VISIBILITY FOR SELF-SIMILAR SETS OF DIMENSION ONE IN THE PLANE

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Abstract. We prove that a purely unrectifiable self-similar set of finite 1-dimensional Hausdorff measure in the plane, satisfying the Open Set Condition, has radial projection of zero length from every point.

1. Introduction

For $a \in \mathbb{R}^2$, let P_a be the radial projection from a:

$$P_a : \mathbb{R}^2 \setminus \{a\} \longrightarrow S^1, \quad P_a(x) = \frac{(x-a)}{|x-a|}.$$

A special case of our theorem asserts that the "four corner Cantor set" of

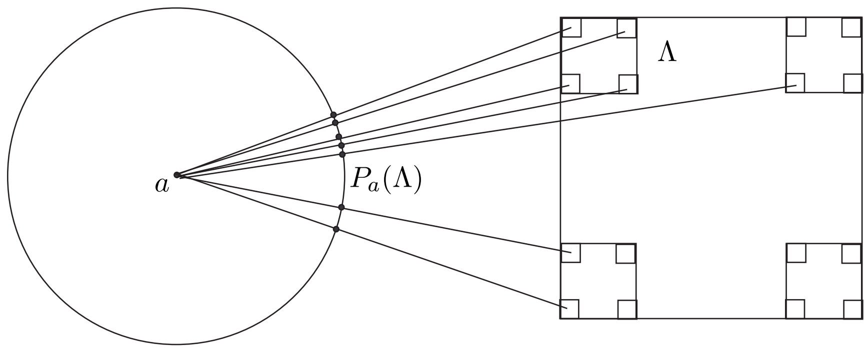

Figure 1. The radial projection of the four corner set

contraction ratio 1/4 has radial projection of zero length from all points $a \in \mathbb{R}^2$.

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See Figure 1 where we show the second-level approximation of the four corner Cantor set and the radial projection of some of its points.

Denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure. A Borel set Λ is a 1-*set* if $0 < \mathcal{H}^1(\Lambda) < \infty$. It is said to be *invisible from* a if $P_a(\Lambda \setminus \{a\})$ has zero length.

Theorem 1.1. *Let Λ be a self-similar 1-set in $\mathbb{R}^2$ satisfying the Open Set Condition, which is not on a line. Then Λ is invisible from every $a \in \mathbb{R}^2$.*

Recall that a nonempty compact Λ is self-similar if $\Lambda = \bigcup_{i=1}^m S_i(\Lambda)$ for some contracting similitudes S_i. This means that

$$S_i(x) = \lambda_i \mathcal{O}_i x + b_i,$$

where $0 < \lambda_i < 1$, $\mathcal{O}_i$ is an orthogonal transformation of the plane, and $b_i \in \mathbb{R}^2$. The Open Set Condition holds if there exists an open set $V \neq \emptyset$ such that $S_i(V) \subset V$ for all i and $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$. For a self-similar set satisfying the Open Set Condition, being a 1-set is equivalent to $\sum_{i=1}^m \lambda_i = 1$.

A Borel set Λ is *purely unrectifiable* (or *irregular*), if $\mathcal{H}^1(\Lambda \cap \Gamma) = 0$ for every rectifiable curve Γ. A set Λ satisfying the assumptions of Theorem 1.1 is purely unrectifiable by Hutchinson [5] (see also [8]). A classical theorem of Besicovitch [2] (see also [4, Theorem 6.13]) says that a purely unrectifiable 1-set has orthogonal projections of zero length on almost every line through the origin. We use it in our proof.

In [10, Problem 12] (see also [9, 10.12]) Mattila raised the following question: Let Λ be a Borel set in $\mathbb{R}^2$ with $\mathcal{H}^1(\Lambda) < \infty$. Is it true that for $\mathcal{H}^1$ almost all $a \in A$, the intersection $\Lambda \cap L$ is a finite set for almost all lines L through a? If Λ is purely unrectifiable, is it true that $\Lambda \cap L = \{a\}$ for almost all lines through a? Our theorem implies a positive answer for a purely unrectifiable self-similar 1-set Λ satisfying the Open Set Condition. The general case of a purely unrectifiable set remains open. On the other hand, M. Csörnyei and D. Preiss proved recently that the answer to the first part of the question is negative [personal communication].

Note that we prove a stronger property for our class of sets, namely, that the set is invisible from *every* point $a \in \mathbb{R}^2$. It is easy to construct examples of non-self-similar purely unrectifiable 1-sets for which this property fails. Marstrand [6] has an example of a purely unrectifiable 1-set which is visible from a set of dimension one.

We do not discuss here other results and problems related to visibility; see [9, Section 6] for a recent survey. We only mention a result of Mattila [7, Th.5.1]: if a set Λ has projections of zero length on almost every line (which could have $\mathcal{H}^1(\Lambda)=\infty$), then the set of points Ξ from which Λ is visible is a purely unrectifiable set of zero 1-capacity. A different proof of this and a characterization of such sets Ξ is due to Csörnyei [3].

2. Preliminaries

We have $S_i(x) := \lambda_i \mathcal{O}_i x + b_i$, where $0<\lambda_i<1$,

$$\mathcal{O}_i = \begin{bmatrix} \cos(\varphi_i) & -\varepsilon_i \sin(\varphi_i) \\ \sin(\varphi_i) & \varepsilon_i \cos(\varphi_i) \end{bmatrix},$$

$\varphi_i \in [0,2\pi)$, and $\varepsilon_i \in \{-1,1\}$ shows whether $\mathcal{O}_i$ is a rotation through the angle φ_i or a reflection about the line through the origin making the angle $\varphi_i/2$ with the x-axis.

Let $\Sigma := \{1,\dots,m\}^{\mathbb{N}}$ be the symbolic space. The natural projection $\Pi : \Sigma \to \Lambda$ is defined by

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} S_{i_1\dots i_n}(x_0), \quad \text{where} \;\; \mathbf{i} = (i_1 i_2 i_3 \dots) \in \Sigma, \tag{1}$$

and $S_{i_1\dots i_n} = S_{i_1}\circ\dots\circ S_{i_n}$. The limit in (1) exists and does not depend on x_0. Denote $\lambda_{i_1\dots i_n} = \lambda_{i_1}\cdots\lambda_{i_n}$ and $\varepsilon_{i_1\dots i_k} = \varepsilon_{i_1}\cdots\varepsilon_{i_k}$. We can write

$$S_{i_1\dots i_n}(x) = \lambda_{i_1\dots i_n}\mathcal{O}_{i_1\dots i_n}x + b_{i_1\dots i_n},$$

where

$$\mathcal{O}_{i_1\dots i_n} := \mathcal{O}_{i_1}\circ\dots\circ\mathcal{O}_{i_n} = \begin{bmatrix} \cos(\varphi_{i_1\dots i_n}) & -\varepsilon_{i_1\dots i_n}\sin(\varphi_{i_1\dots i_n}) \\ \sin(\varphi_{i_1\dots i_n}) & \varepsilon_{i_1\dots i_n}\cos(\varphi_{i_1\dots i_n}) \end{bmatrix},$$

$$\varphi_{i_1\dots i_n} := \varphi_{i_1} + \varepsilon_{i_1}\varphi_{i_2} + \varepsilon_{i_1 i_2}\varphi_{i_3} + \dots + \varepsilon_{i_1\dots i_{n-1}}\varphi_{i_n},$$

and

$$b_{i_1\dots i_n} = b_{i_1} + \lambda_{i_1}\mathcal{O}_{i_1}b_{i_2} + \dots + \lambda_{i_1\dots i_{n-1}}\mathcal{O}_{i_1\dots i_{n-1}}b_{i_n}.$$

Since $\sum_{i=1}^m \lambda_i = 1$, we can consider the probability product measure $\mu = (\lambda_1,\dots,\lambda_m)^{\mathbb{N}}$ on the symbolic space Σ and define the *natural measure* on Λ:

$$\nu = \mu\circ\Pi^{-1}.$$

By a result of Hutchinson [5, Theorem 5.3.1(iii)], as a consequence of the Open Set Condition we have

$$\nu = c\mathcal{H}^1|_\Lambda, \quad \text{where} \;\; c = (\mathcal{H}^1(\Lambda))^{-1}. \tag{2}$$

To $\theta \in [0,\pi)$ we associate the unit vector $e_\theta = (\cos\theta, \sin\theta)$, the line $L_\theta = \{te_\theta : t \in \mathbb{R}\}$, and the orthogonal projection onto L_θ given by $x \mapsto (e_\theta \cdot x)e_\theta$. It is more convenient to work with the signed distance of the projection to the origin, which we denote by p_θ:

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}, \quad p_\theta x = e_\theta \cdot x.$$

Denote $\mathcal{A} := \{1,\dots,m\}$ and let $\mathcal{A}^* = \bigcup_{i=1}^\infty \mathcal{A}^i$ be the set of all finite words over the alphabet $\mathcal{A}$. For $u = u_1 \dots u_k \in \mathcal{A}^k$ we define the corresponding "symbolic" cylinder set by

$$[u] = [u_1 \dots u_k] := \{\mathbf{i} \in \Sigma : \; i_\ell = u_\ell, \, 1 \le \ell \le k\}.$$

We also let

$$\Lambda_u = S_u(\Lambda) = \lambda_u \mathcal{O}_u \Lambda + b_u$$

and call Λ_u the cylinder set of Λ corresponding to the word u. Let d_Λ be the diameter of Λ; then $\operatorname{diam}(\Lambda_u) = \lambda_u d_\Lambda$. For $\rho > 0$ consider the "cut-set"

$$\mathcal{W}(\rho) = \{u \in \mathcal{A}^* : \; \lambda_u \le \rho, \; \lambda_{u'} > \rho\}$$

where u' is obtained from u by deleting the last symbol. Observe that for every $\rho > 0$,

$$\Lambda = \bigcup_{u \in \mathcal{W}(\rho)} \Lambda_u. \tag{3}$$

In view of (2), we have $\nu(\Lambda_u \cap \Lambda_v) = 0$ for distinct $u, v \in \mathcal{W}(\rho)$, hence

$$\nu(\Lambda_u) = \lambda_u \quad \text{for all } u \in \mathcal{A}^*.$$

Denote $\lambda_{\min} := \min\{\lambda_i : \; i \le m\}$; then $\mu(\Lambda_u) = \lambda_u \in (\rho\lambda_{\min}, \rho]$ for $u \in \mathcal{W}(\rho)$.

We identify the unit circle S^1 with $[0, 2\pi)$ and use additive notation $\theta_1 + \theta_2$ understood mod 2π for points on the circle. For a Radon measure η on the line or on S^1, the upper density of η with respect to $\mathcal{H}^1$ is defined by

$$\overline{D}(\eta, t) = \limsup_{r \to 0} \frac{\eta([t-r, t+r])}{2r}.$$

The open ball of radius r centered at x is denoted by $B(x, r)$.

3. Proof of the main theorem

In the proof of Theorem 1.1 we can assume, without loss of generality, that $a \notin \Lambda$, and

$$P_a(\Lambda) \text{ is contained in an arc of length less than } \pi. \tag{4}$$

Indeed, $\Lambda \setminus \{a\}$ can be written as a countable union of self-similar sets Λ_u for $u \in \mathcal{A}^*$, of arbitrarily small diameter. If each of them is invisible from a, then Λ is invisible from a.

Let

$$\Omega := \{\mathbf{i} \in \Sigma : \; \forall\, u \in \mathcal{A}^* \; \exists\, n \text{ such that } \sigma^n \mathbf{i} \in [u]\},$$

that is, Ω is the set of sequences which contain each finite word over the alphabet $\mathcal{A} = \{1, \dots, m\}$. It is clear that every $\mathbf{i} \in \Omega$ contains each finite word infinitely many times and $\mu(\Sigma \setminus \Omega) = 0$.

Lemma 3.1 (Recurrence Lemma)**.** *For every $\mathbf{i} \in \Omega$, $\delta > 0$, and $j_1, \dots, j_k \in \{1, \dots, m\}$, there are infinitely many $n \in \mathbb{N}$ such that*

$$\phi_{i_1 \dots i_n} \in [0, \delta], \;\; \varepsilon_{i_1 \dots, i_n} = 1, \quad \textit{and} \;\; \sigma^n \mathbf{i} \in [j_1 \dots j_k]. \tag{5}$$

If the similitudes have no rotations or reflections, that is, $\phi_i = 0$ and $\varepsilon_i = 1$ for all $i \le m$ (as in the case of the four corner Cantor set), then the conditions on ϕ and ε in (5) hold automatically and the lemma is true by the definition of Ω. The proof in the general case is not difficult, but requires a detailed case analysis, so we postpone it to the next section.

Let

$$\Theta := \{\theta \in [0, \pi) : \; \mathcal{H}^1(p_\theta(\Lambda)) = 0\} \quad \text{and} \quad \Theta' := (\Theta + \pi/2) \cup (\Theta + 3\pi/2)$$

(recall that addition is considered mod 2π). Since Λ is purely unrectifiable, $\mathcal{H}^1([0, \pi) \setminus \Theta') = 0$ by Besicovitch's Theorem [2]. The following proposition is the key step of the proof. We need the following measures:

$$\nu_a := \nu \circ P_a^{-1} \quad \text{and} \quad \nu_\theta := \nu \circ p_\theta^{-1}, \; \theta \in [0, \pi).$$

We also denote $\Lambda' = \Pi(\Omega)$.

Proposition 3.2. *If $\theta' \in P_a(\Lambda') \cap \Theta'$, then $\overline{D}(\nu_a, \theta') = \infty$.*

Proof of Theorem 1.1 assuming Proposition 3.2. By Proposition 3.2 and [9, Lemma 2.13] (a corollary of the Vitali covering theorem), we obtain that $\mathcal{H}^1(P_a(\Lambda')\cap\Theta')=0$. As noted above, Θ' has full $\mathcal{H}^1$ measure in S^1. On the other hand,

$$\mu(\Sigma\setminus\Omega)=0 \;\Rightarrow\; \nu(\Lambda\setminus\Lambda')=0 \;\Rightarrow\; \mathcal{H}^1(\Lambda\setminus\Lambda')=0 \;\Rightarrow\; \mathcal{H}^1(P_a(\Lambda\setminus\Lambda'))=0,$$

and we conclude that $\mathcal{H}^1(P_a(\Lambda))=0$, as desired. □

Proof of Proposition 3.2. Let $x\in\Lambda'$ and $\theta'=P_a(x)\in\Theta'$. Let $\theta:=\theta'-\pi/2$ mod $[0,\pi)$. By the definition of Θ' we have $\mathcal{H}^1(p_\theta(\Lambda))=0$.

First we sketch the idea of the proof. Since $\mathcal{H}^1(p_\theta(\Lambda))=0$, we have $\nu_\theta\perp\mathcal{H}^1$, and this implies that for every $N\in\mathbb{N}$ there exist N cylinders of Λ approximately the same diameter (say, $\sim r$), such that their projections to L_θ are r-close to each other. Then there is a line parallel to the segment $[a,x]$, whose Cr-neighborhood contains all Λ_{u_j}, $j=1,\ldots,N$. By the definition of $\Lambda'=\Pi(\Omega)$, we can find similar copies of this picture near $x\in\Lambda'$ at arbitrarily small scales. The Recurrence Lemma 3.1 guarantees that these copies can be chosen with a small relative rotation. This will give N cylinders of Λ of diameter $\sim r_0 r$ contained in a $C'r_0r$-neighborhood of the ray obtained by extending $[a,x]$. Since a is assumed to be separated from Λ, we will conclude that $\overline{D}(\nu_a,\theta')\geq C''N$, and the proposition will follow. Now we make this precise. The proof is illustrated in Figure 2.

Claim. *For each $N\in\mathbb{N}$ there exists $r>0$ and distinct $u^{(1)},\ldots,u^{(N)}\in\mathcal{W}(r)$ such that*

$$|p_\theta(b_{u^{(j)}}-b_{u^{(i)}})|\leq r,\quad \forall\, i,j\leq N. \tag{6}$$

Indeed, for every $u\in\mathcal{A}^*$,

$$\Lambda_u=\lambda_u\mathcal{O}_u\Lambda+b_u \;\Rightarrow\; \Lambda_u\subset B(b_u,d_\Lambda\lambda_u),$$

hence for every interval $I\subset\mathbb{R}$ and $r>0$,

$$\nu_\theta(I)\leq\sum_{u\in\mathcal{W}(r)}\{\lambda_u:\ \mathrm{dist}(p_\theta(b_u),I)\leq d_\Lambda r\}.$$

If the claim does not hold, then there exists $N\in\mathbb{N}$ such that for every $t\in\mathbb{R}$ and $r>0$,

$$\nu_\theta([t-r,t+r])\leq N(2(1+d_\Lambda)+1)r.$$

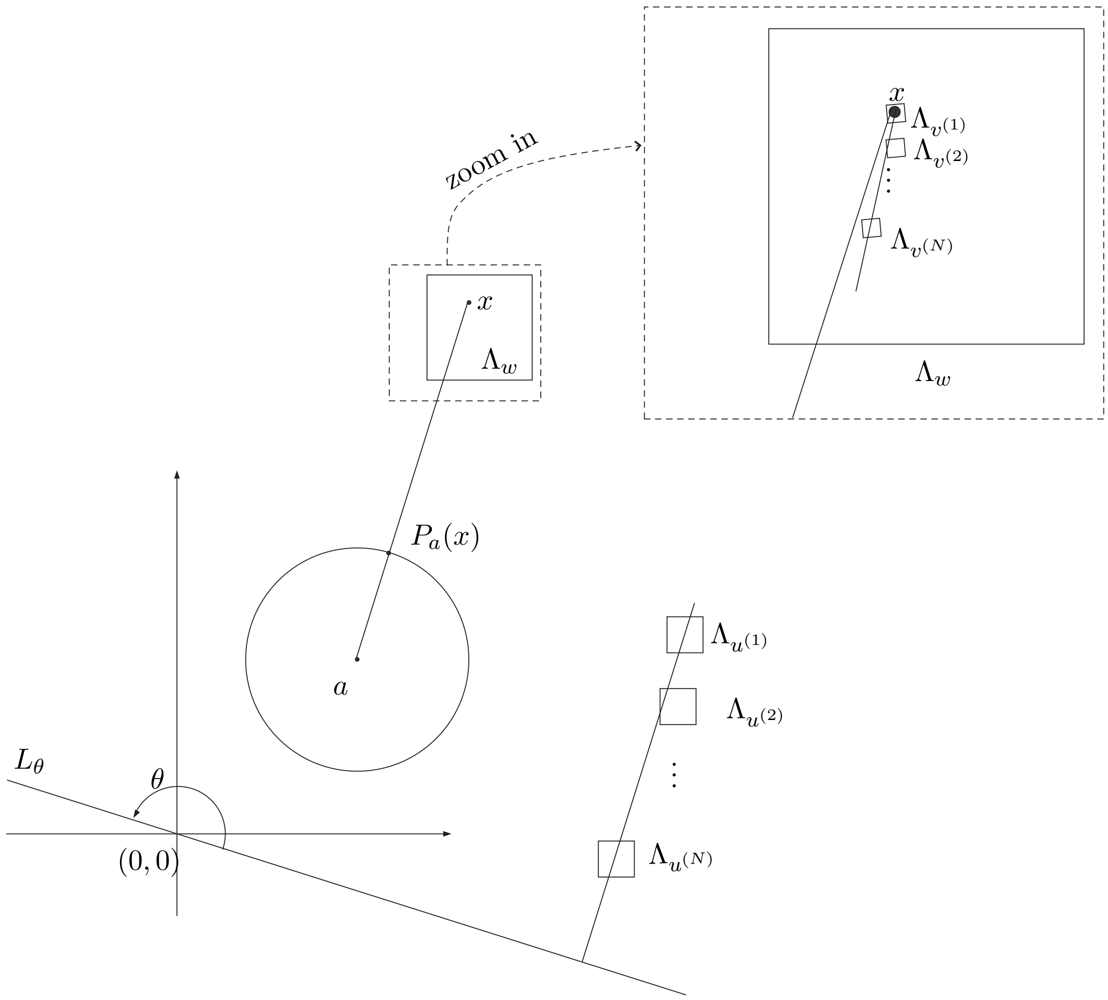

FIGURE 2. The cylinders of Λ causing high density

Then ν_θ is absolutely continuous with respect to $\mathcal{H}^1$, which is a contradiction. The claim is verified. □

We are given that $x \in \Lambda' = \Pi(\Omega)$, which means that $x = \pi(\mathbf{i})$ for an infinite sequence $\mathbf{i}$ containing all finite words. We fix $N \in \mathbb{N}$ and find $r > 0$, $u^{(1)}, \ldots, u^{(N)} \in \mathcal{W}(r)$ from the Claim. Then we apply Recurrence Lemma 3.1 with $j_1 \ldots j_k := u^{(1)}$ and $\delta = r$ to obtain infinitely many $n \in \mathbb{N}$ satisfying (5). Fix such an n. Denote

$$w := i_1 \ldots i_n \quad \text{and} \quad v^{(j)} = wu^{(j)},\ j = 1, \ldots, N.$$

Observe that $\mathbf{i}$ starts with $v^{(1)}$, so $x = \Pi(\mathbf{i}) \in \Lambda_{v^{(1)}}$, hence

$$|p_\theta(x - b_{v^{(1)}})| \le |x - b_{v^{(1)}}| \le d_\Lambda \lambda_{v^{(1)}} \le d_\Lambda \lambda_w r. \tag{7}$$

Here we used that $u^{(1)} \in \mathcal{W}(r)$, so $\lambda_{v^{(1)}} = \lambda_w \lambda_{u^{(1)}} \le \lambda_w r$. We have for $z \in \mathbb{R}^2$,

$$\lambda_{v^{(j)}} \mathcal{O}_{v^{(j)}} z + b_{v^{(j)}} = S_{v^{(j)}}(z) = S_w \circ S_{u^{(j)}}(z) = \lambda_w \mathcal{O}_w (\lambda_{u^{(j)}} \mathcal{O}_{u^{(j)}} z + b_{u^{(j)}}) + b_w,$$

hence

$$b_{v^{(j)}} = \lambda_w \mathcal{O}_w b_{u^{(j)}} + b_w.$$

It follows that

$$p_\theta(b_{v^{(i)}} - b_{v^{(j)}}) = \lambda_w p_\theta \mathcal{O}_w (b_{u^{(i)}} - b_{u^{(j)}}).$$

By (5), we have $\varepsilon_w = 1$ and $\phi := \phi_w \in [0, r)$; therefore, $\mathcal{O}_w = R_\theta$ is the rotation through the angle ϕ. One can check that $p_\theta R_\phi = p_{\theta-\phi}$, which yields

$$|p_\theta(b_{v^{(i)}} - b_{v^{(j)}})| = \lambda_w |p_{\theta-\phi}(b_{u^{(i)}} - b_{u^{(j)}})|. \tag{8}$$

Clearly, $\|p_\theta - p_{\theta-\phi}\| \le |\phi| \le r$, where $\|\cdot\|$ is the operator norm, so we obtain from (6) and (8) that

$$|p_\theta(b_{v^{(i)}} - b_{v^{(j)}})| \le \lambda_w (|b_{u^{(i)}} - b_{u^{(j)}}| r + r) \le \lambda_w (d_\Lambda + 1) r.$$

Recall that $\mathbf{i}$ starts with v_1, so $x = \Pi(\mathbf{i}) \in \Lambda_{v^{(1)}}$, hence for each $j \le N$, for every $y \in \Lambda_{v^{(j)}}$,

$$\begin{aligned} |p_\theta(x - y)| &\le |x - b_{v^{(1)}}| + |p_\theta(b_{v^{(1)}} - b_{v^{(j)}})| + |b_{v^{(j)}} - y| \\ &\le d_\Lambda(\lambda_{v^{(1)}} + \lambda_{v^{(j)}}) + \lambda_w (d_\Lambda + 1) r \le \lambda_w (3 d_\Lambda + 1) r. \end{aligned} \tag{9}$$

Now we need a simple geometric fact: given that

$P_a(x) = \theta'$, $\theta = \theta' + \pi/2 \bmod [0, \pi)$, $|p_\theta(x - y)| \le \rho$, $|y - a| \ge c_1$, and (4) holds,

we have

$$|P_a(y) - \theta'| = |P_a(y) - P_a(x)| = \arcsin \frac{|p_\theta(y - x)|}{|y - a|} \le \frac{\pi}{2c_1}\rho.$$

This implies, in view of (9), that for $c_2 = \pi(3d_\Lambda + 1)/(2c_1)$,

$$\nu_a([\theta' - c_2\lambda_w r, \theta' + c_2 \lambda_w r]) \ge \sum_{j=1}^N \nu(\Lambda_{v^{(j)}}) = \sum_{j=1}^N \lambda_{v^{(j)}} = \lambda_w \sum_{j=1}^N \lambda_{u^{(j)}} \ge \lambda_w N \lambda_{\min} r,$$

where $\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_m\}$, by the definition of $\mathcal{W}(r)$. Recall that n can be chosen arbitrarily large, so λ_w can be arbitrarily small, and we obtain that

$$\overline{D}(\nu_a, \theta') \ge c_2^{-1} \lambda_{\min} N.$$

Since $N \in \mathbb{N}$ is arbitrary, the proposition follows. □

4. Proof of the recurrence lemma 3.1

Let $K \in \{0, \ldots, m\}$ be the number of i for which $\varphi_i \notin \pi\mathbb{Q}$. Without loss of generality we may assume the following: if $K \geq 1$ then $\varphi_1, \ldots, \varphi_K \notin \pi\mathbb{Q}$.

We distinguish the following cases:

A: $\varphi_i \in \pi\mathbb{Q}$ for all $i \leq m$.
B: there exists i such that $\varphi_i \notin \pi\mathbb{Q}$ and $\varepsilon_i = 1$.
C: $K \geq 1$ and $\varepsilon_i = -1$ for all $i \leq K$.
  C1: there exist $i, j \leq K$ such that $\varphi_i - \varphi_j \notin \pi\mathbb{Q}$.
  C2: there exists $r_i \in \mathbb{Q}$ such that $\varphi_i = \varphi_1 + r_i\pi$ for $1 \leq i \leq K$.
    C2a: $K < m$ and there exists $j \geq K+1$ such that $\varepsilon_j = -1$.
    C2b: $K < m$ and for all $j \geq K+1$ we have $\varepsilon_j = 1$.
    C2c: $K = m$.

Denote by R_ϕ the rotation through the angle ϕ. We call it an irrational rotation if $\phi \notin \pi\mathbb{Q}$. Consider the semigroup generated by $\mathcal{O}_i,\ i \leq m$, which we denote by $\mathcal{S}$. We begin with the following observation.

Claim. *Either $\mathcal{S}$ is finite, or $\mathcal{S}$ contains an irrational rotation.*

The semigroup $\mathcal{S}$ is clearly finite in Case A and contains an irrational rotation in Case B. In Case C1 we have $\mathcal{O}_i\mathcal{O}_j = R_{\phi_i - \phi_j}$, which is an irrational rotation. In Case C2a we also have that $\mathcal{O}_i\mathcal{O}_j = R_{\phi_i - \phi_j}$ is an irrational rotation, since $\phi \notin \pi\mathbb{Q}$ and $\phi_j \in \pi\mathbb{Q}$. We claim that in remaining Cases C2b and C2c the semigroup is finite. This follows easily; then $\mathcal{S}$ is generated by one irrational reflection and finitely many rational rotations.

Proof of Lemma 3.1 when $\mathcal{S}$ is finite. A finite semigroup of invertible transformations is necessarily a group. Let $\mathcal{S} = \{s_1, \ldots, s_t\}$. By the definition of the semigroup $\mathcal{S}$ we have $s_i = \mathcal{O}_{w^{(i)}}$ for some $w^{(i)} \in \mathcal{A}^*$, $i = 1, \ldots, t$. For every $v \in \mathcal{A}^*$ we can find $\widehat{v} \in \mathcal{A}^*$ such that $\mathcal{O}_{\widehat{v}} = \mathcal{O}_v^{-1}$. Fix $u = j_1 \ldots j_k$ from the statement of the lemma. Consider the following finite word over the alphabet $\mathcal{A}$:

$$\omega := \tau_1 \ldots \tau_t, \quad \text{where} \;\; \tau_j = (w^{(j)}u)\,(\widehat{w^{(j)}u}),\ j = 1, \ldots, t.$$

Note that $\mathcal{O}_{\tau_j} = I$ (the identity). By the definition of Ω, the sequence $\mathbf{i} \in \Omega$ contains ω infinitely many times. Suppose that $\sigma^\ell \mathbf{i} \in [\omega]$. Since $\mathcal{O}_{\mathbf{i}|\ell} \in \mathcal{S}$, there exists $w^{(j)}$ such that $\mathcal{O}_{w^{(j)}} = \mathcal{O}_{\mathbf{i}|\ell}^{-1}$. Then the occurrence of u in τ_j, the jth factor of ω, will be at the position n such that $\mathcal{O}_{\mathbf{i}|n} = I$, so we will have $\phi_{\mathbf{i}|n} = 0 \in [0, \delta]$ and $\varepsilon_{\mathbf{i}|n} = 1$, as desired.

Proof of Lemma 3.1 when $\mathcal{S}$ is infinite. By the claim above, there exists $w \in \mathcal{A}^*$ such that $\phi_w \notin \pi\mathbb{Q}$ and $\varepsilon_w = 1$. Fix $u = j_1 \dots j_k$ from the statement of the lemma. Let

$$v := \begin{cases} uu, & \text{if } \phi_u \notin \pi\mathbb{Q}; \\ uuw, & \text{if } \phi_u \in \pi\mathbb{Q}. \end{cases}$$

Observe that $\phi_v \notin \pi\mathbb{Q}$ and $\varepsilon_v = 1$. Let $v^k = v \dots v$ (the word v repeated k times). Since ϕ_v/π is irrational, there exists an N such that every orbit of R_{ϕ_v} of length N contains a point in every subinterval of $[0, 2\pi)$ of length δ. Put

$$\omega := \begin{cases} v^N, & \text{if } \varepsilon_i = 1, \ \forall\, i \le m; \\ v^N j^* v^N, & \text{if } \exists\, j^* \text{ such that } \varepsilon_{j^*} = -1. \end{cases}$$

By the definition of Ω, the sequence $\mathbf{i} \in \Omega$ contains ω infinitely many times. Let $\ell \in \mathbb{N}$ be such that $\sigma^\ell \mathbf{i} \in [\omega]$. Suppose first that $\varepsilon_{\mathbf{i}|\ell} = 1$. Then we have, denoting the length of v by $|v|$,

$$\sigma^{\ell + k|v|}\mathbf{i} \in [u], \qquad \phi_{\mathbf{i}|(\ell+k|v|)} = \phi_{\mathbf{i}|\ell} + k\phi_v \pmod{2\pi}, \qquad \varepsilon_{\mathbf{i}|(\ell+k|v|)} = 1, \tag{10}$$

for $k = 0, \dots, N-1$. By the choice of N, we can find $k \in \{0, \dots, N-1\}$ such that $\phi_{\mathbf{i}|(\ell+k|v|)} \in [0, \delta]$, then $n = \ell + k|v|$ will be as desired. If $\varepsilon_{\mathbf{i}|\ell} = -1$, then we replace ℓ by $\ell^* := \ell + N|v| + 1$ in (10), that is, we consider the occurrences of u in the second factor v^N. The orientation will be switched by $\mathcal{O}_{j^*}$ and we can find the desired n analogously. ☐

5. Concluding remarks

Consider the special case when the self-similar set Λ is of the form

$$\Lambda = \bigcup_{i=1}^{m} (\lambda_i \Lambda + b_i), \qquad b_i \in \mathbb{R}^2. \tag{11}$$

In other words, the contracting similitudes have no rotations or reflections, as for the four corner Cantor set. Then the projection $\Lambda^\theta := p_\theta(\Lambda)$ is itself a self-similar set on the line:

$$\Lambda^\theta = \bigcup_{i=1}^{m} (\lambda_i \Lambda^\theta + p_\theta(b_i)), \quad \text{for } \theta \in [0, \pi).$$

Let $\Lambda_i^\theta = \lambda_i \Lambda^\theta + p_\theta(b_i)$. As above, ν is the natural measure on Λ. Let ν_θ be the natural measure on Λ^θ, so that $\nu_\theta = \nu \circ p_\theta^{-1}$.

Corollary 5.1. *Let Λ be a self-similar set of the form (11) that is not on a line, such that $\sum_{i=1}^{m} \lambda_i \leq 1$. If Λ satisfies the Open Set Condition condition, then*

$$\nu_\theta(\Lambda_i^\theta \cap \Lambda_j^\theta) = 0, \; i \neq j, \quad \textit{for a.e. } \theta \in [0, \pi).$$

Proof. Let $s > 0$ be such that $\sum_{i=1}^{m} \lambda_i^s = 1$. By assumption, we have $s \leq 1$. This number is known as the similarity dimension of Λ (and also of Λ^θ for all θ). Suppose first that $s = 1$. Then we are in the situation covered by Theorem 1.1, and ν is just the normalized restriction of $\mathcal{H}^1$ to Λ. Consider the product measure $\nu \times \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure on $[0, \pi)$. Theorem 1.1 implies that

$$(\nu \times \mathcal{L})\{(x, \theta) \in \Lambda \times [0, \pi) : \; \exists\, y \in \Lambda, \; y \neq x, \; p_\theta(x) = p_\theta(y)\} = 0.$$

By Fubini's Theorem, it follows that for $\mathcal{L}$ a.e. θ, for ν_θ a.e. $z \in L^\theta$, we have that $p_\theta^{-1}(z)$ is a single point. This proves the desired statement, in view of the fact that $\nu(\Lambda_i \cap \Lambda_j) = 0$ for Λ satisfying the Open Set Condition.

In the case when $s < 1$ we can use [11, Proposition 1.3], which implies that the packing measure $\mathcal{P}^s(\Lambda^\theta)$ is positive and finite for $\mathcal{L}$ a.e. θ. By self-similarity and the properties of $\mathcal{P}^s$ (translation invariance and scaling), we have $\mathcal{P}^s(\Lambda_i^\theta \cap \Lambda_j^\theta) = 0$ for $i \neq j$. Then we use [11, Corollary 2.2], which implies that ν_θ is the normalized restriction of $\mathcal{P}^s$ to Λ^θ, to complete the proof. □

Remark. In [1, Proposition 2] it is claimed that if a self-similar set $\mathcal{K} = \bigcup_{i=1}^{m} \mathcal{K}_i$ in $\mathbb{R}^d$ has the Hausdorff dimension equal to the similarity dimension, then the natural measure of the "overlap set" $\bigcup_{i \neq j}(\mathcal{K}_i \cap \mathcal{K}_j)$ is zero. This would imply Corollary 5.1, since the Hausdorff dimension of Λ^θ equals s for $\mathcal{L}$ a.e. θ by Marstrand's Projection Theorem. Unfortunately, the proof in [1] contains an error, and it is still unknown whether the result holds [C. Bandt, personal communication]. (It should be noted that [1, Proposition 2] was not used anywhere in [1].)

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