Homework Exercises ♯9-♯10

1. Find the value of $(f \circ g)'$ when

$$f(u) = \left(\frac{u-1}{u+1}\right)^2, \quad u = g(x) = \frac{1}{x^2}, \quad x = -1.$$

2. Suppose that the radius of a soap bubble is 10 cm and increasing smoothly (that is, the radius is a differentiable function of time) at the rate of 1/2 cm/sec. How fast is the volume changing?

3. Assume that each of the equations below defines y as a differentiable function of x in a planar neighborhood of (that is, in a small enough disc around) the given point (x_0, y_0). Check that the point satisfies the equation, so it is really a point on the graph of the implicit function. Find the derivative dy/dx by implicit differentiation and then find the equation of the tangent to the curve at (x_0, y_0). (The tangent line gives the standard linear approximation to the function even when it is difficult or impossible to express y as an explicit elementary function of x.)

   (a) $x^3 - xy + y^3 = 1 \quad x_0 = 1, \quad y_0 = 1.$

   (b) $x^2 - y^2 + y^4 = 0 \quad x_0 = \sqrt{3}/4, \quad y_0 = \sqrt{3}/2;$
   also, $x_0 = \sqrt{3}/4, \quad y_0 = 1/2$. (See the figure-eight-shaped curve in Calculus 3.6, Exercise 41.)

   (c) $x + \sin y = \frac{3}{\pi}xy \quad x_0 = 2, \quad y_0 = \frac{\pi}{2}.$

4. Find the linearization $L(x)$ of the given function at the given point x_0. Use this approximation to evaluate the given quantity and compare the result to the value obtained by calculator; compute the error.

   (a) $f(x) = \sin x, \quad x_0 = 0, \quad \sin 0.1.$

   (b) $f(x) = \sqrt{1+x}, \quad x_0 = 0, \quad \sqrt{1.008}.$

   (c) $f(x) = (1+x)^{10}, \quad x_0 = 0, \quad 1.01^{10}.$

   (d) $f(x) = \frac{1}{1+x}, \quad x_0 = 0, \quad \frac{1}{1.02}.$

5. Assume the power rule for the function $f(x) = (1+x)^\alpha$ with any fixed real number α. Show that the linearization at $x_0 = 0$ is $L(x) = 1 + \alpha x$. Compare it to (b), (c), and (d) of the previous problem .

6. (a) About how accurately must the interior diameter of a 10-m-high cylindrical tank be measured to calculate the tank's volume to within 1% of its true value?

   (b) About how accurately must the tank's exterior diameter be measured to calculate the amount of the paint it will take to paint the side of the tank within 5% of the true amount?

7. (Using linearization to solve an equation) Let $f(x) = \sqrt{x} + \sqrt{1+x} - 4$.

   (a) Find $f(3)$ and $f(4)$ to show by the Intermediate Value Theorem that the equation $f(x) = 0$ has a solution in the interval $[3, 4]$.

   (b) To estimate the solution, replace the square roots by their linearization at $x_0 = 3$ and solve the resulting linear equation.

   (c) Check your estimate in the original equation.

8. Ohm's law states that $V = IR$, where V is the voltage, I is the current in amperes, and R is the resistance in ohms. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

9. Two commercial jets at 40,000 ft are flying at 520 mph along straight-line courses that cross at right angles. How fast is the distance between the planes closing when plane A is 5 mi from the intersection point and plane B is 12 mi from the intersection point?

10. Water runs into a conical tank with vertex downward, radius of the top circle is 5ft, height 10 ft, at the rate 9 ft^3/min. How fast is the level rising when the water is 6 ft deep?

11. Show that the following equations have exactly one solution in the given interval.

    (a) $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1.$

    (b) $2x - \cos x = 0, \quad -\pi \le x \le \pi.$

12. Use the Mean Value Theorem to show that for any real numbers a and b, $|\sin b - \sin a| \le |b - a|$.

13. Suppose that $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$. Using the Mean Value Theorem, give an upper and lower estimate for $f(0.1)$.

14. Show that the function $f(x) = x^4 + x - 3$ has a root between -2 and -1 and also between 1 and 2. Find estimates which are correct to two decimal places of both roots by Newton's method.

15. (Divergence with Newton's method) Apply Newton's method to $f(x) = \sqrt[3]{x}$ with $x_1 = 1$. Show that the resulting sequence $(x_n)_{n=1}^{\infty}$ is divergent. Draw a picture that shows what is going on.

16. (Very slow convergence with Newton's method) Try Newton's method on $f(x) = (x-1)^{40}$ with starting value $x_1 = 2$. Draw a picture that shows what is going on.

17. Find the domain, intercepts, asymptotes, intervals of rise and fall and concavity, and extrema of the following functions. Make a summary table (chart) and draw the graph.

    (a) $f(x) = x^3 - 6x^2 + 9x + 1$;

    (b) $f(x) = x - \cos x, \quad -\pi \leq x \leq \pi$;

    (c) $f(x) = \frac{x^2-4}{x^2-2}$;

    (d) $f(x) = \frac{x^2-x+1}{x-1}$.

18. What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3?

19. Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection. (Hint: A geometric proof is very simple, but try to give an analytic proof as well.)