point. Then sketch ∇f together with the level curve that passes through the point P_0.

$$f(x,y) = \ln(x^2 + y^2), \qquad P_0(1,1)$$

8. Find the derivative of f at P_0 in the direction of $\mathbf{A}$.

$$f(x,y,z) = 3e^x \cos yz, \qquad P_0(0,0,0), \qquad \mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

9. By about how much will

$$f(x,y,z) = 3e^x \cos yz$$

change as the point $P(x,y,z)$ moves from the origin at a distance of $ds = 0.1$ units in the direction of $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$?

10. In what two directions is the derivative of $f(x,y) = xy + y^2$ equal to zero at the point $(3,2)$? (Fig. 13.30)

11. Suppose that the derivative of $f(x,y,z)$ at a given point P is greatest in the direction of the vector $\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction the value of the derivative is $2\sqrt{3}$.

    (a) Find ∇f at P.

    (b) Find the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$.