

Formulas (to differential equations)

Math. A3, Midterm Test I.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)]$$

$$\sin x \sin y = -\frac{1}{2} [\cos(x+y) - \cos(x-y)]$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}, \quad \sinh^2 x = \frac{\cosh 2x - 1}{2}$$

derivatives:

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \ln(a)$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{ar sinh} x)' = \frac{1}{\sqrt{1+x^2}}$$

$$(\operatorname{ar cosh} x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\operatorname{ar tanh} x)' = \frac{1}{1-x^2}$$

$$(\operatorname{ar coth} x)' = \frac{1}{1-x^2}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

differentiation rules:

$$(cu)' = cu' \quad (c \text{ is constant})$$

$$(u+v)' = u' + v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}$$

integration rules:

$$\int cf \, dx = c \int f \, dx \quad (c \text{ is constant})$$

$$\int (f+g) \, dx = \int f \, dx + \int g \, dx$$

$$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + c,$$

where F is antiderivative of f

$$\int f(g(x))g'(x) \, dx = F(g(x)) + c,$$

where F is antiderivative of f

$$\int f^\alpha f' \, dx = \frac{f^{\alpha+1}}{\alpha+1} + c, \text{ if } \alpha \neq -1$$

$$\int \frac{f'}{f} \, dx = \ln |f| + c$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

notable substitutions:

$$R(e^x) \quad e^x = t$$

$$R(\sqrt{ax+b}) \quad \sqrt{ax+b} = t$$

$$R\left(\frac{\sqrt{ax+b}}{\sqrt{cx+d}}\right) \quad \frac{\sqrt{ax+b}}{\sqrt{cx+d}} = t$$

$$R(\sin x, \cos x) \quad \sin x, \cos x, \tan x, \tan \frac{x}{2} = t$$

$$R(x, \sqrt{a^2 - x^2}) \quad x = a \sin t, \quad x = a \cos t$$

$$R(x, \sqrt{a^2 + x^2}) \quad x = a \sinh t$$

$$R(x, \sqrt{x^2 - a^2}) \quad x = a \cosh t$$

antiderivatives:

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$$

$$\int \frac{1}{x} \, dx = \ln |x| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar sinh} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{ar cosh} \frac{x}{a} + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar tanh} \frac{x}{a} + c, \quad \text{if } \left|\frac{x}{a}\right| < 1$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar coth} \frac{x}{a} + c, \quad \text{if } \left|\frac{x}{a}\right| > 1$$

$$\int \tan x \, dx = -\ln |\cos x| + c$$

$$\int \cot x \, dx = \ln |\sin x| + c$$

1.

$$e^{it} = \cos t + i \cdot \sin t, \quad t \in \mathbb{R}.$$

2.

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

its characteristic equation:

$$ar^2 + br + c = 0.$$

3. **Method of Undetermined Coefficients:** If in the equation

$$ay'' + by' + cy = g(t), \quad a \neq 0 \text{ és } t \in I$$

the right-hand side function $g(t)$ has the form

$$g(t) = e^{ut} (A_n(t) \cos(vt) + B_m(t) \sin(vt)),$$

where $A_n(t), B_m(t)$ are polynomials of degree n and m respectively, then the particular solution of the inhomogeneous equation has the form:

$$y_{i,p} = t^s e^{ut} (P_k(t) \cos(vt) + Q_k(t) \sin(vt)),$$

where s is the multiplicity of the root $u + i \cdot v$ among the roots of the characteristic equation; further, $P_k(t)$ and $Q_k(t)$ are polynomials of degree $k = \max(n, m)$.

4. **Variation of Parameters Method:** Consider the inhomogeneous d.e.

$$y'' + p(t)y' + q(t)y = g(t) \quad t \in I$$

and its homogeneous part $Y'' + p(t)Y' + q(t)Y = 0$. If the y_1, y_2 pair is a fundamental solution of the homogeneous d.e., then a particular solution of the inhomogeneous equation is looked for in the form $y_{i,p} = C_1(t) \cdot y_1(t) + C_2(t) \cdot y_2(t)$, where for the derivatives of the unknown functions $C_1(t), C_2(t)$ the following system of equations holds:

$$\begin{aligned} C_1'(t)y_1(t) + C_2'(t)y_2(t) &= 0 \\ C_1'(t)y_1'(t) + C_2'(t)y_2'(t) &= g(t) \end{aligned}$$

5. **Special second order d.e.'s:**

If y is missing, then substitute $p(x) := y'(x)$.

If x is missing, then substitute $q(y) := y'$

6. The first order d.e. $M(x, y)dx + N(x, y)dy = 0$ is **exact**, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To solve the d.e., a function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ has to be found such that $\text{grad}F = (M, N)$. Then the solution of the d.e. is:

$$F(x, y) = \text{Const.}$$