Hausdorff dimension for some hyperbolic attractors with overlaps and without finite Markov partition

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Abstract. Inn this paper we consider some families of self maps of the square which are analogous to the well known tent-map family. These maps are *almost* injective. That is the points of the attractor having more than one inverses from a set of zero measure for all interesting measures. We will compute the Hausdorff dimension of the attractor.

1. Introduction

In this paper our object is to compute the Hausdorff dimension of the attractor of maps

$$\widehat{F}(x,y) = \left(\widehat{T}(x), \widehat{g}(x,y)\right), \tag{1}$$

where $\widehat{T} : [0,1] \to [0,1]$ is a piecewise strictly increasing map such that for its restriction T to some interval $J \subset [0,1]$ we have that

(A1) $T : J \to J$ topologically transitive.

(A2) $1 < \inf_x |T'(x)| < \sup_x |T'(x)| =: L^* < \infty$.

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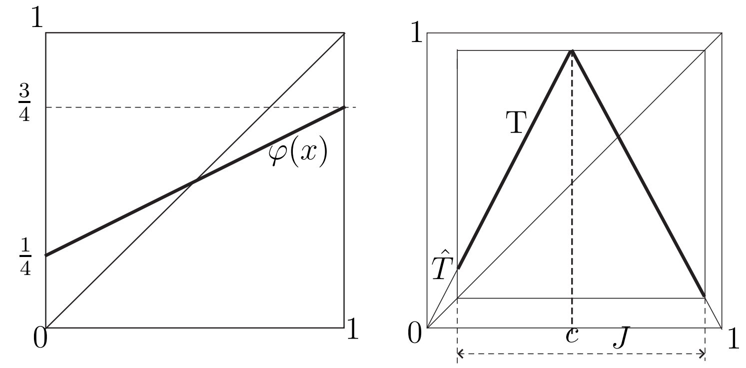

Figure 1: The definition of $\varphi(x)$ and $T := \widehat{T}|_J$

(A3) T is piecewise continuous and piecewise strictly monotonic, this means there exists a finite partition $\mathcal{V}$ of pairwise disjoint open intervals with $\bigcup_{V\in\mathcal{V}} \overline{V} = J$ such that $T\big|_V$ is continuous and strictly monotonic for all $V \in \mathcal{V}$.

(A4) For every $x \in (0,1)$ $\exists n$ such that $\widehat{T}^n(x) \in J$.

Further,

$$\widehat{g}(x,y) := \varphi(x) + \lambda\left(y - \frac{1}{2}\right), \tag{2}$$

where φ is defined by Figure 1.

We assume that

$$\lambda < \min\left\{\frac{1}{m_0^2}, \frac{1}{L^{*2}}\right\}, \tag{3}$$

where m_0 is the number of intervals of monotonicity of T. Note that this implies that

$$\lambda < \exp\{-2 \cdot h_{\mathrm{top}}(T)\} \tag{4}$$

After we have fixed λ, we also fix a number λ_2 such that

$$\sqrt{\lambda} < \lambda_2 < \frac{1}{L^*}. \tag{5}$$

Then $\widehat{F} : [0,1]^2 \to [0,1]^2$. See Figure 2. Let $Q := J \times [0,1]$. We write

$$F := \widehat{F}|_Q.$$

and let $\widehat{\Lambda}$ and Λ be the attractor of $\widehat{F}$ and F respectively. It is immediate that

$$\dim_{\mathrm{H}} \widehat{\Lambda} = \dim_{\mathrm{H}} \Lambda. \tag{6}$$

Our main result is that

Theorem 1. *If the assumptions* (A1)–(A4) *hold and further if the so called transversality condition is satisfied (see Definition* 3 *in Section* 2*) then*

$$\dim_{\mathrm{H}} \widehat{\Lambda} = 1 + \frac{h_{\mathrm{top}}(\widehat{T})}{-\log \lambda} = 1 + \frac{h_{\mathrm{top}}(T)}{-\log \lambda}. \tag{7}$$

The most important application is when $\widehat{T}$ is a (not necessarily fully developed) tent map (see Figure 1). In this case the map $\widehat{F}$ is a natural two dimensional analogue of the tent map $\widehat{T}$ and $\widehat{F}$ satisfies the assumptions of our Theorem 1.

It follows from [11] that

$$t := 1 + \frac{h_{\mathrm{top}}(T)}{-\log \lambda}$$

is an upper bound on the dimension of the attractor. The difficulty to prove that t is also a lower bound comes from two sources: on the one hand the cylinders (in case of the tent map $\widehat{F}\left([0,c] \times [0,1]\right)$ and $\widehat{F}\left([c,1] \times [0,1]\right)$) intersect each other, which may cause drop of the dimension in comparison with t. We handle this difficulty with the so-called transversality condition (see Section 4). On the other hand we do not have finite Markov partition for neither $\widehat{T}$ nor T. And so, we do not have finite Markov partition neither for $\widehat{F}$ nor for F which cause a substantial technical difficulty. To handle this in Section 3 we prove a theorem (Proposition 1) which is interesting on its own.

Our main example is the tent map family. For $a \in (1,2]$ define

$$F_a(x,y) := \left(T_a(x), \varphi(x) + \lambda\left(y - \frac{1}{2}\right)\right), \tag{8}$$

where

$$\widehat{T}_a(x) := \begin{cases} ax, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ a - ax, & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

$J := \left[\widehat{T}_a(1/2), \widehat{T}_a^2(1/2)\right]$ and $T_a : J \to J$, $T_a := \widehat{T}_a\big|_J$. Then for $a \in \left[\sqrt{2}, 2\right]$ the properties (A1)–(A4) and the transversality condition (see Lemma 4) hold. The research in this paper was motivated by a paper of M.V. Jakobson [6].

2. Notation

From now on we always assume that $T : J \to J$, where $J \subset [0,1]$ is a closed subinterval satisfying assumptions (A1)–(A3).

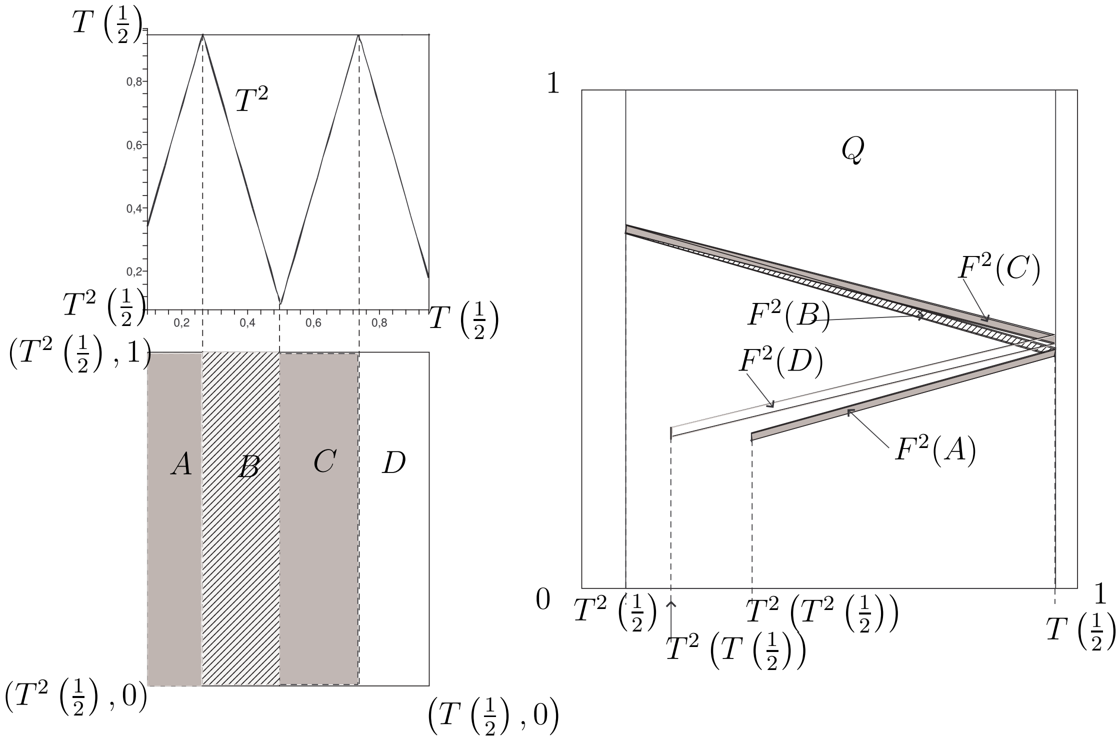

Figure 2: $A, B, C, D \in \mathcal{Z}_1$ and $F^2(A), F^2(B), F^2(C), F^2(D) \in \mathcal{I}_2^F$.

Definition 1. (1) We denote the projection $(x, y) \in J \times [0,1]$ to $x \in \mathbb{R}$ by π_1. That is $\pi_1(x, y) = x$. Similarly $\pi_2(x, y) := y$.

(2) For an $a \in J$ and $A \subset Q$ we write $A_a := \{y \in [0,1] : (a, y) \in A\}$.

(3) We define $\mathcal{V}_0 = \mathcal{V}$ the partition of J by the intervals of monotonicity of T. Let $c_0 < \cdots < c_{m_0}$ be the critical points of T (including the endpoints of J). We write
$$V_i := [c_{i-1}, c_i]$$
for $i = 1, \ldots, m_0$. That is
$$\mathcal{V} = \{V_1, \ldots, V_{m_0}\}.$$
Let int J_i be the interior of (J_i).

(4) The partition of J by the intervals of monotonicity of T^{n+1} is called $\mathcal{V}_n$. That is $\mathcal{V}_n := \bigvee_{i=0}^n T^{-i}(\mathcal{V})$.

(5) Let $\mathcal{Z} := \pi_1^{-1}(\mathcal{V})$ and $\mathcal{Z}_n := \bigvee_{i=0}^n F^{-i}\mathcal{Z} = \pi_1^{-1}\mathcal{V}_n$ is the set of vertical n-cylinders .

(6) For $n \geq 1$ let $\mathcal{I}_n^F := F^n \mathcal{Z}_{n-1}$. Then $\mathcal{I}_n^F$ is the set of horizontal n-cylinders. See Figure 2.

(7) $\mathcal{I}_\infty^F := \left\{\bigcap_{n\geq 0} I_n^F \neq \emptyset : I_n^F \in \mathcal{I}_n^F\right\}$. The elements of $\mathcal{I}_\infty^F$ are smooth curves on which π_1 is one-to-one. Their slopes are uniformly separated from the vertical direction. We call them unstable curves of Λ^F. Clearly $\Lambda^F = \bigcup \left\{I : I \in \mathcal{I}_\infty^F\right\}$. $I_{\infty,x}^F := \left\{I \in \mathcal{I}_\infty^F : x \in \pi_1 I\right\}$, $\mathcal{I}_{n,x}^F := \left\{I \in \mathcal{I}_n^F : x \in I\right\}$.

(8) $\mathcal{Z}_{k,l} := \{H : \exists Z \in \mathcal{Z}_k \text{ and } I \in \mathcal{I}_l \text{ such that } H = Z \cap I\}$. We call $\mathcal{Z}_{k,l}$ the collection of (k,l)-cylinders.

(9) For a continuous $f : Q \to \mathbb{R}$ we write

$$S_n f(x) := \sum_{i=0}^{n-1} f(F^i x), \quad \text{and} \quad p_n(F,f) := \sum_{Z \in \mathcal{Z}_{n-1}} \sup_{x \in Z} e^{S_n f(x)}.$$

Then the pressure is defined by:

$$p(F,f) := \lim_{n\to\infty} \frac{1}{n} \log p_n(F,f).$$

The limit in the last definition exists because of the sub additivity of the sequence $p_n(F,f)$. Let $f(x,y) := \log \lambda$. Consider the function:

$$t \to p(F, t \cdot f). \tag{9}$$

The zero of this function is clearly $\frac{h_{\text{top}}(T)}{-\log \lambda}$.

3. Approximation of the one-dimensional map by Markov maps

We introduce $D_k \subset J$, a "big" Cantor-set on which T is transitive and for every $x \in D_k$ and for every $n \geq 0$ the length of any n-th cylinder containing x is bigger than some constant to the power n. We will consider the restriction of Λ to $D_k \times [0,1]$ that we will call Λ_k. This is going to be a Cantor set in Q. Later in the paper we will prove that as k tends to infinity the dimension of Λ_k tends to t.

For $k \geq 1$ set

$$N_k := \bigcup \{V : V \in \mathcal{V}_{k-1}, \exists i \in \{0, \dots, m_0\}, c_i \in V\}. \tag{10}$$

Note that the assumption $c_i \in V$ above means that c_i is an endpoint of V. Now we define

$$C_k := \{x \in J : T^n(x) \notin \operatorname{int} N_k, \forall n \geq 0\}.$$

Note that $T(C_k) \subset C_k$.

In what follows we always write $V_l(x)$ for the l-cylinder containing x.

Proposition 1. *Under the assumptions* (A1)–(A3) *for every $\varepsilon > 0$ there exists a k such that there exists a perfect set $D_k \subset J$ having the following properties:*

(a) $T(D_k) = D_k$,

(b) $T\big|_{D_k}$ *is topologically transitive on* D_k,

(c) $h_{\text{top}}\left(T\big|_{D_k}\right) > h_{\text{top}}(T) - \varepsilon$,

(d) $\dim_{\mathrm{H}}(D_k) > 1 - \varepsilon$,

(e) *For every $\beta \in \left(0, \frac{1}{\alpha}\right)$ there is a $c > 0$ such that for every $x \in D_k$ and for every $n \geq 0$ we have*

$$|V_n(x)| \geq c \cdot \beta^n, \tag{11}$$

(f) *We can find a set $N_k' \subset J$ which is the union of $(k-1)$-cylinders such that $N_k \subset N_k'$ and*

$$D_k = \{x \in J : T^n(x) \notin \operatorname{int} N_k', \forall n \geq 0\}. \tag{12}$$

Moreover, for every $(k-1)$-cylinder V satisfying $V \cap D_k \neq \emptyset$ there are $(k-1)$-cylinders $W_1, \dots, W_l$ such that

$$T(V \cap D_k) = T(V) \cap D_k = \bigcup_{j=1}^{l} (W_j \cap D_k). \tag{13}$$

In the rest of this section we prove this proposition via a number of lemmas.

Lemma 1. *For every $n \geq k-1$ with $n \geq 1$ we have that*

$$V_n(x) \cap \operatorname{int} N_k = \emptyset \text{ implies that } T(V_n(x)) = V_{n-1}(T(x)). \tag{14}$$

Proof. Given $x \notin N_k$ and $n \geq k-1$. Then there exist $V_{i_0}, \ldots, V_{i_n} \in \mathcal{V}$ such that

$$V_n(x) = \bigcap_{l=0}^{n} T^{-l} V_{i_l} = V_{i_0} \cap T^{-1}(\widetilde{V}), \tag{15}$$

where $\widetilde{V} = \bigcap_{l=0}^{n-1} T^{-l} V_{i_{l+1}}$. Then

$$T(V_n(x)) = TV_{i_0} \cap \widetilde{V}. \tag{16}$$

Since $\widetilde{V}$ is an $n-1$-cylinder containing Tx it is enough to prove that

$$\widetilde{V} \subset TV_{i_0}. \tag{17}$$

To verify this, we argue with contradiction. Assume that $\widetilde{V} \not\subset TV_{i_0}$. By (16) $TV_{i_0} \cap \widetilde{V} \neq \emptyset$. Since TV_{i_0} and $\widetilde{V}$ are intervals, $\widetilde{V}$ must contain an endpoint of TV_{i_0}. Therefore $T^{-1}\widetilde{V}$ must contain an endpoint c of V_{i_0}, which is (by definition) a critical point. Then by (15) c must be an endpoint of $V_n(x)$. As $V_n(x) \subset V_{k-1}(x) \subset V_{i_0}$, the critical point c must be an endpoint of $V_{k-1}(x)$. This is a contradiction to $x \notin N_k$. $\square$

Lemma 2. *For $k > 1$, the set of those $(k-1)$-cylinders whose interior intersect C_k forms a Markov partition for C_k. This means that for every $(k-1)$-cylinder V with $V \cap \operatorname{int} N_k = \emptyset$ there are $(k-1)$-cylinders $W_1, \ldots, W_l$ such that*

$$T(V \cap C_k) = T(V) \cap C_k = \bigcup_{j=1}^{l} (W_j \cap C_k).$$

Proof. Let V be a $(k-1)$-cylinder with $V \cap \operatorname{int} N_k = \emptyset$. Then there is an $x \in C_k$ with $V = V_{k-1}(x)$. By Lemma 1 $TV = TV_{k-1}(x) = V_{k-2}(Tx)$, hence TV is a $(k-2)$-cylinder. Since every $(k-2)$-cylinder is a union of $(k-1)$-cylinders, there are $(k-1)$-cylinders $W_1, \ldots, W_l$ with $TV = \bigcup_{j=1}^{l} W_j$. Now the definition of C_k implies $T(V \cap C_k) = T(V) \cap C_k = \bigcup_{j=1}^{l} (W_j \cap C_k)$, as $V \cap \operatorname{int} N_k = \emptyset$. $\square$

Lemma 3. *For every $k > 1$ and for every $\beta \in \left(0, \frac{1}{\alpha}\right)$ there is a $c > 0$ such that for every $x \in C_k$ and for every $n \geq 0$ we have*

$$|V_n(x)| \geq c \cdot \beta^n. \tag{18}$$

Proof. There exists n_0 such that $\sup_{y\in J}\sqrt[n]{|(T^n)'(y)|}<\frac{1}{\beta}$ for all $n\geq n_0$. Set $c_1:=\min\left\{\frac{|V_l(x)|}{\beta^l}:l\in\{0,1,\dots,n_0\}\right\}$. Since $T^n(x)\notin\operatorname{int}N_k$ for all $n\geq 0$, using Lemma 1 we have $T(V_n(x))=V_{n-1}(T(x))$, $T^2(V_n(x))=T(V_{n-1}(T(x)))=V_{n-2}(T^2(x))$, $\dots$, $T^{n-k}(V_n(x))=V_k(T^{n-k}(x))$. Set $\tilde{c}:=\inf\{|V|:V\in\mathcal{V}_k\}$. The mean value theorem implies that for $n\geq n_0+k$ we have

$$\begin{aligned}|V_n(x)| &\geq |V_k(T^{n-k}(x))|\frac{1}{\sup_{y\in J}|\left(T^{n-k}\right)'(y)|}\\ &\geq \tilde{c}\beta^{n-k}=(\tilde{c}\beta^{-k})\beta^n.\end{aligned}$$

Choosing $c:=\min\left\{\tilde{c}\beta^{n-k},c_1\right\}$ completes the proof of the lemma. $\square$

Proof of Proposition 1. By Theorem 1 in [4] there exists an absolutely continuous invariant measure ϱ. Set $\psi:=-\log|T'|$, and let $\varepsilon\in(0,1)$. Obviously $\psi\in L^1(\varrho)$. As $\inf_{x\in J}|T'x|>1$ we get $\sup_{x\in J}\psi(x)<0$, and hence $\chi_\varrho:=-\int\psi\,d\varrho>0$. Now Theorem 1 of [5] gives $\frac{h_\varrho(T)}{\chi_\varrho}=\dim_{\mathrm{H}}(\varrho)=1$, implying $h_\varrho(T)+\int\psi\,d\varrho=0$. Set $t:=1-\varepsilon$. As $\int\psi\,d\varrho=-\chi_\varrho<0$ we obtain $h_\varrho(T)+\int t\psi\,d\varrho>0$. Hence $h_\varrho(T)>0$ and $p(T,t\psi)>0$.

Note that each N_k is a union of m_0 intervals. Moreover for any $\delta>0$ there is a k_δ such that for any $k\geq k_\delta$, and for any endpoint c of an interval of monotonicity the endpoints of the maximal subinterval of N_k containing c differ at most by δ from c. This means we have exactly the situation described in [9].

Since $\psi<0$ (hence $\lim_{n\to\infty}\frac{1}{n}\sup_{x\in J}\sum_{j=0}^{n-1}t\psi(T^jx)<0<p(T,t\psi)$) Theorem 1 in [9] implies that there exists a k such that $h_{\mathrm{top}}\left(T\big|_{C_k}\right)>h_{\mathrm{top}}(T)-\varepsilon$ and $p(T\big|_{C_k},t\psi\big|_{C_k})>0$. The partition of $(k-1)$-cylinders forms a Markov partition on C_k by Lemma 2. As the corresponding matrix is finite, there exists an irreducible subset $\mathcal{J}$ of the set of $(k-1)$-cylinders such that the restriction of the matrix to $\mathcal{J}$ has the same spectral radius as the original one. Define D_k as the set of all x satisfying $T^nx\in\bigcup_{V\in\mathcal{J}}V$ for all n. By Theorem 7 of [3] and Lemma 6 of [8] we obtain $h_{\mathrm{top}}\left(T\big|_{D_k}\right)=h_{\mathrm{top}}\left(T\big|_{C_k}\right)>h_{\mathrm{top}}(T)-\varepsilon$ and $p(T\big|_{D_k},t\psi\big|_{D_k})=p(T\big|_{C_k},t\psi\big|_{C_k})>0$. This shows (c).

It follows from the proof of Theorem 4 in [3] that $T\big|_{D_k}$ is topologically transitive. Hence (b) holds. Moreover, this also implies $T(D_k)=D_k$, showing (a). From the definitions we obtain (f). Lemma 3 implies (e).

There exists an invariant measure τ with $h_\tau(T)+\int t\psi\,d\tau>0$. Obviously $\psi\in L^1(\tau)$, and as $\sup_{x\in J}\psi(x)<0$ we get $\int\psi\,d\tau<0$. Since $h_\tau(T)=0$

would contradict $h_\tau(T) + \int t\psi\, d\tau > 0$ we obtain $h_\tau(T) > 0$. By Theorem 1 of [5] this implies $\dim_{\mathrm{H}}(D_k) \geq \dim_{\mathrm{H}}(\tau) = \frac{h_\tau(T)}{-\int \psi\, d\tau} > \frac{-t\int \psi\, d\tau}{-\int \psi\, d\tau} = t = 1 - \varepsilon$. $\square$

4. Transversality condition

To handle the overlapping between the horizontal cylinders we need the so-called transversality condition. In order to state it we need some symbolic representation of the horizontal cylinders.

Define

(19)
$$\mathcal{H} := \left\{ (\ldots, i_{-2}, i_{-1}, i_0) : V_{i_{-n}} \cap T^{-1} V_{i_{-(n-1)}} \cap \cdots \cap T^{-n} V_{i_0} \neq \emptyset \ \forall n \geq 0 \right\}.$$

For every $\mathbf{i} \in \mathcal{H}$ we write

$$p(\mathbf{i}) := \bigcap_{k=0}^{\infty} F^{k+1} Z_{i_{-k}}, \tag{20}$$

where $Z_l := V_l \times [0,1]$ for $l \in \{1, \ldots, m_0\}$. Then p is a homeomorphism between $\mathcal{H}$ and $\mathcal{I}_\infty^F$. Set

$$S_n^\lambda(x) := \varphi(T^n(x)) + \lambda \varphi(T^{n-1}x) + \cdots + \lambda^{n-1}\varphi(Tx) + \lambda^n \varphi(x). \tag{21}$$

Now we define the shift $\sigma : \mathcal{H} \to \mathcal{H}$ by

$$\sigma(\ldots, i_{-2}, i_{-1}, i_0) := (\ldots, i_{-3}, i_{-2}, i_{-1}). \tag{22}$$

From the definition of F we get

$$F^{n+1}(x,y) = \left(T^{n+1}(x), S_n^\lambda(x) + \lambda^{n+1} y + q_n \right), \tag{23}$$

where $q_n := -\frac{1}{2}\left(\lambda^{n+1} + \cdots \lambda\right)$. Let $I = p(\mathbf{i})$ and for an $N \in \mathbb{N}$ we call I_N the horizontal N-cylinder containing I. Then

$$I_N = F^{N+1}\left(V_{i_{-N}} \cap T^{-1}(V_{i_{-(N-1)}}) \cap \cdots \cap T^{-N} V_{i_0} \times [0,1] \right). \tag{24}$$

That is I_N is a nearly horizontal strip of height λ^{N+1}. For any $n \geq 0$ denote by $T_{i_0 \ldots i_{-n}}^{-(n+1)} : T^{(n+1)} W \to W$ the inverse of $T^{(n+1)}\big|_W : W \to T^{(n+1)} W$, where $W = V_{i_{-n}} \cap T^{-1} V_{i_{-(n-1)}} \cap \cdots \cap T^{-n} V_{i_0}$. We introduce the new variable $t = T^{N+1}x$ for $x \in V_{i_{-N}} \cap T^{-1} V_{i_{-(N-1)}} \cap \cdots \cap T^{-N} V_{i_0}$. For

$$t \in T^{N+1}\left(V_{i_{-N}} \cap T^{-1} V_{i_{-(N-1)}} \cap \cdots \cap T^{-N} V_{i_0} \right)$$

put

$$f_{\mathbf{i}}^N(t) := \varphi(T_{i_0}^{-1}t) + \lambda\varphi(T_{i_0 i_{-1}}^{-2}t) + \cdots + \lambda^N\varphi(T_{i_0\dots i_{-(n)}}^{-(N+1)}t) + q_N. \tag{25}$$

Then we get that I_N is bounded from below by the curve

$$\gamma_N^l(\mathbf{i})(t) := \big(t, f_{\mathbf{i}}^N(t)\big) \tag{26}$$

and I_N is bounded from above by the curve

$$\gamma_N^u(\mathbf{i})(t) := \big(t, f_{\mathbf{i}}^N(t) + \lambda^{n+1}\big) . \tag{27}$$

Definition 2. Let $I, I' \in \mathcal{I}_\infty^F$ and $I = p(\mathbf{i}), I' = p(\mathbf{j})$. We say that I and I' have different n-cylinders if $(i_{-(n-1)}, \dots, i_0) \neq (j_{-(n-1)}, \dots, j_0)$.

Definition 3 (Transversality condition). Using all the notation of Definition 2 we say that the transversality condition holds if there exists a $c_1 > 0$ such that for all $I, I' \in \mathcal{I}_\infty^F$ having different first cylinders, and for every N we have

$$|\pi_1(I_N \cap I'_N)| \leq c_1 \cdot \lambda^N, \tag{28}$$

where I_N and I'_N are the horizontal N-cylinders containing I and I' respectively.

Since our main example is the tent map, we prove that the transversality condition holds for these maps.

Lemma 4. *Let $a \in \big[\sqrt{2}, 2\big]$. If $a > 2\lambda$, then the transversality condition holds for the tent map F_a (see* (8) *for the definition).*

Proof. Let $\mathcal{V} = \{V_0, V_1\}$ be the partition of J by the intervals of monotonicity of T_a. Then the slope of T_a is $\pm a$. Let $\varepsilon_l := \operatorname{sign}(T'_a\big|_{V_l})$. Then $T'_a\big|_{V_l} = \varepsilon_l a$. Using that $\varphi' \equiv \frac{1}{2}$ we obtain from the definition (25) of $f_{\mathbf{i}}^N(t)$ that

$$\left(f_{\mathbf{i}}^N\right)'(t) \equiv \frac{\varepsilon_{i_0}}{2a}\left(1 + \sum_{k=1}^{N} \varepsilon_{i_{-1}} \cdots \varepsilon_{i_{-N}} \left(\frac{\lambda}{a}\right)^k\right) \tag{29}$$

for every t in the domain of $f_{\mathbf{i}}^N$. Since $\frac{\lambda}{a} < \frac{1}{2}$ this derivative is uniformly (in N) separated from zero and its sign is ε_{i_0}. Since I is the graph of the function $f_{\mathbf{i}}$ defined by $f_{\mathbf{i}}(t) := \lim_{N\to\infty} f_{\mathbf{i}}^N(t)$ for $t \in \pi_1(I)$ and the convergence of the derivatives is uniform, we get that the sign of the slope of the derivative of I is given by i_0 and separated uniformly from zero which means that the transversality condition holds. □

Lemma 5. *If the transversality condition holds then for every $k \geq 1$ there exists $c(k) > 0$ such that for all $I, I' \in \mathcal{I}_\infty^F$ having different k-cylinders, and for every $N \geq k$ we have*

$$|\pi_1(I_N \cap I'_N)| \leq c(k) \cdot \lambda^N, \tag{30}$$

where I_N and I'_N are the horizontal N-cylinders containing I and I' respectively.

Proof. Let $I = p(\mathbf{i})$ and $I' = p(\mathbf{j})$. From the assumption $(i_{-(k-1)}, \dots, i_0) \neq (j_{-(k-1)}, \dots, j_0)$. Let $l := \min\{n : i_{-n} \neq j_{-n}\}$. We know from (24) that

$$I_N = F^{N+1}\left(V_{i_{-N}} \cap T^{-1}V_{i_{-(N-1)}} \cap \dots \cap T^{-N}V_{i_0} \times [0,1]\right) \quad \text{and}$$
$$I'_N = F^{N+1}\left(V_{j_{-N}} \cap T^{-1}V_{j_{-(N-1)}} \cap \dots \cap T^{-N}V_{j_0} \times [0,1]\right).$$

Using that F^{N+1} is injective on the domains on the right hand side of the previous equalities, we obtain that

$$I_N \subset F^l \underbrace{\left(F^{N+1-l}\left(\left[V_{i_{-N}} \cap \dots \cap T^{-(N-l)}V_{i_{-l}}\right] \times [0,1]\right)\right)}_{(N-l)\text{-horizontal cylinder for } p(\sigma^l \mathbf{i})} \quad \text{and} \tag{31}$$

$$I'_N \subset F^l \underbrace{\left(F^{N+1-l}\left(\left[V_{j_{-N}} \cap \dots \cap T^{-(N-l)}V_{j_{-l}}\right] \times [0,1]\right)\right)}_{(N-l)\text{-horizontal cylinder for } p(\sigma^l \mathbf{j})}. \tag{32}$$

Now using the transversality condition for the two $(N-l)$-horizontal cylinders in the previous two equations and also using the definition of L^* (see condition (A2)) we obtain $|\pi_1(I_N \cap I'_N)| \leq \left(\frac{L^*}{\lambda}\right)^l \cdot c_1 \cdot \lambda^N$. As $\lambda < 1 < L^*$ and $l \leq k$ this immediately implies the desired result with the choice of $c(k) = c_1 \cdot (L^*)^k \cdot \lambda^{-k}$. □

5. Overlapping

In this section we always assume that the transversality condition and the assumptions (A1)–(A3) hold. Now we construct an injective self map G of Q which is practically the natural extension of F.

Choose the numbers $\lambda < q_1 < \dots < q_{m_0} < 1 - \lambda$ such that $q_i - q_{i-1} > \lambda$ for $i = 1, \dots, m$. This is possible by the choice of λ in (3). For $x \in V_i$ let

$$G(x, y) := \left(T(x), q_i + \lambda\left(y - \frac{1}{2}\right)\right). \tag{33}$$

Obviously, for every $(x, y) \in Q$ we have

$$\pi_1(F(x, y)) = \pi_1(G(x, y)). \tag{34}$$

Let $\Lambda^G := \bigcap_{n=0}^{\infty} G^n Q$ be the attractor of G. Clearly Λ^G consists of uncountably many horizontal segments. We can define $\mathcal{I}_n^G, \mathcal{I}_\infty^G, \mathcal{I}_{\infty,x}^G$ in a similar way for G as $\mathcal{I}_n^F, \mathcal{I}_\infty^F, \mathcal{I}_{\infty,x}^F$ was defined for F. Observe that geometrically the elements of $\mathcal{I}_n^G$ are rectangles with sides parallel to coordinate axes and the elements of $\mathcal{I}_\infty^G$ are horizontal segments.

Next we define a natural projection from Λ^G onto Λ^F which will be called Φ.

Definition 4 (Definition of Φ).

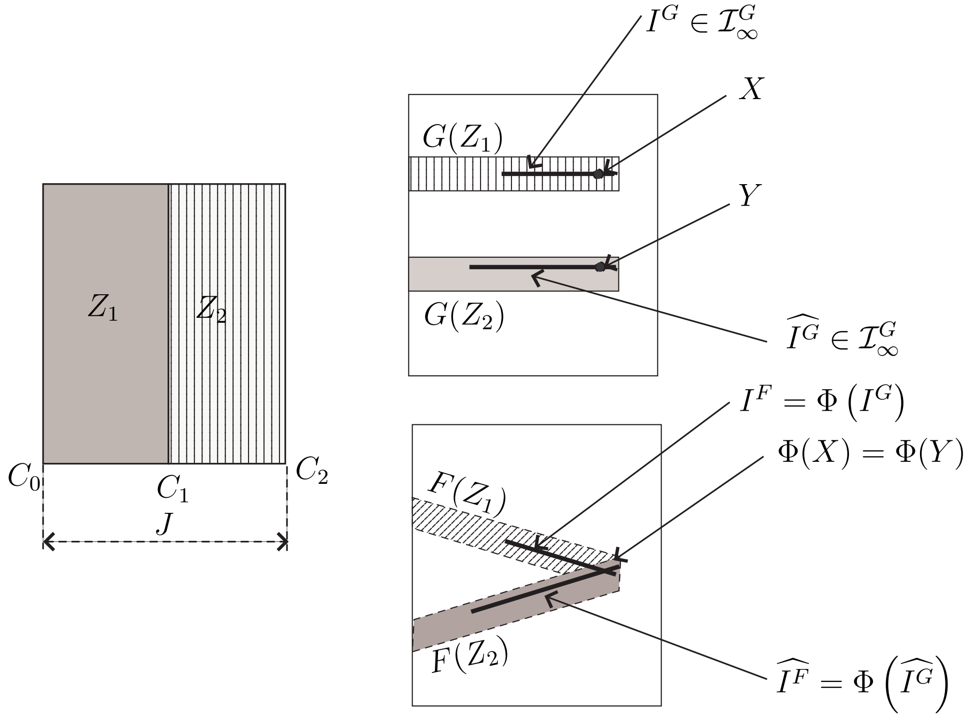

Figure 3: The definition of Φ.

For any $I^G \in \mathcal{I}_n^G$ there exists a $Z \in \mathcal{Z}_{n-1}$ such that $I^G = G^n(Z)$. The corresponding horizontal $(n-1)$-cylinder for the map F is $I^F = F^n(Z)$. Note that $\pi_1(F^n(Z)) = \pi_1(G^n(Z))$. For every $I^F \in \mathcal{I}_\infty^F$ we can find a sequence $\{Z^n\}_{n=1}^{\infty}$, $Z^n \in \mathcal{Z}_{n-1}$ such that $I^F = \bigcap_{n=1}^{\infty} F^n(Z^n)$. In this way there is a corresponding element $I^G \in \mathcal{I}_\infty^G$ defined by $I^G = \bigcap_{n=1}^{\infty} G^n(Z^n)$. Let $(a, b) \in \Lambda^G$ and let $I^G \in \mathcal{I}_\infty^G$ such that and $(a, b) = \Lambda_a^G \cap I^G$. Further, let $(a, b') := \Lambda_a^F \cap I^F$. The natural projection $\Phi : \Lambda^G \to \Lambda^F$ is defined by $\Phi(a, b) = (a, b')$.

It is easy to see that Φ is a Lipschitz map and the following diagram is commutative:

$$
\begin{array}{ccc}
\Lambda^G & \xrightarrow{G} & \Lambda^G \\
{\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\
\Lambda^F & \xrightarrow[F]{} & \Lambda^F
\end{array}
\tag{35}
$$

We say $I, I' \in \mathcal{I}_\infty^G$ *have different first cylinder* if a statement analogous to Definition 2 holds with replacing F with G everywhere. The main result of Steinberger's paper [11] immediately implies the following result.

Theorem 2 (Steinberger).

$$
\dim_{\mathrm{H}} \Lambda^G = 1 + \frac{h_{\mathrm{top}}(T)}{-\log \lambda}. \tag{36}
$$

Since the map Φ is Lipschitz, we obtain the following result.

Corollary 1. $\dim_{\mathrm{H}} \Lambda^F \leq 1 + \frac{h_{\mathrm{top}}(T)}{-\log \lambda}$.

In the rest of the paper we will prove that this upper bound is in fact the Hausdorff dimension of Λ^F.

To this end we need to prove that on the one hand the set of those points of Λ^F which are covered by at least two elements of $\mathcal{I}_\infty^F$ (Lemma 6) ("set of bad points") is small. On the other hand, we need to prove something about the size of the small neighborhood of these points. We can prove the first below. The second is much more difficult and requires a proposition which uses a number of results proved earlier.

Let us denote the "bad" points of Λ^G that is the ones whose image with Φ in Λ^F have at least two preimages by

$$
B := \left\{ x \in \Lambda^G : \#\Phi^{-1}(\Phi x) > 1 \right\}.
$$

Lemma 6. *We have* $\dim_{\mathrm{H}}(\pi_1(\Phi(B))) = \dim_{\mathrm{H}}(\pi_1(B)) < 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda} < 1$.

Proof. Let $I, I' \in \mathcal{I}_\infty^{\mathcal{G}}$ with different first cylinders. We define

$$
B_{I,I'} := \left\{ x \in \Lambda^G : x \in \Phi(I) \cap \Phi(I') \right\}.
$$

Further let

$$
B_1 := \bigcup B_{I,I'},
$$

where the union is taken for $I, I' \in \mathcal{I}_\infty^{\mathcal{G}}$ with different first cylinders. It follows from the transversality condition that for such I, I' and their horizontal N-cylinders I_N, I'_N we have

$$|\pi_1 (I_N \cap I'_N)| \leq c_1 \cdot \lambda^N. \tag{37}$$

Note that

$$\pi_1 B_1 = \bigcap_{N \geq 1} \bigcup_{I,I' \text{ with different first cylinders}} \pi_1(\Pi(I_N) \cap \Pi(I'_N)) \tag{38}$$

Let $0 < s < 1$ such that $\lambda^s < \exp\{-2h_{\mathrm{top}}(T)\}$. Then

$$\lim_{N\to\infty} c_1 (\#\mathcal{V}_{N-1})^2 (\lambda^s)^N = 0.$$

Therefore it follows from (37) and (38) that $\dim_{\mathrm{H}} B_1 < 1$. Using that $B = \bigcup_{n=1}^{\infty} G^n(B_1)$ completes the proof. $\square$

6. An invariant subset Λ_{k_0} of Λ^F

To prove that in some sense the n-th level neighborhood of the "bad" points is small we need a result which provides us with a "large" subset of J on which T is conjugated to a Bernoulli shift and all k-cylinders of this subset are "long".

Fix $k_0 \in \mathbb{N}$. Let $J_1, \dots, J_h \in \mathcal{V}_{k_0-1}$ closed intervals such that

$$D_{k_0} = \left\{ x \in J : T^n(x) \in \bigcup_{k\geq 1}^{h} J_k,\ \forall n \geq 0 \right\},$$

where D_{k_0} is as in Proposition 1. Set

$$s := s_{k_0} = \frac{h_{\mathrm{top}}(T|_{D_{k_0}})}{-\log\lambda}. \tag{39}$$

We define the $h\times h$ matrices $A = (a_{i,j})_{1\leq i,j\leq h}$ and $D = (d_{i,j})_{1\leq i,j\leq h}$ as follows:

$$a_{i,j} := \begin{cases} \lambda^s, & \text{if } T(J_i) \supset J_j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad d_{i,j} := \begin{cases} 1, & \text{if } T(J_i) \supset J_j; \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\Sigma_D := \left\{ \mathbf{i} \in \{1, \dots, h\}^{\mathbb{Z}} : \forall l :\ d_{i_l, i_{l+1}} = 1 \right\}$$

and let σ be the left shift on Σ_D. Then by Theorem 11 in [3] the spectral radii are

$$\rho(A) = 1, \qquad \rho(D) = \exp\left(h_{\text{top}}(T\big|_{D_{k_0}})\right). \tag{40}$$

Since $T|_{D_{k_0}}$ is topologically transitive by Proposition 1 we have that both of the matrices A and D are irreducible. Then we know that there exist $\mathbf{u} = (u_1, \dots, u_h)$; $\mathbf{v} = (v_1, \dots, v_h)$; $\mathbf{w} = (w_1, \dots, w_h) \in \mathbb{R}^h$ (each of them depend on k_0) such that for all i, $u_i, v_i > 0$, $\mathbf{u} \cdot A = \mathbf{u}$, $A \cdot \mathbf{v} = \mathbf{v}$, for all i we have $w_i = u_i \cdot v_i$, and $\sum_{i=1}^{h} w_i = 1$. Moreover, the $h \times h$ matrix $P = (p_{i,j})_{1 \le i,j \le h}$ defined by

$$p_{i,j} = a_{i,j}\frac{v_j}{v_i}$$

is a stochastic matrix and

$$\mathbf{w} \cdot P = \mathbf{w}.$$

Let μ be the corresponding Markov measure on Σ_D. These immediately imply that there exist constants c_1, c_2 (depending only on k_0) such that for all $l \in \mathbb{Z}$ and $j \ge 1$ and for all admissible word $[i_l, i_{l+1}, \dots, i_{l+j}]$ (according to the matrix D) we have

$$c_1 \cdot \lambda^{js} < \mu([i_l, i_{l+1}, \dots, i_{l+j}]) < c_2 \cdot \lambda^{js}. \tag{41}$$

Furthermore, set

$$Q_{k_0} := D_{k_0} \times [0,1] \quad \text{and} \quad F_{k_0} := F\big|_{Q_{k_0}}.$$

We know that $F_{k_0}(Q_{k_0}) \subset Q_{k_0}$. Now we define

$$\Lambda_{k_0} := \bigcap_{i=0}^{\infty} F_{k_0}^i(Q_{k_0}).$$

Then it is obvious that

$$\Lambda_{k_0} \subset \Lambda^F \tag{42}$$

holds for all k_0. For every $l \in \{1, \dots, h\}$ we define

$$\Delta_l := (J_l \cap D_{k_0}) \times [0,1].$$

These are the first level vertical cylinders of Λ_{k_0}. For an admissible word $(i_1, \dots, i_q) \in \{1, \dots, h\}^q$ (according to the matrix D) the corresponding q-level vertical cylinder is

$$\Delta_{i_1,\dots,i_q} := \Delta_{i_1} \cap F_{k_0}^{-1}\Delta_{i_2} \cap \dots \cap F_{k_0}^{q-1}\Delta_{i_q}.$$

Let $p \geq 1$. For an admissible word $(i_{-(p-1)}, \ldots, i_0)$ (according to the matrix D) the corresponding "horizontal" (actually almost horizontal) cylinder of Λ_{k_0} is

$$\widetilde{S}_{i_{-(p-1)},\ldots,i_0} := F_{k_0}^p \underbrace{\left(\Delta_{i_{-(p-1)}} \cap F_{k_0}^{-1} \Delta_{i_{-(p-2)}} \cap \cdots \cap F_{k_0}^{-(p-1)} \Delta_{i_0}\right)}_{\Delta_{i_{-(p-1)},\ldots,i_0}}$$

For an admissible word $\left(i_{-(p-1)}, \ldots, i_0, i_1, \ldots, i_q\right)$ we call the set

$$\widetilde{S}_{i_{-(p-1)},\ldots,i_0} \cap \Delta_{i_1,\ldots,i_q}$$

a (p,q)-cylinder of Λ_{k_0}. As we let $p \to \infty, q \to \infty$ for every $\mathbf{i} \in \Sigma_D$ these (p,q) cylinders converge to a point. This point is called the natural projection of $\mathbf{i} \in \Sigma_D$ to Λ_{k_0}. That is

$$\Pi(\mathbf{i}) := \bigcap_{p=1}^{\infty} \bigcap_{q=0}^{\infty} \widetilde{S}_{i_{-(p-1)},\ldots,i_0} \cap \Delta_{i_1,\ldots,i_q}.$$

In this case we will write

$$\left[i_{-(p-1)}, \ldots, i_0\right]_0 := \{\mathbf{j} : j_{-k} = i_{-k} \text{ for all } 0 \leq k \leq p-1\} \tag{43}$$

and we say that $\left[i_{-(p-1)}, \ldots, i_0\right]_0$ is a *non-positive p-cylinder* of Σ_D.

It is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} \Sigma_D & \xrightarrow{\sigma} & \Sigma_D \\ \Pi \downarrow & & \downarrow \Pi \\ \Lambda_{k_0} & \xrightarrow{F_{k_0}} & \Lambda_{k_0} \end{array} \tag{44}$$

The measure μ gives rise to a measure $\Pi_*\mu$ on $J \times [0,1]$ defined by

$$\Pi_*\mu(A) := \mu\left(\Pi^{-1}(A \cap \Lambda_{k_0})\right) . \tag{45}$$

Note that $\Pi_*\mu$ depends on k_0. However, for every k_0, $\Pi_*\mu$ is a probability measure concentrated on Λ_{k_0}.

7. Dimension of Λ_{k_0}

Note that $2 < \frac{\log \lambda}{\log \lambda_2}$ by (5). Therefore Proposition 1 implies that the properties (46) and (47) are satisfied for all sufficiently large k_0.

Proposition 2. *Assuming that k_0 is so large that*

$$\dim_{\mathrm{H}} D_{k_0} > \dim_{\mathrm{H}}(\pi_1(\Phi B)) \quad \textit{and} \tag{46}$$

$$2h_{\mathrm{top}}(T) < \frac{\log \lambda}{\log \lambda_2} \cdot h_{\mathrm{top}}(T\big|_{D_{k_0}}). \tag{47}$$

Then we have that

$$\dim_{\mathrm{H}} \Lambda_{k_0} \geq \dim_{\mathrm{H}}(D_{k_0}) + \frac{h_{\mathrm{top}}(T\big|_{D_{k_0}})}{-\log \lambda}, \tag{48}$$

The proof of this proposition is divided into several lemmas. For the rest of this section we fix k_0 satisfying (46) and (47) and we write

$$s := s_{k_0} = \frac{h_{\mathrm{top}}(T|_{D_{k_0}})}{-\log \lambda} \quad \text{and} \quad d := \dim_{\mathrm{H}}(D_{k_0}).$$

Lemma 7. *Under the assumptions of Proposition* 2 *we have*

$$\Pi_* \mu \left(\pi_1(\Phi(B)) \times [0,1]\right) = 0.$$

Proof. Using the notations of the proof of Lemma 6 it is enough to prove that for every $u \in \mathbb{N}$ we have

$$\Pi_* \mu \left((T^u \pi_1(B_1)) \times [0,1]\right) = 0 \tag{49}$$

Let $N > u$. Then by (37) and (38) the set $T^u(\pi_1(B_1))$ can be covered by $\mathrm{const}_1 \cdot (\#\mathcal{V}_{N-1})^2$ intervals of length $\mathrm{const}_2 \cdot (L^*)^u \lambda^N$. We choose $q = q(N)$ such that $\lambda_2^q = \lambda^N$. Since by Proposition 1 every $[q]$-level cylinder of D_{k_0} has length at least $\mathrm{const}_3 \cdot \lambda_2^{[q]}$ we obtain that $T^u(\pi_1(B_1)) \cap D_{k_0}$ can be covered by at most $\mathrm{const}_4 \cdot (\#\mathcal{V}_{N-1})^2 \cdot (L^*)^u$, $[q]$-level cylinders of D_{k_0}. It follows from (41) that the μ-measure of such a $[q]$-level cylinder is at most $\mathrm{const}_5 \cdot \lambda^{[q]s}$. Hence

$$\Pi_* \mu \left((T^u \pi_1(B_1)) \times [0,1]\right) \leq \mathrm{const}_6 \cdot (\#\mathcal{V}_{N-1})^2 \cdot (L^*)^u \lambda^{s \cdot N \cdot \log \lambda / \log \lambda_2}. \tag{50}$$

Using that $\lambda^s = \exp\left(-h_{\mathrm{top}}(T|_{D_{k_0}})\right)$ and also using that

$$\lim_{n \to \infty} \frac{1}{n} \log \#\mathcal{V}_{n-1} = h_{\mathrm{top}}(T)$$

by (47) we obtain that the right hand side of (50) tends to zero. $\square$

Corollary 2. *Under the assumptions of Proposition 2 we have*

$$\Pi_*\mu(\Phi(B)) = 0.$$

Now we are ready to prove a finer result. We have seen that the set of those point which have not unique coding is a set of $\Pi_*\mu$ zero measure. Now we will point out that most of those points which have only one code do have only the same code of the first n non-positive coordinates in a neighborhood of size close the the size of a horizontal n-cylinder.

Let $x \in \Lambda_{k_0} \setminus \Phi(B)$. Then there exists a unique $\mathbf{i} \in \Sigma_D$ such that $x = \Pi(\mathbf{i})$. We define

$$L(x) := \min \{\operatorname{dist}(x, \Pi(\mathbf{j})) : i_k = j_k \forall k \geq 0 \text{ and } i_0 \neq j_0\}.$$

The meaning of $L(x)$ is as follows: Let l be the vertical line which contains x. Then the symbolic representation of all points in the open interval on l of radius $L(x)$ centered at x has the same zero coordinate as the zero coordinate of $\Pi^{-1}(x)$.

It follows from Corollary 2 that the function L is defined for $\Pi_*\mu$-almost all points of Λ_{k_0}.

For an $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$O_n(\varepsilon) := \left\{\mathbf{i} \in \Sigma_D \setminus \Pi^{-1}\Phi(B) : L\left(\Pi(\sigma^{-n}\mathbf{i})\right) < e^{-\varepsilon n}\right\}.$$

Note that $\Sigma_D \setminus \Pi^{-1}\Phi(B)$ is the set of those $\mathbf{i} \in \Sigma_D$ for which there is a unique $I \in \mathcal{I}_\infty^F$ such that $\Pi(\mathbf{i}) \in I$. So, there is no overlap in Λ^F at $\Pi(\mathbf{i})$.

Lemma 8. *For every $\varepsilon > 0$ there exists an $0 < \eta < 1$, and a $c_* > 0$ and for every n there exists a set $R_n(\varepsilon) \subset \Sigma_D$ such that*

(a) $O_n(\varepsilon) \subset R_n(\varepsilon)$,

(b) $R_n(\varepsilon)$ *is the union of non-positive* $\left[\left(1 + \frac{\varepsilon}{-\log\lambda}\right) n\right]$*- cylinders*

(c) $\mu(R_n(\varepsilon)) < c_*\eta^n$.

Proof. Let

$$Q_n(\varepsilon) := \left\{\mathbf{i} \in \Sigma_D \setminus \Pi^{-1}\Phi(B) : L(\Pi\mathbf{i}) < e^{-\varepsilon n}\right\}.$$

Clearly,

$$O_n(\varepsilon) \subset \sigma^n(Q_n(\varepsilon)). \tag{51}$$

We define m by

$$m := \left[\frac{\varepsilon}{-\log \lambda} n \right] .$$

Let $\mathcal{I}_m$ be the pairs of non-positive m-cylinders of Σ_D with different zero coordinates. That is

$$\mathcal{I}_m := \big\{ (\tau, \omega) : \tau = [\tau_{-(m-1)}, \dots, \tau_0], \omega = [\omega_{-(m-1)}, \dots, \omega_0] \text{ admissible words, } \tau_0 \neq \omega_0 \big\} .$$

The corresponding horizontal m-cylinders of Λ_{k_0} are

$$S_\tau := \Pi(\tau) \text{ and } S_\omega := \Pi(\omega).$$

Put

$$X_n(\omega) := \big\{ y \in \Lambda_{k_0} : \exists x \in S_\omega \text{ such that } \pi_1(y) = \pi_1(x), \operatorname{dist}(x, y) < e^{-\varepsilon n} \big\} .$$

Then

$$Q_n(\varepsilon) \subset \bigcup_{(\omega,\tau) \in \mathcal{I}_m} \Pi^{-1} \left[X_n(\omega) \cap S_\tau \right] . \tag{52}$$

It follows from Lemma 5 that

$$|\pi_1 \left(X_n(\omega) \cap S_\tau \right)| < \left(c(k_0) + 2 \right) \lambda^m. \tag{53}$$

Now we define k by

$$k := \left[\frac{\log \lambda}{\log \lambda_2} m \right] . \tag{54}$$

Then

$$\begin{aligned} k &\le \frac{\log \lambda}{\log \lambda_2} \cdot m \le k + 1 \ , \\ m &\le \frac{\varepsilon}{-\log \lambda} \cdot n \le m + 1 \ , \text{ and} \\ k &\le \frac{\varepsilon}{-\log \lambda_2} \cdot n \le k + 1 + \frac{\log \lambda}{\log \lambda_2} \ . \end{aligned} \tag{55}$$

This implies that $k > m$. Also from (53) we obtain that

$$|\pi_1 \left(X_n(\omega) \cap S_\tau \right)| < \left(c(k_0) + 2 \right) \cdot \lambda_2^k. \tag{56}$$

Let

$$\mathcal{V}_{k-1}(\omega,\tau) := \{V \in \mathcal{V}_{k-1} : V \cap \pi_1 \left(X_n(\omega) \cap S_\tau\right) \neq \emptyset\}.$$

Using that the length of every k-cylinder of D_{k_0} is at least $c\lambda_2^k$ by Proposition 1 and that $\pi_1 \left(X_n(\omega) \cap S_\tau\right) \subset D_{k_0}$ we obtain from (53) that

$$\#\mathcal{V}_{k-1}(\omega,\tau) \leq c(k_0) + 2. \tag{57}$$

For an $V \in \mathcal{V}_{k-1}$ we can find $v_1,\dots,v_k \in \{1,\dots,h\}$ such that $V \cap D_{k_0} = \pi_1\Pi\left([v_1,\dots,v_k]_k\right)$. Similarly $S_\omega = \Pi[\tau_{-(\omega-1)},\dots,\tau_0]_0$. It follows from (41) that

$$\mu[\tau_{-(m-1)},\dots,\tau_0,v_1,\dots,v_k]_k \leq c_2 \cdot \lambda^{(k+m)s}. \tag{58}$$

We introduce the set $P_n(\varepsilon) \subset \Sigma_D$ defined by

$$P_n(\varepsilon) := \bigcup_{(\omega,\tau)\in\mathcal{I}_m} \bigcup_{\substack{V\in\mathcal{V}_{k-1}(\omega,\tau)\\ V\cap D_{k_0}=\pi_1\Pi([v_1,\dots,v_k]_k)}} [\tau_{-(m-1)},\dots,\tau_0,v_1,\dots,v_k]_k.$$

Then clearly

$$Q_n(\varepsilon) \subset P_n(\varepsilon). \tag{59}$$

Using (55) it follows that the set

$$R_n(\varepsilon) := \sigma^n P_n(\varepsilon) = \\ = \bigcup_{(\omega,\tau)\in\mathcal{I}_m} \bigcup_{\substack{V\in\mathcal{V}_{k-1}(\omega,\tau)\\ V\cap D_{k_0}=\pi_1\Pi([v_1,\dots,v_k]_k)}} [\tau_{-(m-1)},\dots,\tau_0,v_1,\dots,v_k]_{k-n}$$

can be represented as horizontal (non-positive coordinates) $n(1+\frac{\varepsilon}{-\log\lambda}) = n+m$ cylinders. Further (51) and (59) imply that $O_n(\varepsilon) \subset R_n(\varepsilon)$. In this way the only thing to verify remains that

$$\mu(R_n)(\varepsilon) = \mu(P_n)(\varepsilon) \to 0 \text{ exponentially fast.} \tag{60}$$

To see this note that it follows from (58) and (55) that

$$\begin{aligned} \mu(P_n(\varepsilon)) &\leq \sum_{(\omega,\tau)\in\mathcal{I}_m} \sum_{V\in\mathcal{V}_{k-1}(\omega,\tau)} \mu\left([\tau_{-(m-1)},\dots,\tau_0,v_1,\dots,v_k]_k\right) \\ &\leq c_2 \left(\#\mathcal{V}_{m-1}\right)^2 \lambda^{s(k+m)} \\ &\leq c_2 \left(\#\mathcal{V}_{m-1}\right)^2 \lambda^{sm(1+\log\lambda/\log\lambda_2)}. \end{aligned} \tag{61}$$

Therefore

$$\begin{aligned}
&\frac{1}{n}\log\mu(P_n(\varepsilon)) - \frac{1}{n}\log c_2 \leq \\
\leq &\frac{\varepsilon}{-m\log\lambda}\left[2\log\#\mathcal{V}_{m-1} + sm(1+\log\lambda/\log\lambda_2)\log\lambda\right] = \\
= &\frac{\varepsilon}{-\log\lambda}\cdot\left[\frac{2}{m}\cdot\log\#\mathcal{V}_{m-1} - h_{\mathrm{top}}(T|_{D_{k_0}})\,(1+\log\lambda/\log\lambda_2)\right].
\end{aligned}$$

Thus as n and consequently m tends to infinity we obtain that

$$\lim_{n\to\infty}\frac{1}{n}\log\mu(P_n(\varepsilon)) = \frac{\varepsilon}{-\log\lambda}\left(-h_{\mathrm{top}}(T|_{D_{k_0}})\right) < 0,$$

which completes the proof of the Lemma. □

Proof of Proposition 2. For the rest of the proof we fix an $\varepsilon > 0$. We need some further notation. For an $a \in D_{k_0}$ we define the segment $l_a := \{a\}\times[0,1]$. For an $\mathbf{i} \in \Sigma_D$ we write

$$\mathbf{i}^- := \{\mathbf{j}\in\Sigma_D : j_k = i_k \text{ for every } k \leq 0\}.$$

This corresponds to the unstable fibre which contains $\Pi(\mathbf{i})$. Geometrically the projection $\Pi(\mathbf{i}^-)$ is the almost horizontal line above D_{k_0} of the attractor which contains $\Pi(\mathbf{i})$. Analogously we define

$$\mathbf{i}^+ := \{\mathbf{j}\in\Sigma_D : j_k = i_k \text{ for every } k \geq 1\}.$$

This corresponds to the stable fibre containing $\Pi(\mathbf{i})$ and also geometrically this means the vertical line containing $\Pi(\mathbf{i})$.

We need the following very easy fact.

Lemma 9. *For every $a \in D_{k_0}$ we have*

$$\mu\left\{\mathbf{i}\in\Sigma_D : a \in \pi_1\Pi(\mathbf{i}^-)\right\} > 0. \tag{62}$$

Proof. Let $a = \pi_1\Pi(\dots, j_0, j_1, j_2, \dots)$. Then

$$\left\{\mathbf{i}\in\Sigma_D : a\in\pi_1\Pi(\mathbf{i}^-)\right\} = \left\{\mathbf{i}\in\Sigma_D : d_{i_0,j_1} = 1\right\},$$

which a set of positive μ measure. □

Let

$$G := \left\{\mathbf{i}\in\Sigma_D\setminus\Pi^{-1}\Phi(B) : \exists N = N(\mathbf{i}) \text{ such that } \forall n\geq N,\ \Pi(\mathbf{i})\notin R_n(\varepsilon)\right\},$$

where $R_n(\varepsilon)$ is as in Lemma 8. It follows from Lemma 8 and from the Borel Cantelli Lemma that

$$\mu(G) = 1. \tag{63}$$

Fix an arbitrary $a \in D_{k_0} \setminus \pi_1(\Phi(B))$. Set

$$G_a := \left\{ \mathbf{i} \in G : a \in \pi_1 \Pi(\mathbf{i}^-) \right\}.$$

Using Lemma 8 (b) we obtain that for any $\mathbf{i}, \mathbf{j} \in \Sigma_D$ with $\mathbf{i}^- = \mathbf{j}^-$ we have that either both $\mathbf{i}$ and $\mathbf{j}$ are contained in G_a or neither of them are contained in G_a. Now we obtain from (62) that

$$\mu(G_a) > 0. \tag{64}$$

We define the measure ν_a on the segment l_a as follows: for an $H \subset [0,1]$

$$\nu_a(H) := \mu \left\{ \mathbf{i} \in G_a : \Pi(\mathbf{i}^-) \cap l_a \in H \right\}.$$

We have seen in (64) that $\nu_a([0,1]) > 0$. Fix an arbitrary $\mathbf{i} \in G_a$ such that $a = \pi_1(\Pi \mathbf{i})$. Put $x := \Pi(\mathbf{i})$. Then by the definition of G we have that there exists a N such that for all $n \geq N$ we have $\mathbf{i} \notin R_n(\varepsilon)$. Set

$$\rho_n := \max \left\{ r : B(x,r) \cap l_a \cap \Lambda_{k_0} \subset \Pi[i_{-(n-1)}, \ldots, i_0]_0 \right\}.$$

Then for every n we have $\rho_n > 0$ by the choice of a. In this way

$$\nu_a \left(B(x, \rho_n) \right) \leq \mu([i_{-(n-1)}, \ldots, i_0]). \tag{65}$$

Now we estimate the magnitude of ρ_n. Let

$$C_3 = C_3(x) := \min_{0 \leq k \leq N} L(F^{-k} x).$$

Since $\mathbf{i} \notin \Pi^{-1}\Phi(B)$ we have $C_3 > 0$. Thus we can find $N_1 > N$ such that if $n > N_1$ then

$$\min_{k \leq n} L(F^{-k}(x)) > e^{-\varepsilon n} \tag{66}$$

holds. Hence by definition

$$\rho_n \geq e^{-\varepsilon n} \lambda^n, \ \forall n \geq N_1. \tag{67}$$

It follows from the Frostman's Lemma that ν_a-ess sup $\liminf_{r\to 0} \frac{\log \nu_a(B(x,r))}{\log r}$ is a lower bound for the Hausdorff dimension of $\Lambda_{k_0} \cap l_a$. To use this we observe that for ν_a typical $x \in l_a$ we have:

$$\lim_{n\to\infty} \frac{\log \nu_a(B(x,\rho_n))}{\log \rho_n} \geq \lim_{n\to\infty} \frac{-\log \mu[i_{-(n-1)} \dots i_0]}{\varepsilon n - n \log \lambda} =$$
$$= \lim_{n\to\infty} \frac{\frac{-\log \mu[i_{-(n-1)} \dots i_0]}{-n\log\lambda}}{1 - \frac{\varepsilon}{\log \lambda}} \geq \frac{s}{1-\frac{\varepsilon}{\log\lambda}} \ .$$

This implies that

$$\dim_{\mathrm{H}}(\Lambda_{k_0} \cap l_a) \geq s \tag{68}$$

holds for all $a \in D_{k_0} \setminus \pi_1(\Phi(B))$. Using (46) it follows from Theorem 5.8 in [1] that

$$\dim_{\mathrm{H}} \Lambda_{k_0} \geq \dim_{\mathrm{H}} D_{K_0} + s. \tag{69}$$

This completes the proof of the proposition. □

8. The proof of Theorem 1

Proof of Theorem 1. The upper estimate follows immediately from Corollary 1. Using that for every k we have $\dim_{\mathrm{H}} \Lambda^F > \dim_{\mathrm{H}} \Lambda_k$ we obtain by Proposition 2 that $\dim_{\mathrm{H}} \Lambda^F \geq \dim_{\mathrm{H}}(D_k) + \frac{h_{\mathrm{top}}(T|_{D_k})}{-\log\lambda}$ for every k. By **(c)** and **(d)** of Proposition 1 $\lim_{k\to\infty} \dim_{\mathrm{H}}(D_k) + \frac{h_{\mathrm{top}}(T|_{D_k})}{-\log\lambda} = 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$, completing the proof. □

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