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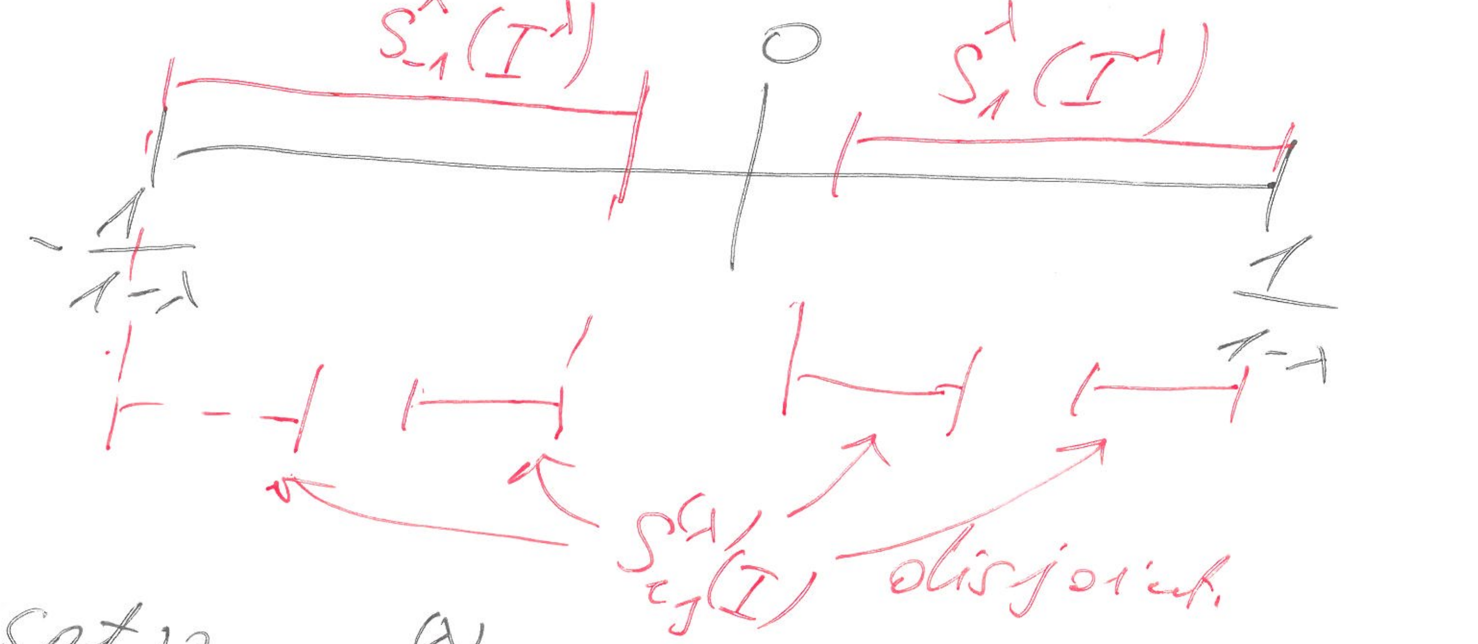
$Y_\lambda := \sum_{n=0}^{\infty} \pm \lambda^n$, $0<\lambda<1$; $+,-$ sign chosen with probability $1/2$, $\frac{1}{2}$ independently, let $\nu_\lambda(E) := \mathbb{P}(Y_\lambda \in E)$, the distribution of Y_λ.

$S_i^{(\lambda)}(x) := \lambda x + i$, $i \in \{-1, 1\}$, and $p = \{\frac{1}{2}, \frac{1}{2}\}$ then ν_λ is the self similar measure. If $\lambda \in (0, \frac{1}{2})$ then for $I^\lambda := \left[-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right]$ we have

$S_{-1}^{(\lambda)}(I^\lambda) \cap S_1^{(\lambda)}(I^\lambda) = \emptyset$.



$$\Lambda^{(\lambda)} := \bigcap_{n=1}^{\infty} \bigcup_{|\underline{i}|=n} S_{\underline{i}}^{(\lambda)}(I^\lambda)$$

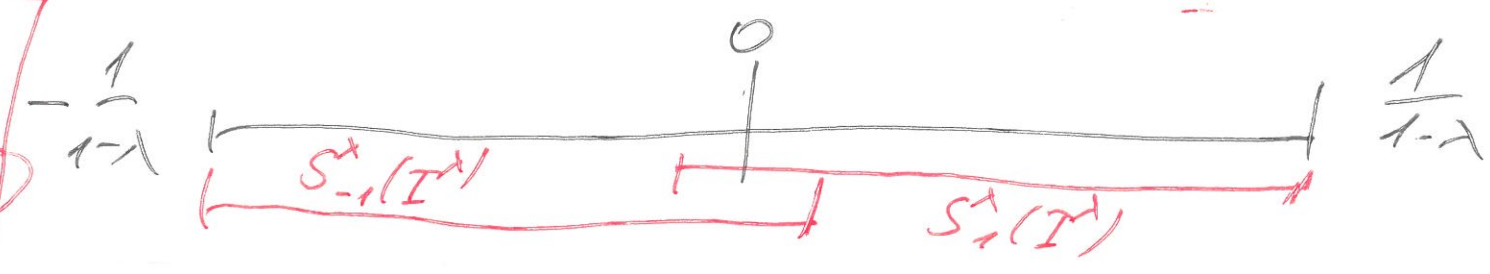
then Λ^λ is a Cantor set and $\operatorname{spt} \nu_\lambda = \Lambda^{(\lambda)}$ so $\nu_\lambda \perp$ Leb.

If $\lambda = \frac{1}{2}$ $\Lambda^{(\lambda)} = I_\lambda$. and ν_λ is a scaled copy of Leb$|_{I}$.

The interesting case is when $\lambda \in (\frac{1}{2}, 1)$. Then

$\operatorname{spt} \nu_\lambda = I^\lambda$.

$\boxed{\text{Problem: Asked by P. Erdős in 1930's: Is it true that } \nu_\lambda \ll \text{Leb for Leb a.e. } \lambda \in (\tfrac{1}{2}, 1)?}$



2

Thm Erdős: (a) If λ^{-1} is a PV number ($\lambda > 1$ but its conjugates are < 1 in modulus) then $\nu_\lambda \perp \text{Leb}$.

(b) $\exists \delta > 0$ Leb. a.e. $\lambda \in (1-\delta, 1)$ $\nu_\lambda \ll \text{Leb}$.

Thm Solomyak: $\nu_\lambda \ll \text{Leb}$ for Leb. a.e. $\lambda \in (\frac{1}{2}, 1)$

Here we present a simplified proof by Yuval Peres & Boris Solomyak.

Proof: We work on $\mathbb{R}$ so $B(x,r) = [x-r, x+r]$.

$$\underline{D}(\nu_\lambda, \text{Leb}, x) = \varliminf_{r \downarrow 0} \frac{\nu_\lambda(B(x,r))}{2r}.$$ We learned that

$$\nu_\lambda \ll \text{Leb} \iff \underline{D}(\nu_\lambda, \text{Leb}, x) < \infty \text{ for } \nu_\lambda\text{-a.e. } x.$$

So in order to prove that $\nu_\lambda \ll \text{Leb}$ for Leb. a.e. $\lambda \in J \subset (\frac{1}{2}, 1)$ it is enough to prove that

$$(*) \quad S := \int_{\lambda \in J} \int_{x \in \mathbb{R}} \underline{D}(\nu_\lambda, \text{Leb}, x) \, d\nu_\lambda(x) \, d\lambda < \infty$$

J will be specified later. $J = [\lambda_0, \lambda_1]$

Fatou's Lemma

$$S = \int_{\lambda \in J} \int_{x \in \mathbb{R}} \varliminf_{r \to 0} \frac{\nu_\lambda(B(x,r))}{2r} \, d\nu_\lambda(x) \, d\lambda \leq \varliminf_{r \downarrow 0} \frac{1}{2r} \int_{\lambda \in J} \int_{x \in \mathbb{R}} \nu_\lambda(B(x,r)) \, d\nu_\lambda(x) \, d\lambda.$$

$\Pi^\lambda(\underline{i}) := \lim_{n \to \infty} S_{i_1 \dots i_n}(0)$, $\underline{i} = (i_1, i_2, \dots) \in \{-1, 1\}^{\mathbb{N}} =: \Sigma$

$\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ on Σ. $\boxed{\nu_\lambda = \Pi^\lambda_* \mu}$

Change of variables formula

$$= \varliminf_{r \downarrow 0} \frac{1}{2r} \int_{\lambda \in J} \int_{\underline{i} \in \Sigma} \boxed{\nu_\lambda(B(\Pi^\lambda(\underline{i}), r))} \, d\mu(\underline{i}) \, d\lambda$$

$$\int_{\underline{j} \in \Sigma} \mathbb{1}_{B(\Pi^\lambda(\underline{i}), r)}(\Pi^\lambda(\underline{j})) \, d\mu(\underline{j})$$

$$S \le \lim_{r\downarrow 0} \frac{1}{2r} \int_{\lambda\in J} \int_{\underline{i}\in\Sigma_1} \int_{\underline{j}\in\Sigma_1} 1\!\!1_{B(\Pi^{1}(\underline{i}),r)}(\underline{j})\, d\mu(\underline{j})\, d\mu(\underline{i})\, d\lambda \quad ; \qquad \int 1\!\!1_{B(\Pi^1(\underline{i}),r)}(\underline{j})\, d\lambda = \mathrm{Leb}\{\lambda\in J : |\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})| \le r\}$$

$$S \le \lim_{r\downarrow 0} \frac{1}{2r} \int_{\underline{i}\in\Sigma_1} \int_{\underline{j}\in\Sigma_1} \mathrm{Leb}\{\lambda\in J : |\underbrace{\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})}_{\phi_{\underline{i},\underline{j}}(\lambda)}| < r\}\, d\mu(\underline{j})\, d\mu(\underline{i}).$$

$$\phi_{\underline{i},\underline{j}}(\lambda) := \Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j}) = \sum_{n=0}^{\infty} (i_n - j_n)\lambda^n$$

$i_n - j_n \in \{-2, 0, 2\}$.

$k = k(\underline{i},\underline{j}) := |\underline{i} \wedge \underline{j}| = \min\{n : i_n \ne j_n\}$.

$$\boxed{\phi_{\underline{i},\underline{j}}(\lambda) = 2\cdot\lambda^k g(\lambda),} \qquad g(\lambda) = 1 + \sum_{n=1}^{\infty} b_n \lambda^n, \quad b_n \in \{-1,0,1\} \quad (**)$$

We may assume that $\boxed{j_k \le i_k}$

Let $\mathcal{G} := \{g \text{ of the form } (**)\}$.

To estimate this integral we assume the

δ-transversality condition for some $\delta > 0$:

$$\forall g\in\mathcal{G}\ \forall x\in J, \ \boxed{\text{if } g(x) < \delta \Rightarrow g'(x) < -\delta}$$

$$\Downarrow$$

$$\forall \rho > 0 \quad \mathrm{Leb}\{\lambda\in J : |g(\lambda)| \le \rho\} \le 2\cdot\delta^{-1}\rho \quad (***)$$

$$\Downarrow \ \boxed{\rho = \lambda_0^{-k} r/2}$$

$|\phi_{\underline{i},\underline{j}}(\lambda)| \le r \Rightarrow |g(\lambda)| \le \lambda_0^{-k}\cdot r/2$; $\lambda\in J = [\lambda_0, \lambda_1]$

$:\mathrm{Leb}\{\lambda\in J : |\phi_{\underline{i},\underline{j}}(\lambda)| \le r\} \le \delta^{-1}\lambda_0^{-k} r.$ So

$$S \le \lim_{r\downarrow 0} \frac{1}{2r} \int_{\underline{i}\in\Sigma} \int_{\underline{j}\in\Sigma} \overbrace{\mathrm{Leb}\{\lambda\in J : |\phi_{\underline{i},\underline{j}}(\lambda)| < r\}}^{\delta^{-1}\lambda_0^{-k} r}\, d\mu(\underline{j})\, d\mu(\underline{i}) \le$$

$$S \le \delta^{-1} \lim_{r \downarrow 0} \frac{1}{2r} \int_{\underline{i} \in \Sigma_1} \int_{\underline{j} \in \Sigma_1} \lambda_0^{-|\underline{i} \wedge \underline{j}|} \cdot r \, d\mu(\underline{j})\, d\mu(\underline{i}) = (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} (\mu \times \mu)\{(\underline{i},\underline{j}) : |\underline{i} \wedge \underline{j}| = k\}$$ 4

$$= (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} \cdot 2^{-k-1} < \infty.$$

$\lambda_0 > \frac{1}{2}$; $\boxed{2\lambda_0 > 1}$ $\square$

That is δ transversality implies absolute continuity.

Establishing transversality:

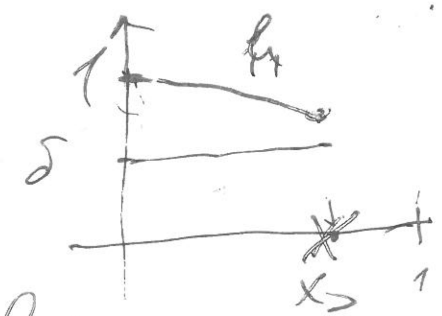
Definition: $h(x)$ is a ⊛ function if $\exists\ k \ge 1$ and $a_k \in [-1, 1]$ s.t.

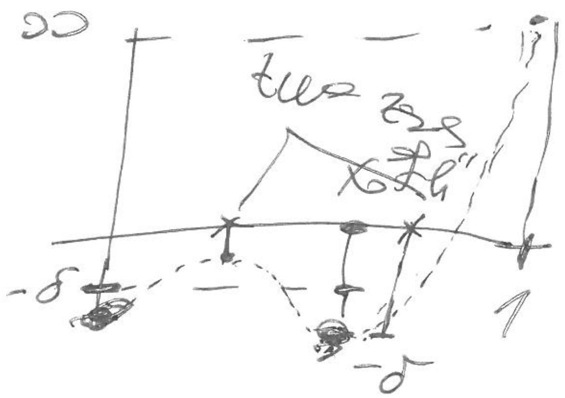
$$\boxed{h(x) = 1 - \sum_{i=1}^{k-1} x^i + a_k x^k + \sum_{i=k+1}^{\infty} x^i}$$

Lemma: Assume the h is a ⊛-function & $\boxed{\exists\, \delta \in (0,1] \text{ and } x_0 \in (0,1) \text{ s.t.}}$ $h(x_0) > \delta$ and $h'(x_0) < -\delta$.
Then the transversality condition holds.

proof: Observe that $\forall\ x \in [0, x_0)$: $h(x) > \delta$ & $h'(x) < -\delta$.
Namely h'' has at most one zero on $(0,1)$ since it has at most one coefficient sign change. So, by Descarte rule of sign change h'' has at most one zero on $(0,1)$. $h'(0) = -1$ if $k > 1$ & $h'(0) = a_1$ if $k = 1$, $h'(0) < -\delta$ $\left(\text{if } k=1 \quad h'(0) = a_1 < \underbrace{a_1 + \sum_{i=2}^{\infty} i x_0^{i-1}}_{h'(x_0) < -\delta} \text{ and} \right)$

Using that $h'(1-) = \infty$, $\forall x \in (0, x_0)$: $h'(x) < -\delta$ otherwise h'' has ≥ 2 zeros.





So h is decreasing on $(0, x_0)$ and $h(x_0) > \delta \Rightarrow h(x) \geq h(x_0) > \delta$.

Let $f(x) = g(x) - h(x)$, where $g \in \mathcal{G}$. Then

$$f(x) = \sum_{i=1}^{l} c_i x^i - \sum_{i=l+1}^{\infty} c_i x^i, \quad \text{where } c_i \geq 0, \; l = \begin{cases} k-1 \\ k \end{cases}$$

$-h' > \delta$

$0 > f' = g' - h' > g' + \delta \Rightarrow \boxed{g' < -\delta}$

$\forall x \in [0, x_0]$: $\quad g(x) < \delta \Rightarrow f(x) < 0 \Rightarrow f'(x) < 0 \Rightarrow g'(x) < -\delta$

$$f(x) < 0 \Rightarrow \sum_{i=1}^{l} c_i x^i < \sum_{i=l+1}^{\infty} c_i x^i \Rightarrow \sum_{i=1}^{l} c_i \cdot i \cdot x^{i-1} < \sum_{i=l+1}^{\infty} c_i \cdot i \cdot x^{i-1} \Rightarrow f'(x) < 0$$

So δ transversality holds for every $g \in \mathcal{G}$.

First we establish δ-transversality on $[\frac{1}{2}, 2^{-2/3}]$.

$$h(x) = 1 - x - x^2 - x^3 + 0.5x^4 + \sum_{i=5}^{\infty} x^i \qquad h(2^{-2/3}) > 0.07$$

$$h'(2^{-2/3}) < -0.09$$

So δ transversality holds on $[0, 2^{-2/3}]$, the argument above works for $\lambda_0 > \frac{1}{2}$. So $\nu_\lambda \ll$ Leb for Leb a.e. $\lambda \in (\frac{1}{2}, 2^{-2/3})$.

There is power series with double zero around 0.68!

Sketch of the rest of the proof: $Z_\lambda := \sum_{i \neq 2+3j} \pm \lambda^i$. $\tilde{\mu}$: on $\{-1, 0, 1\}$ s.t. every $2+3j$ th term is zero and the others are ± 1 with $\frac{1}{2}, \frac{1}{2}$ prob.

$\underline{i} = [i_1, i_2, 0, i_4, i_5, 0, i_7, i_8, 0]$, $\mu[\underline{i}] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

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$$\hat{\nu}_\lambda(\xi) = \int_{-\infty}^{\infty} e^{it\xi}\, d\nu_\lambda(t); \quad \hat{\nu}_\lambda(\xi) = \prod_{n=0}^{\infty} \cos(\lambda^n \xi); \quad \boxed{\hat{\nu}_\lambda(\xi) = \hat{\nu}_{\lambda^2}(\xi) \cdot \hat{\nu}_{\lambda^2}(\lambda\xi)}$$

abs. cont.

assume that $\nu_\lambda \ll \text{Leb}$ for $\lambda \in [2^{-1}, 2^{-\frac{1}{2}}]$

then for $\lambda \in [2^{-\frac{1}{2}}, 2^{-\frac{1}{4}}]$ we have $\lambda^2 \in [2^{-1}, 2^{-\frac{1}{2}}]$ so $\boxed{\nu_{\lambda^2}(\xi) \ll \text{Leb}}$ ν_λ

$$Z_\lambda = \sum_{n=0}^{\infty} \pm \lambda^{2n}; \quad T_\lambda = \lambda \sum_{n=0}^{\infty} \pm \lambda^{2n} = \lambda \cdot Z_\lambda$$

$Z_\lambda + T_\lambda$

distr.

ν_{λ^2}

ξ_λ

$\nu_{\lambda^2} * \xi_\lambda$

Fourier tr.

$\hat{\nu}_{\lambda^2}(\xi)$

Fourier tr.

$\hat{\xi}_\lambda = \hat{\nu}_{\lambda^2}(\lambda \cdot \xi)$

Fourier tr.

$\boxed{\hat{\nu}_{\lambda^2}(\xi) \cdot \hat{\nu}_{\lambda^2}(\lambda\xi)}$