Homework Exercises $\sharp 1$

In the exercises 1-4 try to use Boolean algebra instead of working with Venn diagrams or "inclusion" arguments.

1. Show that $\bar{\bar{A}} = A$ for any set A.
2. Show that $A \cap A = A$ for any set A.
3. Show De Morgan's second rule: $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
4. Simplify the expression $((A \setminus B) \cap (B \setminus C)) \cup (C \setminus A)$.
5. Show that the following implication is always true, for any logical statements A, B, and C:

   $$(A \wedge B) \vee (B \wedge C) \vee (C \wedge A) \Rightarrow A \vee B \vee C.$$

   (Here you may use truth tables or Boolean algebra or "common sense".)

   In the following two exercises determine the compositions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$ together with their domains and ranges.

6. $f(x) = -x$ and $g(x) = \sqrt{x}$.
7. $f(x) = \sin(x)$ and $g(x) = 1 + x^2$.
8. Show that the following relation between rational numbers is an equivalence relation on $\mathbb{Q}$:

   $$\frac{m}{n} \sim \frac{p}{q} \iff mq = pn.$$

9. Show that the following relation in $\mathbb{R}^2$ is a (complete) ordering (the so-called lexicographic ordering):

   $$(x, y) < (u, v) \iff (x < u) \vee (x = u \wedge y < v).$$

10. Show that the countable is the smallest infinite cardinality: any infinite subset of a countable set is also countable. (Hint: You only have to define a one-to-one mapping from the natural numbers onto this subset.)