DIMENSION OF SLICES THROUGH THE SIERPINSKI CARPET

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Abstract. For Lebesgue typical (θ, a), the intersection of the Sierpinski carpet F with a line $y = x \tan\theta + a$ has (if non-empty) dimension $s - 1$, where $s = \log 8 / \log 3 = \dim_{\mathrm{H}} F$. Fix the slope $\tan\theta \in \mathbb{Q}$. Then we shall show on the one hand that this dimension is strictly less than $s - 1$ for Lebesgue almost every a. On the other hand, for almost every a according to the angle θ-projection ν^θ of the natural measure ν on F, this dimension is at least $s - 1$. For any θ we find a connection between the box dimension of this intersection and the local dimension of ν^θ at a.

1. Introduction

Let F denote the Sierpinski carpet [14, p.81] in the unit square $[0,1] \times [0,1] = I^2$ in the plane $\mathbb{R}^2$ and let $E_{\theta,a} := \{(x,y) \in F : y - x\tan\theta = a\}$ denote its intersection with the line segment $\mathrm{Line}_{\theta,a}$ across I^2 of slope θ through $(0,a)$. (We only consider $\theta \in \left[0, \frac{\pi}{2}\right)$ because the case $\theta \in \left[\frac{\pi}{2}, \pi\right)$ is equivalent under the transformation $(x,y) \mapsto (x, 1-y)$.) We shall study the dimension of $E_{\theta,a}$ for $a \in I^\theta := [-\tan\theta, 1]$, as a subset of the y-axis. The angle θ projection of the unit square $[0,1]^2$ to the y-axis is

$$\mathrm{proj}^\theta(x,y) := (-\tan\theta, 1) \cdot (x,y). \tag{1}$$

When $\tan\theta \in \mathbb{Q}$, we shall show as our main result, in Theorem 9, that for Lebesgue almost every a

$$\dim_{\mathrm{H}} E_{\theta,a} < \log 8 / \log 3 - 1.$$

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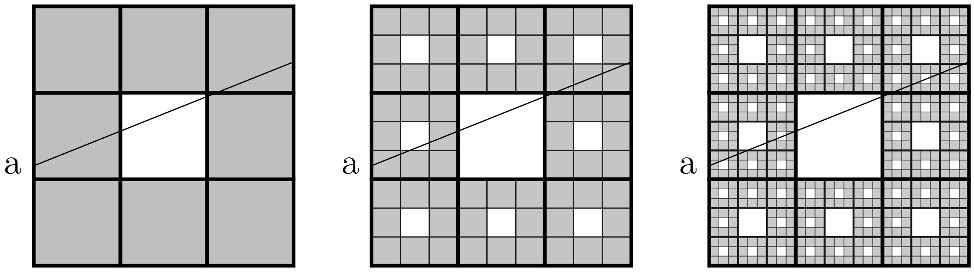

FIGURE 1. The intersection of the Sierpinski carpet with the line $y = \frac{2}{5}x + a$ for some $a \in [0,1]$.

This behavior for all rational slopes is atypical because, as an easy consequence of some results of Marstrand (see [12, Chapter 10]), we shall prove in Lemma 2 that, for Lebesgue almost all (θ, a), $\dim(E_{\theta,a}) = s-1$, where $s = \log 8/\log 3 = \dim_{\mathrm{H}} F$.

When $\tan\theta \in \mathbb{Q}$ the opposite behavior also occurs because, for ν^θ-almost all $a \in I^\theta$, we shall show $\dim(E_{\theta,a}) \geq s-1$. Here ν^θ is defined as follows.

We order the vectors $(u,v) \in \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}$ in lexicographic order and write $\mathbf{t}_i$ for the i-th vector, $i = 1,\dots,8$. The Sierpinski carpet F is the attractor of the IFS

$$\mathcal{G} := \left\{ g_i(x,y) = \frac{1}{3}(x,y) + \frac{1}{3}\mathbf{t}_i \right\}_{i=1}^{8}.$$

Let $\Sigma_8 := \{1,\dots,8\}^{\mathbb{N}}$ and write $\sigma : \Sigma_8 \to \Sigma_8$ for the left shift. We write $\Pi : \Sigma_8 \to F$ for the natural projection

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} g_{i_1\dots i_n}(0)$$

and $\nu := \Pi_*\mu_8$ for the natural measure on F, where μ_8 is the Bernoulli measure on Σ_8 given by $\left\{\frac{1}{8},\dots,\frac{1}{8}\right\}^{\mathbb{N}}$. Then $\nu^\theta := \mathrm{proj}_*^\theta(\nu)$.

Feng and Hu proved [4, Theorem 2.12] that every self-similar measure η is exact dimensional. That is, the local dimension of the measure η, given by

$$d(\eta,x) = \lim_{r\to 0} \frac{\log \eta(B(x,r))}{\log r} \tag{2}$$

is defined and constant for η-almost all x. It was shown by Young [19] that this constant must be the Hausdorff dimension of the measure η. That is

$$\text{for } \eta \text{ a.a. } x,\ d(\eta,x) = \dim_{\mathrm{H}}(\eta) := \inf\{\dim_{\mathrm{H}}(U) : \eta(U) = 1\}. \tag{3}$$

We will apply this result to for the measure ν^θ, which is a self similar measure for the IFS

$$\Phi := \left\{ \varphi_i^\theta(t) = \frac{1}{3} \cdot t + \frac{1}{3} \cdot \mathrm{proj}^\theta(\mathbf{t}_i) \right\}_{i=1}^{8}$$

with equal weights. That is, for every Borel set B,

$$\nu^\theta(B) = \sum_{k=1}^{8} \frac{1}{8} \nu^\theta \left(\left(\varphi_k^\theta\right)^{-1}(B) \right).$$

A simple calculation shows that whenever $\tan\theta \in \mathbb{Q}$ then Φ satisfies the so called Weak Separation Property (WSP) (see [10]). This means that there exists $c_* > 0$ such that for every n and for every $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{j} = (j_1, \dots, j_n)$ the level n cylinder intervals $I_{\mathbf{i}}^\theta := \varphi_{\mathbf{i}}^\theta(I^\theta)$ and $I_{\mathbf{j}}^\theta := \varphi_{\mathbf{j}}^\theta(I^\theta)$ are either the same $I_{\mathbf{i}}^\theta = I_{\mathbf{j}}^\theta$ or they do not overlap too much in the sense that $\mathcal{L}\mathrm{eb}(I_{\mathbf{i}}^\theta \setminus I_{\mathbf{j}}^\theta)/|I_{\mathbf{i}}^\theta| > c_*$. From the WSP it follows by the results of [10] that ν^θ is singular with respect to the Lebesgue measure.

2. Statement of results

2.1. **Behavior for typical slope.** A special case of a Theorem of Marstrand [13] (also see [12, Theorem 10.11]) is

Proposition 1 (Marstrand). *For ν-almost all $z \in F$ and for Lebesgue almost all $\theta \in [0, \pi)$ we have*

$$\dim\left(F \cap (z + W^\theta)\right) = s - 1 \text{ and } \mathcal{H}^{s-1}(F \cap (z + W^\theta)) < \infty,$$

where W^θ is the straight line of angle θ through the origin.

Lemma 2. *For Lebesgue almost all $(\theta, a), a \in I^\theta, \theta \in [0, \pi/2)$*

$$\dim\left(E_{\theta,a}\right) = s - 1, \tag{4}$$

where $\dim$ *denotes either* $\dim_{\mathrm{H}}$ *or* $\dim_{\mathrm{B}}$.

Proof. This follows from [12, Theorem 10.10] using a density point argument. □

2.2. **The general case.** We shall prove that for all θ the various dimensions are ν^θ-almost everywhere constant functions.

Proposition 3. *Fix an arbitrary $\theta \in [0, \pi/2)$. Then there exist non-negative numbers d_H^θ and $d_{\underline{B}}^\theta, d_{\overline{B}}^\theta$ such that for ν^θ-almost all $a \in I^\theta$ we have*

$$\dim_{\mathrm{H}}(E_{\theta,a}) = d_H^\theta, \underline{\dim}_{\mathrm{B}}(E_{\theta,a}) = d_{\underline{B}}^\theta \text{ and } \overline{\dim}_{\mathrm{B}}(E_{\theta,a}) = d_{\overline{B}}^\theta. \tag{5}$$

Proposition 4. *For all $\theta \in [0,\pi/2)$ and $a \in I^\theta$ if either of the two limits*

$$\dim_{\mathrm{B}}(E_{\theta,a}) = \lim_{n\to\infty} \frac{\log N_{\theta,a}(n)}{\log 3^n}, \quad d(\nu^\theta, a) = \lim_{\delta\to 0} \frac{\log(\nu^\theta[a-\delta, a+\delta])}{\log \delta} \tag{6}$$

exists then the other limit also exists, and, in this case,

$$\dim_{\mathrm{B}}(E_{\theta,a}) + d(\nu^\theta, a) = s. \tag{7}$$

Theorem 5. *For every $\theta \in [0,\pi/2)$ and for ν^θ-almost all $a \in I^\theta$ we have*

$$\dim_{\mathrm{B}}(E_{\theta,a}) = s - \dim_{\mathrm{H}}(\nu^\theta) \geq s - 1.$$

The assertion includes that the box dimension exists.

2.3. **The case of absolutely continuous ν^θ.** It is well known (see [12, Theorem 9.7]) that $\nu^\theta \ll \mathcal{L}\mathrm{eb}$ for Lebesgue almost all $\theta \in [0,\pi/2)$.

Theorem 6. *Suppose that θ satisfies $\nu^\theta \ll \mathcal{L}\mathrm{eb}$.*

- **(a):** *For $\mathcal{L}\mathrm{eb}$-almost all $a \in I^\theta$, there exist $0 < c_3(\theta,a) < c_4(\theta,a) < \infty$ such that*

$$\forall n, \qquad c_3(\theta,a)\cdot\left(\frac{8}{3}\right)^n < N_{\theta,a}(n) < c_4(\theta,a)\cdot\left(\frac{8}{3}\right)^n. \tag{8}$$

- **(b):** *In particular, for Lebesgue almost all $a \in I^\theta$, $\dim_{\mathrm{B}}(E_{\theta,a})$ exists and is equal to $s-1$.*

2.4. **Behavior for rational slope.** Recently Liu, Xi and Zhao proved

Theorem 7. [11] *Let $\tan\theta \in \mathbb{Q}$. For Lebesgue almost all $a \in I^\theta$ we have*

$$\dim_{\mathrm{B}}(E_{\theta,a}) = \dim_{\mathrm{H}}(E_{\theta,a}) = d^\theta(\mathcal{L}\mathrm{eb}) \leq \frac{\log 8}{\log 3} - 1,$$

where $d^\theta(\mathcal{L}\mathrm{eb})$ is a constant depending only on θ.

They conjectured the strict inequality.

If $\tan\theta \in \mathbb{Q}$ then the dimensions in Proposition 3 are equal.

Proposition 8. *If $\tan\theta \in \mathbb{Q}$ then there is a constant $d^\theta(\nu^\theta)$ such that*

$$d^\theta(\nu^\theta) := d_H^\theta = d_{\underline{B}}^\theta = d_{\overline{B}}^\theta. \tag{9}$$

Our main theorem asserts the strict inequality.

Theorem 9. *If $\tan\theta \in \mathbb{Q}$ then, for Lebesgue almost all $a \in I^\theta$, we have*

$$d^\theta(\mathcal{L}\mathrm{eb}) := \dim_{\mathrm{B}}(E_{\theta,a}) = \dim_{\mathrm{H}}(E_{\theta,a}) < \frac{\log 8}{\log 3} - 1. \tag{10}$$

It now follows from Proposition 4 that the local dimension of the self-similar measure ν^θ is Lebesgue almost everywhere equal to a constant which is bigger than 1.

Corollary 10. *If* $\tan\theta \in \mathbb{Q}$ *then, for Lebesgue almost all* $a \in I^\theta$*, we have*

$$d(\nu^\theta, a) = s - d^\theta(\mathcal{L}\mathrm{eb}) > 1. \tag{11}$$

There are many slices which do not have the small dimension of Theorem 9.

Theorem 11. *If* $\tan\theta \in \mathbb{Q}$ *then, for* ν^θ*-almost all* $a \in I^\theta$*,*

$$\dim_{\mathrm{H}}(E_{\theta,a}) = s - \dim_{\mathrm{H}}(\nu^\theta) \geq s - 1.$$

Organization of the paper: In Section 3 we develop our method of symbolic dynamics and prove Theorem 9. Then in Section 4 we prove the remaining results by using some notation and Proposition 18 from Section 3.

3. The proof of our main result

We make the standing hypothesis that $\tan\theta = \frac{M}{N} \geq 0$ where the natural numbers M, N are coprime.

If $3|N$ then $3 \nmid M$ and, using the isometry $(x, y) \mapsto (y, x)$, we may consider $\tan\theta = N/M$ instead; thus we may assume that $3 \nmid N$. Because of the isometry $(x, y) \mapsto (1-x, 1-y)$ we do not need to consider $a \in \left[-\frac{M}{N}, 0\right)$.

The idea of the proof of Theorem 7 was as follows: The authors found three non-negative integer matrices A_0, A_1, A_2 such that corresponding to the first n digits $(a_1, \ldots, a_n) \in \{0,1,2\}^n$ of the base 3 expansion of $a \in [0,1]$ they could approximate the minimal number of squares of size $1/3^n$ one needs to cover the n-th approximation of F by the norm of $A_{a_1} \cdots A_{a_n}$.

Our method is similar but we use different matrices A_0, A_1, A_2 which carry more geometric meaning. In Proposition 12 the subadditive ergodic theorem is used to express Hausdorff dimension in terms of the average of the logarithm of the norm of products of A_0, A_1, A_2. In Proposition 18 we show that one of these products B_1 has each row either all zero or all positive (which requires extra care in §3.3 if N is even). In §3.4 we consider the action of A_0, A_1, A_2 on the right on the simplex Δ of positive vectors as an Iterated Function System and, after studying powers of B_1, show that this IFS is contracting on average. In §3.5 our Theorem 9 is proved using the property that there

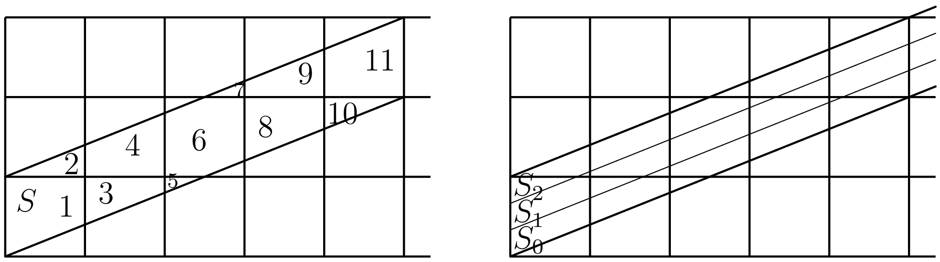

FIGURE 2. Tiling S when $(N, M) = (5, 2),\ K = 11$

is a positive measure subset of the invariant set of the IFS where the various products do not all expand by the same amount and a theorem of Furstenberg about the integral representation of the Lyapunov exponent of a random matrix product.

3.1. **Our transition matrices.** Now the interior of the strip

$$S := \{(x, y) \in \mathbb{R}^2 : x \in [0, N], y - x \tan\theta \in [0, 1]\}$$

meets the $2N + M - 1 (= K$, say), unit squares $I^2 + z$ for z in

$$\left\{ \left(q, \left[q\frac{M}{N}\right]\right), \left(q, \left[q\frac{M}{N}\right] + 1\right) : 0 \le q < N \right\} \cup \left\{ \left(\left[r\frac{N}{M}\right], r + 1\right) : 0 < r < M \right\}.$$

In the tiling of S by its intersection with these squares we number the tiles

$$Q_i := ((q, r) + I^2) \cap S,\ 1 \le i \le K \tag{12}$$

in increasing order of q and, for given q, in increasing order of r. Let us call the (q, r) in (12) as (q_i, r_i). That is

$$Q_i = ((q_i, r_i) + I^2) \cap S, \text{ and } Q_i \cap \text{int}(S) \neq \emptyset. \tag{13}$$

Figure 2 illustrates the case $M/N = 2/5,\ K = 11$.

Consider the three parallel narrower infinite strips S_0, S_1, S_2 with

$$S_t := \{(x, y) \in \mathbb{R}^2 : y - x \tan\theta - t/3 \in [0, 1/3]\}$$

and the expanding maps

$$\psi_t : S_0 \cup S_1 \cup S_2 \to S_t, \quad \psi_t(x, y) := (x/3, (y + t)/3), \quad t = 0, 1, 2.$$

Then

$$E_{\theta,a} = F \cap \bigcap_{n=1}^{\infty} \psi_{a_{n-1}} \circ \cdots \circ \psi_{a_1} S_{a_n}$$

expresses $E_{\theta,a}$ as the intersection with F of strips of vertical height 3^{-n} chosen according to the expansion $0.a_1 a_2 a_3 \ldots$ of a in base 3.

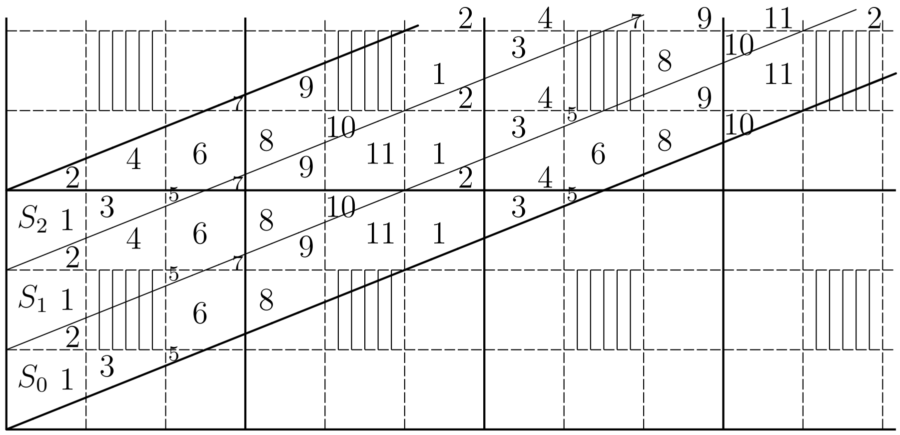

FIGURE 3. Tiling each strip S_t when $(N, M) = (5, 2)$

Consider the intersection of S_t with the eight squares of side 3^{-1} that cover F and, correspondingly, of $3S_t$ with $(q', r') + I^2$ where q', r' are not both congruent to $1 \mod 3$. We define, for $t \in \{0, 1, 2\}$, the $K \times K$ transition matrix A_t, with entries 0 and 1 so that its (i, j)-th entry is 1 if and only if $3(Q_i \cap S_t)$ contains $(\ell N, t+\ell M)+Q_j$ for some $\ell \in \{0, 1, 2\}$ with $q_j + \ell N, r_j + t + \ell M$ not both congruent to $1 \mod 3$. That is

(14)

$$A_t(i,j) = 1 \Leftrightarrow \left\{ \begin{array}{l} \exists \ell \in \{0,1,2\}, Q_i \cap S_t \supset 3^{-1}\left((\ell N, t+\ell M) + Q_j\right) \text{ and} \\ \text{either } q_j + \ell N \not\equiv 1 \bmod 3 \text{ or } r_j + t + \ell M \not\equiv 1 \bmod 3. \end{array} \right\}$$

In figure 3 the label j is marked in $3^{-1}((\ell N, t + \ell M) + Q_j)$ and this illustrates, for example, that the first three rows of these transition matrices are

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \cdots \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \cdots \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots \end{pmatrix}$$

The involution $(x, y) \mapsto (N, M+1)-(x, y)$ sends S_1 to itself, exchanges S_0 and S_2, and sends Q_i to Q_{K+1-i}. Therefore these matrices exhibit the symmetries

$$\forall i,j \in \{1,\dots,K\}\ A_0(i,j) = A_2(K+1-i, K+1-j), \tag{15}$$

$$A_1(i,j) = A_1(K+1-i, K+1-j). \tag{16}$$

Consider transitions from $Q_i = ((q,r)+I^2)\cap S$ to $Q_j = ((q',r')+I^2)\cap(S+(0,t))$. For the nine cases $\ell, t \in \{0,1,2\}$, $A_t(i,j)=1$ when $Q_i = (([(q'+\ell N)/3], [(r'+t+\ell M)/3]) + I^2)\cap S$ except for any case where

$$q' + \ell N \equiv r' + t + \ell M \equiv 1 \bmod 3, \tag{17}$$

which corresponds to a square deleted from F.

Because $3 \nmid N$, given j, (17) determines (ℓ, t), and so each of A_0, A_1, A_2 has each column sum equal to 2 or 3 and

$$A_s := A_0 + A_1 + A_2 \text{ has each column sum } 8. \tag{18}$$

If $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{1}$ denote the row vectors in $\mathbb{R}^K$ for which each entry of $\mathbf{1}$ is 1 and each A_ℓ has the row vector of its column sums equal to $3.\mathbf{1} - \mathbf{a}_t$, then each entry of $\mathbf{a}_t$ is 0 or 1 and $\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{1}$.

Now $A_t(i,j) = 1$ whenever $Q_i \cap S_t$ contains a 3^{-1}-sized copy of Q_j, and $A_u(j,k) = 1$ when $Q_j \cap S_u$ contains a 3^{-1}-sized copy of Q_k, so the (i,k)-th entry of $A_t A_u$ is the number of the 8^2 squares of side 3^{-2} covering F that meet $Q_i \cap S_t \cap \psi_t(S_u)$ in a 3^{-2}-sized copy of Q_k. By induction, the $(1,k)$-th entry in the product matrix $A_{a_1}A_{a_2}\dots A_{a_n}$ is the number of the 8^n squares of side 3^{-n} covering F that meet $Q_1 \cap S_{a_1} \cap \psi_{a_1}(S_{a_2}) \cap \dots \cap (\psi_{a_1} \circ \dots \circ \psi_{a_{n-1}})(S_{a_n})$ in a 3^{-n}-sized copy of Q_k. Thus the first row of $A_{a_1}A_{a_2}\dots A_{a_n}$ counts the elements of a $(3^{-n}\sqrt{2})$-cover of $E_{\theta,a}$. (In the cases $(N,M) = (1,1)$ and $(2,1)$, certain matrix entries are 2, and the elements of the cover are still counted correctly.) Thus we get:

Proposition 12. *For every $a \in [0,1]$, $a = \sum_{i=1}^{\infty} a_i 3^{-i}$ we have*

$$\overline{\dim}_{\mathrm{B}} E_{\theta,a} \le \frac{1}{\log 3} \limsup_{n\to\infty} \frac{1}{n} \log \|A_{a_1\dots a_n}\|_1, \tag{19}$$

where $A_{a_1\dots a_n}$ denotes $A_{a_1}\cdots A_{a_n}$ and $\|\cdot\|_1$ denotes the sum of the moduli of the entries. For almost all $a = \sum_{i=1}^{\infty} a_i \cdot 3^{-i}$ the right hand side gives the same value and we can replace lim sup *by* lim. *Consider the random product of the matrices A_0, A_1, A_2 each taken with probability* 1/3 *independently in every step. Then the Lyapunov exponent γ of this random matrix product is the almost sure value of the limit above. That is*

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \log \|A_{a_1\dots a_n}\|_1, \text{ for a.a. } (a_1, a_2, \dots). \tag{20}$$

Similarly,

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{i_1\dots i_n} \frac{1}{3^n} \log \|A_{i_1\dots i_n}\|_1. \tag{21}$$

Further,

$$\gamma \in [\log 2, \log 3]. \tag{22}$$

As an easy consequence of Theorem 7, the definition of γ and the translation invariance of the Lebesgue measure we obtain the following proposition.

Proposition 13. *If* $\tan\theta \in \mathbb{Q}$ *then for Lebesgue almost all* $a \in I^\theta$ *we have*

$$\dim_B(E_{\theta,a}) = \dim_H(E_{\theta,a}) = \frac{\gamma}{\log 3}. \tag{23}$$

We will use the following definitions.

Definition 14. *The symbolic space to code the translation parameter* $a \in [0,1]$ *is* $\Sigma := \{0,1,2\}^{\mathbb{N}}$*. Let* $\pi_y : \Sigma \to [0,1]$ *be defined by*

$$\pi_y(\mathbf{i}) := \sum_{k=1}^{\infty} i_k \cdot 3^{-k}, \qquad \mathbf{i} = (i_1, i_2, \dots).$$

We denote the uniform distribution on Σ *by* $\mathbb{P} := \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}^{\mathbb{N}}$ *. The push down measure* $(\pi_y)_*$ *of* $\mathbb{P}$ *by* π_y *is the Lebesgue measure* $\mathcal{L}\mathrm{eb}$ *on* $[0,1]$*. The measure* $\mathbb{P}$ *is ergodic with respect to* (Σ, σ)*, where* σ *is the left shift on* Σ*.*

As an easy case consider first the intersection of F with horizontal lines.

Proposition 15. *For almost every* a $\dim_H(E_{0,a}) = \frac{1}{3}\log 18/\log 3 < \log 8/\log 3 - 1$.

Proof. If $\tan\theta = 0/1$ then the above construction gives $K = 2N + M - 1 = 1$ and 1×1 matrices $A_0 = A_2 = (3), A_1 = (2)$. Then $E_{0,a}$ is covered by $A_{a_1}\dots A_{a_n}$ squares of side 3^{-n}. This leads us to study the function on $\Sigma := \{0,1,2\}^{\mathbb{N}}$ taking the value $\log 2$ where the first symbol is 1 and $\log 3$ elsewhere, whose integral with respect to $\mathbb{P}$ is $\frac{1}{3}\log 18$. □

So, from now on we may assume that $M/N \neq 0$. By symmetry, without loss of generality we may assume that $M/N > 0$.

To prove Proposition 12 we need the following simple observation which will also be used later.

Fact 16. *Consider the non-negative $K \times K$ matrices A, B. Let*

$$\mathbf{c}_A(j) := \sum_{i=1}^{K} A(i,j) \text{ and } \mathbf{r}_B(i) := \sum_{j=1}^{K} B(i,j) \tag{24}$$

be the j-th column sum and the i-th row sum of the matrices A, B respectively. Then

$$\|A \cdot B\|_1 = \sum_i \mathbf{c}_A(i) \cdot \mathbf{r}_B(i). \tag{25}$$

Proof. The proof of the Fact is a simple calculation. □

Now we turn to the proof of Proposition 12.

Proof of Proposition 12. The inequality in (19) immediately follows from the discussion right above the proposition. The fact that we can replace lim sup by lim in (19) is an immediate corollary of the sub-additive ergodic theorem (see [18, p. 231]). For non-negative matrices it is easy to check that $\|\cdot\|_1$ is submultiplicative. Indeed

$$\|A \cdot B\|_1 = \sum_i \mathbf{c}_A(i) \cdot \mathbf{r}_B(i) \leq \sum_{i=1}^{K} \sum_{j=1}^{K} \mathbf{c}_A(i) \cdot \mathbf{r}_B(j) = \|A\|_1 \cdot \|B\|_1.$$

This implies that the sequence of the bounded functions $f_k : \Sigma \to \mathbb{R}$

$$f_k(\mathbf{i}) := \log \|A_{i_k}^T \cdots A_{i_1}^T\|_1 = \log \|A_{i_1 \dots i_k}\|_1 \tag{26}$$

is sub-additive. Using the ergodicity of $\mathbb{P}$ and the sub-additive ergodic theorem [18, p. 231] we obtain that the limit

$$\gamma := \lim_{n \to \infty} \frac{1}{n} \cdot f_n(\mathbf{i}) = \lim_{n \to \infty} \frac{1}{n} \cdot \log \|A_{i_1} \cdots A_{i_n}\|_1 \tag{27}$$

exists for $\mathbb{P}$-almost all $(a_1, a_2, \dots) \in \Sigma$ and gives the same value. Further, using

$$\frac{1}{n} \int f_n(\mathbf{i}) d\mathbb{P}(\mathbf{i}) = \frac{1}{n} \sum_{i_1 \dots i_n} \frac{1}{3^n} \log \|A_{i_1 \dots i_n}\|_1,$$

the sub-additive ergodic theorem implies that (21) holds.

To verify (22) we make the following observation: each factor in the matrix product $A_{i_1} \cdots A_{i_n}$ has each column sum either 2 or 3. This implies that all the column sums of the product matrix are between 2^n and 3^n. This yields

$$\forall [i_1, \dots, i_n] \text{ we have: } K \cdot 2^n \leq \|A_{i_1} \cdots A_{i_n}\|_1 \leq K \cdot 3^n. \tag{28}$$

□

The inequality (29) below is an immediate corollary of (19) and (20).

Corollary 17. *The following holds:*

$$\overline{\dim}_{\mathrm{B}} E_{\theta,a} \leq \frac{\gamma}{\log 3}. \tag{29}$$

3.2. **Positive rows in some products of our matrices.** Recall from Section 1 that

$$3 \nmid N. \tag{30}$$

In this section and the next we prove

Proposition 18. *There exists n_0 and $(a_1 \ldots a_{n_0}) \in \{0,1,2\}^{n_0}$ such that the rows of the matrix $A_{a_1 \ldots a_{n_0}}$ are vectors with either all positive or all zero elements.*

Our approach is as follows. A row will be positive when labels 1 to K appear in the intersection of each square with the strip in the n_0th version of Figure 3. The labelling depends on the relation between N and 3. When $M/N < 1$ we shall find the labels near the left or right edge of the square in positions corresponding to squares in the n_0th stage in the construction of F. When $M/N \geq 1$ the strip may not be near either edge, so we shall search instead in small rectangles with diagonal of slope near M/N and carefully chosen horizontal and vertical coordinates.

3.2.1. *An IFS leaving S invariant.* First we introduce some notation. For $t, \ell \in \{0,1,2\}$ we define the contraction:

$$\psi_t^\ell(x,y) := \frac{1}{3} \cdot \left[x + \ell \cdot N, y + \ell \cdot M + t\right].$$

To compute the n-fold compositions we introduce the following notation:

$$\ell_{1,n} := \ell_1 3^{n-1} + \cdots + \ell_{n-1} \cdot 3 + \ell_n \tag{31}$$

and

$$a_{1,n} := a_1 3^{n-1} + \cdots + a_{n-1} \cdot 3 + a_n. \tag{32}$$

If we write $\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}$ for $\psi_{a_1}^{\ell_1} \circ \cdots \circ \psi_{a_n}^{\ell_n}$ and $S_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}$ for $\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(S)$ then we have

$$\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(x,y) = \frac{1}{3^n} \cdot \left[x + N \cdot \ell_{1,n}, y + M \cdot \ell_{1,n} + a_{1,n}\right] \tag{33}$$

and

$$S = \bigcup_{(a_1 \ldots a_n), (\ell_1 \ldots \ell_n) \in \{0,1,2\}^n} S_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}, \tag{34}$$

where the sets on the right hand side have disjoint interior. The set $S^{\ell_1\ldots\ell_n}_{a_1\ldots a_n}$ is the intersection of the slope θ level n strip

$$S_{a_1\ldots a_n} := \left\{(x,y) : 0 \le x \le N \text{ and } 0 \le y - (x\cdot\tan\theta + \frac{a_1}{3} + \cdots + \frac{a_n}{3^n}) \le \frac{1}{3^n},\right\}$$

with the level n vertical strip

$$V^{\ell_1\ldots\ell_n} := \left\{(x,y) : N\cdot\left(\frac{\ell_1}{3} + \cdots + \frac{\ell_n}{3^n}\right) \le x < N\cdot\left(\frac{\ell_1}{3} + \cdots + \frac{\ell_n}{3^n} + \frac{1}{3^n}\right)\right\},$$

for all $(\ell_1\ldots\ell_n),(a_1\ldots a_n) \in \{0,1,2\}^n$. See Figure 4.

Earlier in (13) we defined the $K = 2N+M-1$ different level zero squares which together cover S. Similarly, here we define the level n square of shape j in $S^{\ell_1\ldots\ell_n}_{a_1\ldots a_n}$ by

$$Q^{\ell_1\ldots\ell_n}_{a_1\ldots a_n}(j) := \psi^{\ell_1\ldots\ell_n}_{a_1\ldots a_n}(Q_j). \tag{35}$$

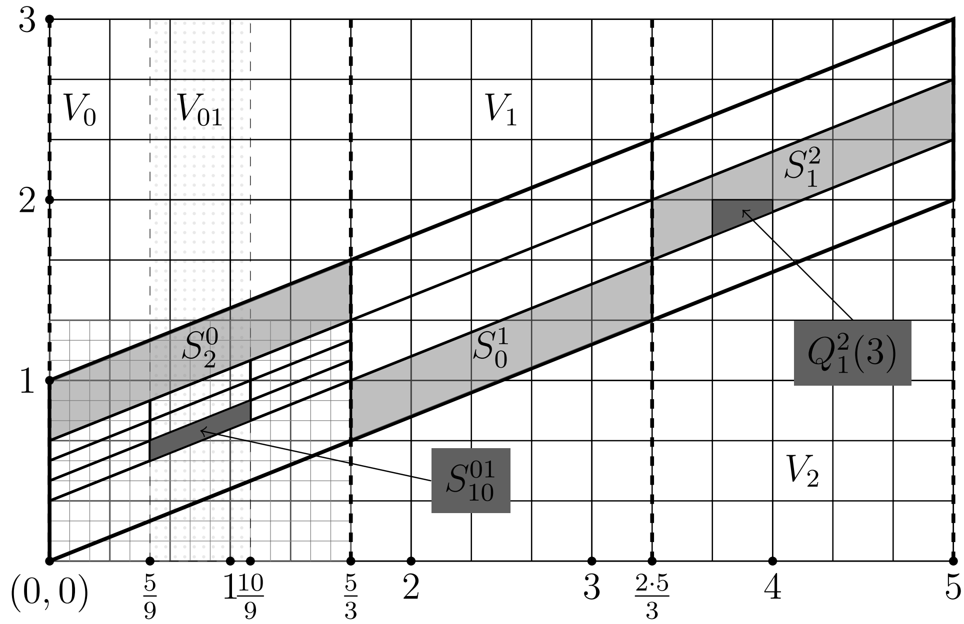

FIGURE 4. The subsets defined in (34) and (35) when $M/N = 2/5$.

3.2.2. *The n-th approximation of the translated copies of the Sierpinski carpet.*

Definition 19. *Let $\widetilde{F}$ be the union of the translated copies of the Sierpinski carpet to those unit squares that intersect S. That is*

$$\widetilde{F} := \bigcup_{i=1}^{K} ((q_i, r_i) + F).$$

Let $\widetilde{F}^n$ be the level n approximation of $\widetilde{F}$. Put

$$U_i^n = \left\{u \in [q_i, q_i+1) : \exists (u_1, \dots, u_n) \in \{0,1,2\}^m \text{ s.t. } u = q_i + \sum_{m=1}^{n} u_m \cdot 3^{-m}\right\}$$
$$V_i^n = \left\{v \in [r_i, r_i+1) : \exists (v_1, \dots, v_n) \in \{0,1,2\}^m \text{ s.t. } v = r_i + \sum_{m=1}^{n} u_m \cdot 3^{-m}\right\}$$

and say that u_m, v_m are the m-th ternary (that is base 3) digits of u, v respectively. For some $n_1 \geq n_2$ and for $u \in U_i^{n_1}$ and $v \in V_i^{n_2}$ we define the so called level (n_1, n_2) grid rectangle in the square $(q_i, r_i) + I^2$:

(36)
$$R_i(u,v) := (q_i, r_i) + \left(\sum_{m=1}^{n_1} u_m \cdot 3^{-m}, \sum_{m=1}^{n_2} v_m \cdot 3^{-m}\right) + \left[0, 3^{-n_1}\right] \times \left[0, 3^{-n_2}\right].$$

The collection of all level (n_1, n_2) grid rectangles in $(q_i, r_i) + I^2$ is called $\mathcal{R}_i(n_1, n_2)$. That is

$$\mathcal{R}_i(n_1, n_2) := \{R_i(u,v) : u \in U_i^{n_1} \text{ and } v \in V_i^{n_2}\}.$$

When $n_1 = n_2$ then the elements of

$$\mathcal{C}_i(n) := \mathcal{R}_i(n_1, n_2), \qquad n = n_1 = n_2$$

are called level n grid squares in $(q_i, r_i) + I^2$. *Those level n grid squares that are contained in $\widetilde{F}^n$ are called n-cylinder squares.*

For a given level n grid square we can decide if it is an n-cylinder square using the following Fact, whose proof follows immediately from the observation that all the elements (x, y) of the Sierpinski carpet F can be represented as $(x,y) = \sum_{k=1}^{\infty} \left(\frac{u_k}{3^k}, \frac{v_k}{3^k}\right)$ such that for all k either $u_k \in \{0,2\}$ or $v_k \in \{0,2\}$.

Fact 20. *The level n grid square $R_i(u,v) \in \mathcal{C}_i(n)$ with $u = q_i + \sum_{m=1}^{n} u_m \cdot 3^{-m}$ and $v = r_i + \sum_{m=1}^{n} v_m \cdot 3^{-m}$ is an n-cylinder square if and only if*

(37)
$$\forall 1 \leq p \leq n, \qquad \text{either } u_p \in \{0,2\} \text{ or } v_p \in \{0,2\}.$$

Fact 21. *If $Q_{a_1 \dots a_n}^{\ell_1 \dots \ell_n}(j) \subset (q_i, r_i) + I^2$ then the inclusion*

(38)
$$Q_{a_1 \dots a_n}^{\ell_1 \dots \ell_n}(j) \subset \widetilde{F}^n \cap ((q_i, r_i) + I^2)$$

is equivalent to the following assertion:

(Assertion): *Let* $(u_1,\dots,u_n),(v_1,\dots,v_n)\in\{0,1,2\}^n$ *be defined by*

$$\psi^{\ell_1\dots\ell_n}_{a_1\dots a_n}(q_j,r_j)=(q_i,r_i)+\frac{1}{3^n}\left(\sum_{\ell=1}^{n}3^{n-\ell}\cdot u_\ell,\sum_{\ell=1}^{n}3^{n-\ell}\cdot v_\ell\right). \tag{39}$$

Then for every $1\le p\le n$ *we have*

$$\text{either } u_p\in\{0,2\} \text{ or } v_p\in\{0,2\}. \tag{40}$$

Lemma 22. *There is* m_0 *such that* $\left\{3^{m_0}+3^{2m_0}\cdots+3^{km_0}\right\}_{k=1}^{N}$ *is a full residue system modulo* N.

Proof. Clearly we can find $k<\ell$ such that $3^k\equiv 3^\ell \mod N$. Let $m_0:=\ell-k$. Then

$$3^{m_0}\equiv 1 \mod N \tag{41}$$

holds (since we assumed that $3\not| N$). Thus $3^{m_0}+3^{2m_0}\cdots+3^{km_0}\equiv k \mod N$. □

Definition 23. **(a):** *First we define* k_0 *as the smallest non-negative integer satisfying* $M/N<3^{k_0}$. *That is if* $M/N\ge 1$ *then*

$$3^{k_0-1}\le M/N<3^{k_0}. \tag{42}$$

On the other hand if $M/N<1$ *then* $k_0:=0$.

(b): *We fix* m_0 *which satisfies (41).*

(c): *Finally, we introduce the equivalence relation* $\sim$ *on* $\{0,\dots,N-1\}$ *as follows:*

- **if** N **is odd:** *then* $k\sim\ell$ *holds for all* k,ℓ
- **if** N **is even:** *then* $k\sim\ell$ *holds iff either both* k *and* ℓ *are even or both of them are odd.*

(d): *Assume that* $M/N<1$. *We shall argue later in proving Proposition 8 that for all shapes* Q_i, $1\le i\le K$, *the region in* Q_i *with first coordinate in the interval*

$$J^n_0(i)=[q_i,q_i+(3^{m_0\cdot N+1})\cdot 3^{-n}) \text{ or } J^n_2(i)=[q_i+1-\left(3^{m_0\cdot N+1}+1\right)\cdot 3^{-n},q_i+1-3^{-n}) \tag{43}$$

contains an image of Q_j *by* $\psi^{\ell_1\dots\ell_n}_{a_1\dots a_n}$ *for an appropriately chosen* $\ell_1\dots\ell_n$.

The definition of the intervals in (43) will be much more complicated when $M/N\ge 1$.

3.2.3. *The definition of the intervals* $J^n_0(i),J^n_2(i)$ *in the general case.* First we have to place some restrictions on $(a_1,\dots,a_n)\in\{0,1,2\}^n$ for the strip $S_{a_1,\dots,a_n}$ considered. To do that we divide the set $\{1,2,\dots,n\}$ into four regions:

$$\mathcal{I}_1:=\{1,\dots,2k_0\},\qquad \mathcal{I}_2:=\{2k_0+1,\dots,n_*\},$$

where $n_* := n - (m_0N+1) - 4N3^{2k_0}$ and

$$\mathcal{I}_3 := \{n_*+1,\dots,n^*\}, \qquad \mathcal{I}_4 := \{n^*+1,\dots,n\},$$

where $n^* := n_* + 4N3^{2k_0} = n - (m_0N+1)$. When $M/N < 1$ then $k_0 = 0$ and in that case $\mathcal{I}_1 = \emptyset$.

Assumption (A1):

(a): $a_n = 0$

(b): $\forall i, \forall u \in U_i^{2k_0}$, $v \in V_i^{k_0}$ we assume that both the top left and bottom right corners of the rectangle $R_i(u,v)$ are farther from $S_{a_1\dots a_n}$ than 3^{-n_*}. This can be arranged by excluding $NM3^{3k_0} \ll 3^{n_*}$ grid intervals of level $n_*-(k_0+2)$ when selecting the level n grid interval determined by $(a_1,\dots,a_n)$ from the interval $[0,1]$. So, this is a restriction implemented by excluding some intervals whose indices are from $\mathcal{I}_1 \cup \mathcal{I}_2$ and their total length is less than $NM3^{3k_0} \cdot 2 \cdot 3^{-n_*} \cdot 3^{k_0+2}$ so now we assume that n is large enough that this is $\ll 1$.

(c): Let us denote the bottom edge of $S_{a_1\dots a_n}$ by $\mathrm{Bottom}_{a_1\dots a_n}$. See Figure 5. We define $\underline{y}^\alpha$ ($\overline{y}^\alpha$) for $1 \le \alpha \le N3^{2k_0}$ ($0 \le \alpha \le N3^{2k_0}-1$) as the second coordinate of the intersection of the line $\mathrm{Bottom}_{a_1\dots a_n}$ with the vertical line $\underline{x}^\alpha := \alpha \cdot 3^{-2k_0} - 2\cdot 3^{-(n_*+4\alpha-1)}$ ($\overline{x}^\alpha := \alpha \cdot 3^{-2k_0} + 3^{-(n_*+4\alpha+1)}$) respectively. Note that all these $\underline{x}^\alpha, \overline{x}^\alpha$ lie in $[0,N]$. We assume that $(a_1,\dots,a_n)$ is chosen in such a way that

- **c1:** for all $1 \le \alpha \le N3^{2k_0}$, both the $(n_*+4\alpha-1)$-th, and the $(n_*+4\alpha)$-th digits of the ternary expansion of $\underline{y}^\alpha$ are zero and
- **c2:** for all $0 \le \alpha \le N3^{2k_0}-1$, both the $(n_*+4\alpha+1)$-th and $(n_*+4\alpha+2)$-th digits of the ternary expansion of $\overline{y}^\alpha$ are zero.

Now we prove that there is a positive proportion (independent of n) of all possible $(a_1,\dots,a_n) \in \{0,1,2\}^n$ for which assumptions **(c1)** and **(c2)** hold. First, for having a more convenient notation we write

$b_\alpha := n_*+4\alpha-1$, $1 \le \alpha \le N3^{2k_0}$ and $f_\alpha := n_*+4\alpha+1$, $0 \le \alpha \le N3^{2k_0}-1$.

(Here f_α refers to forward and b_α refers to backward relative to $\alpha 3^{-2k_0}$ see Figure 5.)

Fact 24. *There are* 3^{n_*} *possible choices of* $(a_1,\dots,a_n) \in \{0,1,2\}^n$ *such that the following holds:*

$$0 = \underline{y}^\alpha_{b_\alpha} = \underline{y}^\alpha_{b_\alpha+1}, \qquad \forall \alpha \in \left\{1,\dots,N3^{2k_0}\right\} \text{ and}$$
$$0 = \overline{y}^\alpha_{f_\alpha} = \overline{y}^\alpha_{f_\alpha+1}, \qquad \forall \alpha \in \left\{0,\dots,N3^{2k_0}-1\right\}, \tag{44}$$

where $\underline{y}^\alpha_k$ ($\overline{y}^\alpha_k$) *is the* k*-th ternary digit of* $\underline{y}^\alpha$ ($\overline{y}^\alpha$) *respectively.*

Proof. Observe that

$$\underline{y}^{\alpha} = \sum_{k=1}^{n} a_k 3^{-k} + \underline{z}^{\alpha} \text{ and } \overline{y}^{\alpha} = \sum_{k=1}^{n} a_k 3^{-k} + \overline{z}^{\alpha}, \tag{45}$$

where

$$\underline{z}^{\alpha} := \frac{M}{N} \cdot \underline{x}^{\alpha} \text{ and } \overline{z}^{\alpha} := \frac{M}{N} \cdot \overline{x}^{\alpha}.$$

We prove that there is a way to choose the elements $a_k \in \{0,1,2\}$, for $k \in \mathcal{I}_3 \cup \mathcal{I}_4$ such that (44) holds for all possible choice of $a_k,\ k \in \mathcal{I}_1 \cup \mathcal{I}_2$. We construct these values a_k for $k \in \mathcal{I}_3 \cup \mathcal{I}_4$ by mathematical induction starting from $k = n$ and moving towards to smaller values of k. Namely, we define $a_k := 0$ for all $k \in \mathcal{I}_4$. Fix an arbitrary $k' \in \mathcal{I}_3$. We assume that we have already defined a_k for all $k' \le k \le n$. Clearly, we can either find an $1 \le \alpha' \le N3^{2k_0}$ such that $k' \in \{b_{\alpha'}, b_{\alpha'} + 1\}$ or we can find an $0 \le \alpha' \le N3^{2k_0} - 1$ such that $k' \in \{f_{\alpha'}, f_{\alpha'} + 1\}$. For symmetry without loss of generality we may assume that we are in the latter case and $k' = f_{\alpha'} + 1$. Then we compute the overflow $o_{k'}$ from the $(k'+1)$-th ternary place to the k'-th ternary place when adding up $\sum_{k=k'+1}^{n} a_k \cdot 3^{-k}$ and $\sum_{k=k'+1}^{\infty} \overline{z}_k^{\alpha'} \cdot 3^{-k}$. That is if $\sum_{k=k'+1}^{n} a_k \cdot 3^{-k} + \sum_{k=k'+1}^{\infty} \overline{z}_k^{\alpha'} \cdot 3^{-k} > 3^{-k'}$ then there is an overflow to the k'-th ternary place and then $o_{k'} := 1$ otherwise there is no overflow and $o_{k'} := 0$. Observe that the value of $o_{k'}$ depends only on the ternary digits a_k for $k' < k \le n$ (which have been determined at this stage of the mathematical induction) and $z_k^{\alpha'}$ for $k' < k$ which are given numbers. So, we can compute the number $o_{k'}$. It follows from (45) that $\overline{y}_{k'}^{\alpha} = a_{k'} + \overline{z}_{k'}^{\alpha} + o_{k'}$. Then for $a_{k'} := -(\overline{z}_{k'}^{\alpha} + o_{k'}) \bmod 3$ we obtain that $\overline{y}_{k'}^{\alpha} = 0$. We continue this process with doing the same first for $k' - 1$ then $k' - 2$ and so on for all $k \ge n_* + 1$.

□

Fact 25. *Let $J \subset [0,1]$ be a non empty interval. Whenever n is big enough we can choose $(a_1, \dots, a_n) \in \{0,1,2\}^n$ which satisfies the requirements of Assumption (**A1**) and*

$$\left[\sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{n} a_k 3^{-k} + 3^{-n}\right] \subset J. \tag{46}$$

Proof. The three parts of the assumption posed restrictions for indices in different regions, so these restrictions cannot conflict. Let G be the biggest grid interval, say level g which is contained in J. This means that we need to fix the first g ternary digits. In parts **(a)** and **(c)** of assumption **(A1)**, we fixed the last $4N3^{2k_0} + m_0 N + 1$ ternary digits of

$(a_1,\dots,a_n)$. So, from now we need to fix $g+4N3^{2k_0}+m_0N+1$ ternary digits to provide that (46) and parts **(a)**, **(c)** of assumption **(A1)** hold. In this way we restrict ourselves to a set of level n grid intervals with a total length of at least $3^{-(g+4N3^{2k_0}+m_0N+1)}$ (which does not depend on n) among which only an amount of total length of $NM3^{3k_0}\cdot 3^{-(n_*-(k_0+2))}$ (which tends to zero as $n\to\infty$) is lost for part **(b)**. So, if n is big enough then we find an $(a_1,\dots a_n)$ satisfying the assumptions **(A1)** and (46). □

The reason for part **(c)** of assumption **(A1)** is as follows:

Remark 26. *We consider $\alpha\cdot 3^{-2k_0}$ as the end point of two level $2k_0$ grid intervals:*

(47)
$$I_L(\alpha):=\left[(\alpha-1)3^{-2k_0},\alpha 3^{-2k_0}\right] \text{ and } I_R:=\left[\alpha 3^{-2k_0},(\alpha+1)3^{-2k_0}\right]$$

Assume that the corresponding ternary digits of these intervals are $(u_1^L,\dots,u_{2k_0}^L)$ and $(u_1^R,\dots,u_{2k_0}^R)$. Let

(48)
$$n_{\alpha,L}:=\#\left\{1\le \ell\le 2k_0 : u_\ell^L=1\right\},\ n_{\alpha,R}:=\#\left\{1\le \ell\le 2k_0 : u_\ell^R=1\right\}$$

Then we define the level n^* grid intervals

(49)
$$J_L(\alpha):=\begin{cases} L_1(\alpha), & \text{if } n_{\alpha,L} \text{ is odd;}\\ L_2(\alpha), & \text{otherwise.}\end{cases}\qquad J_R(\alpha):=\begin{cases} R_1(\alpha), & \text{if } n_{\alpha,L} \text{ is odd;}\\ R_0(\alpha), & \text{otherwise.}\end{cases},$$

where the level n^* grid intervals $L_1(\alpha),L_2(\alpha),R_0(\alpha),R_1(\alpha)$ are defined in Figure 5. In this way $J_L(\alpha)$ and $J_R(\alpha)$ are level n^* grid intervals contained in $I_L(\alpha)$ and $I_R(\alpha)$ respectively. For $V\in\{L,R\}$ we obtain the ternary digits of $J_V(\alpha)$ as the concatenation of $(u_1^V,\dots,u_{2k_0}^V)$ and a vector of n^*-2k_0 components of all zeros or twos if $n_{\alpha,V}$ is an even number. If $n_{\alpha,V}$ is an odd number then the ternary digits of $J_V(\alpha)$ are obtained in the same way with the difference that we have digit one in the b_α-th place (f_α-th place) if $V=L$ (V=R) respectively. In this way for both $J_L(\alpha)$ and $J_R(\alpha)$ the number of ones among the ternary digits is an even number.

Definition 27. *We say that $S_{a_1\dots a_n}$* ***is an* n*-good strip*** *if $(a_1,\dots,a_n)$ satisfies Assumption* ***(A1)****.*

From now on we fix n and an n-good strip $S_{a_1\dots a_n}$. For this strip $Q_i=((q_i,r_i)+I^2)\cap S$ is a ***relevant shape*** *if $\mathrm{int}(S_{a_1\dots a_n}\cap Q_i)\neq\emptyset$.*

We remark that, in the case when $M/N<1$, we do not use part **(c)** of Assumption **(A1)**.

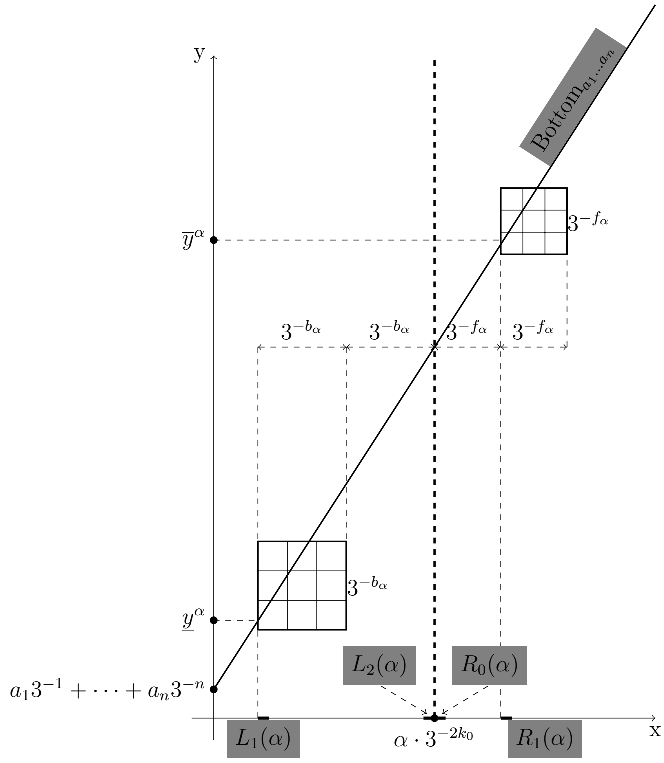

FIGURE 5. $L_1(\alpha), L_2(\alpha), R_0(\alpha), R_1(\alpha)$ are level $n^* \gg 2k_0$ grid intervals.

We recall that in Definition 23 we have already defined the intervals $J_0^n(i) = [q_i, q_i + 3^{-(n^*-1)})$ and $[q_i + 1 - 3^{-(n^*-1)} - 3^{-n}, q + 1 - 3^{-n})$. Now we extend this definition to the case when $M/N \geq 1$ and Q_i is a relevant shape for the strip $S_{a_1 \dots a_n}$. The idea of the construction is as follows: Using the notation of Definition 19, for every $1 \leq i \leq K$ we shall define, by an inductive procedure, $R_i(u^{k_0+\ell}, v^\ell)$, for $0 \leq \ell \leq k_0$ such that

(C1): $S_{a_1 \dots a_n} \cap R_i(u^{k_0+\ell}, v^\ell) \neq \emptyset$.
(C2): $u^{k_0+\ell} \in U_i^{k_0+\ell}$ and $v^\ell \in V_i^\ell$ with

$$u^{k_0+\ell} = q_i + \sum_{m=1}^{k_0+\ell} u_m \cdot 3^{-m}, \; v^\ell = r_i + \sum_{m=1}^{\ell} v_m \cdot 3^{-m}. \tag{50}$$

(C3): $u_{k_0+1}, \dots, u_{k_0+\ell} \in \{0, 2\}$ and $v_1, \dots, v_\ell \in \{0, 2\}$.

In this way we will obtain

$$R_i^* := R_i(u^{2k_0}, v^{k_0}) \in \mathcal{R}(2k_0, k_0). \tag{51}$$

It is clear by the definition that the following two assertions hold:

Remark 28. (1) *Using Fact 20, all the 3^{k_0} level $2k_0$ grid squares contained in R_i^* are in $\widetilde{F}^{2k_0}$.*

(2) *The slope of the increasing diagonal of the rectangle R_i^* is 3^{k_0} which is slightly bigger than M/N. Thus every line of slope M/N that enters the rectangle via the bottom horizontal line will leave R_i^* through its Eastern side. The rectangle R_i^* corresponds, in the simpler case $M/N < 1, k_0 = 0$, to the whole square $(q_i, r_i) + I^2$ for which every line of slope M/N that meets it must cross its Western or Eastern side.*

Definition 29. *Using the previous notation of this section especially (50) and (49) we define the intervals*

$$J_0^n(i) = J_R(u^{2k_0}3^{2k_0}), \qquad J_2^n(i) = J_L(u^{2k_0}3^{2k_0} + 1).$$

The properties of these intervals, which are summarized in the following lemma, are immediate consequences of the definition.

Lemma 30. **(a):** *The orthogonal projection of $S_{a_1 \dots a_n} \cap R_i^*$ (the rectangle R_i^* was defined in (51)) to the x-axis contains at least one of the level n^* grid intervals $J_0^n(i)$ or $J_2^n(i)$. If the line* $\mathrm{Bottom}_{a_1 \dots a_n}$ *enters the rectangle R_i^* on its Western side then $J_0^n(i)$ is such an interval. If the line* $\mathrm{Bottom}_{a_1 \dots a_n}$ *enters the rectangle R_i^* on its Southern side then $J_2^n(i)$ is such an interval.*

(b): *Using the notation of (50), by Fact 20, all the $3^{n^*-k_0}$ level n^* grid squares both in $J_0^n(i) \times [v^{k_0}, v^{k_0} + 3^{-k_0}]$ and in $J_2^n(i) \times [v^{k_0}, v^{k_0} + 3^{-k_0}]$ are contained in $\widetilde{F}^{n^*}$.*

Now we present the inductive construction of rectangle R_i^*. Fix an $1 \le i \le K$ such that Q_i is a relevant shape for $S_{a_1 \dots a_n}$. We recall that the line $\mathrm{Bottom}_{a_1 \dots a_n}$ was defined as the bottom edge of the strip $S_{a_1 \dots a_n}$. Using the notation of Definition 19 we construct a nested sequence of rectangles $\left\{R_i(u^{k_0+\ell}, v^\ell)\right\}_{\ell=0}^{k_0}$ satisfying the conditions **(C1)**, **(C2)** and **(C3)** on page 18. To construct $R_i(u^{k_0}, v^0)$ note that it follows from (42) that we can find an $0 \le m' \le 3^{k_0}$ such that the line $\mathrm{Bottom}_{a_1 \dots a_n}$ intersects the vertical line segments $\left\{(x,y) : x = q_i + m'3^{-k_0},\ q_i < y < q_i + 1\right\}$. Let us call this segment l_0 and the point where l_0 intersects $\mathrm{Bottom}_{a_1 \dots a_n}$ A_0. See Figure 6. Observe that $q_i + m'3^{-k_0}$ is the (left or right) end point of two level k_0 grid intervals. At least one of these two intervals lies in $[q_i, q_i + 1]$. We call this level k_0 grid interval I_0 (if there are two such intervals then we pick one). Note that the left end point of I_0 (which is either $q_i + m'3^{-k_0}$ or $q_i + m'3^{-k_0} - 3^{-k_0}$) is defined as $u^{k_0} = q_i + \sum\limits_{m=1}^{k_0} u_m 3^{-m}$. Let $v^0 := r_i$. Note that A_0 lies on one of the vertical sides of $\mathrm{Box}_0 := R_i(u^{k_0}, v^0)$. If the first ternary digit of the y-coordinate of A_0 is either 0 or 2 then we call it v_1 and define $\text{Box}_1 \subset \text{Box}_0$ as that 1/3 scaled copy of Box_0 which contains A_0 on one of its vertical sides. Clearly, the orthogonal projection I_1 of Box_1 to the x-axis is a level k_0+1 grid interval which has a level k_0 end point therefore the last digit, called u_{k_0+1}, of the ternary representation of I_1 is either 0 or 2. In this way we defined $R_i(u^{k_0+1}, v^1) = \text{Box}_1$. On the other hand, if the first ternary digit of the y-coordinate of A_0 is equal to 1 then the definition of u_{k_0+1}, v_1 is more complicated. Namely, in this case we define Box_1 as follows: Without lost of generality we may assume that Box_0 is on the left hand side of l_0 (as in Figure 6). In this case we define $v_1 := 0$ (otherwise we would have chosen $v_1 = 2$). We divide the bottom third part of Box_0 into three equal vertical strips corresponding to the level k_0+1 grid intervals contained in I_0 (Figure 6). It follows from (42) that one of the vertical sides of one of these three $3^{-(k_0+1)} \times 3^{-1}$ rectangles, which is different from the middle one, intersects $\text{Bottom}_{a_1\dots a_n}$. Let us call this point A_1 and the non-middle positioned $3^{-(k_0+1)} \times 3^{-1}$ grid rectangle which contains A_1 on one of its vertical sides is called Box_1 and the projection of Box_1 to the x-axis is called I_1. Clearly, $I_1 \subset I_0$ is a level 3^{k_0+1} grid interval and its k_0+1 ternary digit is different from 1. This follows from the non-middle position of Box_1 as mentioned above. So, the rectangle $R_i(u^{k_0+1}, v^1) := \text{Box}_1$ satisfies the requirements **(C1)**-**(C3)** on page 18. We continue the construction $R_i(u^{k_0+\ell}, v^\ell) := \text{Box}_\ell$ for all $\ell \leq k_0$ exactly in the same way.

3.2.4. *The proof of Proposition 18.* Level n shapes are labeled using $N \cdot \ell_{1,n}$ and which of these appear in $\tilde{F}^n$ is affected by $u_{1,n}$ so we compare these, using (33) and Fact 21, in the following lemma, which is the key step in proving Proposition 18.

Lemma 31. *Fix any $i, j \in \{1, \dots, K\}$ satisfying $q_i \sim q_j$. Let $n \geq m_0 \cdot N + 1$. Assume that we are given*

$$u_1, \dots, u_{n-(m_0 N+1)} \in \{0,1,2\} \tag{52}$$

in such a way that

$$u_1 + \dots + u_{n-(m_0 N+1)} \text{ is an even number.} \tag{53}$$

We can choose $u_{n-m_0 N}, \dots, u_n \in \{0,1,2\}$ and $(\ell_1, \dots, \ell_n) \in \{0,1,2\}^n$ such that

$$q_j + N \cdot \ell_{1,n} = 3^n \cdot q_i + \sum_{k=1}^{n} 3^{n-k} \cdot u_k. \tag{54}$$

Proof. Clearly we can choose $(\ell'_1, \dots, \ell'_n) \in \{0,1,2\}^n$ such that for

$$p := N(\ell'_1 \cdot 3^{n-1} + \dots + \ell'_n) - 3^n \cdot q_i + q_j - \left(3^{n-1} \cdot u_1 + \dots + 3^{m_0 \cdot N+1} \cdot u_{n-(m_0 \cdot N+1)}\right)$$

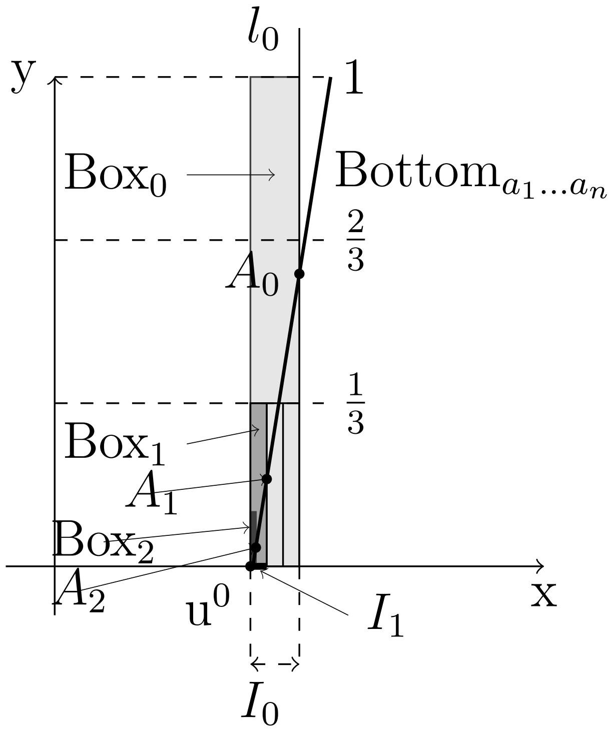

FIGURE 6. The inductive definition of Box_ℓ.

we have

$$0 \le p \le N - 1. \tag{55}$$

Now we distinguish two cases based on the parity of N.

N **is odd:** Then it follows from (41) that

$$\left\{2 \cdot \left(3^{m_0} + 3^{2m_0} + \cdots + 3^{km_0}\right)\right\}_{k=1}^{N}$$

is a complete residue system modulo N since $\{2k\}_{k=1}^{N}$ is such a system. So we can find integers $1 \le k \le N$ and $v \in \mathbb{N}$ such that

$$2\left(3^{m_0} + \cdots + 3^{km_0}\right) = v \cdot N + p. \tag{56}$$

Choose $(\ell_1, \ldots, \ell_n) \in \{0, 1, 2\}^n$ such that

$$N(\ell_1' \cdot 3^{n-1} + \cdots + \ell_n') + vN = N(\ell_1 \cdot 3^{n-1} + \cdots + \ell_n).$$

Then

(57)
$$N(\ell_1 \cdot 3^{n-1} + \cdots + \ell_n) - 3^n \cdot q_i + q_j - \left(3^{n-1} \cdot u_1 + \cdots + 3^{m_0 \cdot N+1} \cdot u_{n-(m_0 \cdot N+1)}\right)$$
$$= v \cdot N + p = 2\left(3^{m_0} + \cdots + 3^{km_0}\right),$$

which immediately implies the assertion of the lemma.

N **is even:** In this case

$$\left\{2 \cdot \left(3^{m_0} + 3^{2m_0} + \cdots + 3^{km_0}\right)\right\}_{k=1}^{N}$$

is not a complete residue system but contains (actually twice) all the even number residues. On the other hand, using $q_i \sim q_j$ and (53), we see that p is an even number. This follows from (49) and Definition 29. So, as above, we can find integers $1 \leq k \leq N$ and $v \in \mathbb{N}$ such that (56) holds. The rest of the proof is the same as in the case when N is odd.

□

This implies

Corollary 32. *Fix an arbitrary $(a_1, \ldots, a_n) \in \{0, 1, 2\}^n$ and also fix $1 \leq i, j \leq K$ such that $q_i \sim q_j$. For any $u \in \{0, 2\}$ we can find $(\ell_1, \ldots, \ell_n) \in \{0, 1, 2\}^n$ such that for the first component function $(\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n})_1$ of $\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}$ we have*

- **(a):** $(\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n})_1(q_j) \in J_u^n(i)$ *and*
- **(b):** $Q_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(j) \subset \widetilde{F}^n$.

Proof. Using Fact 21 and Lemma 31 the proof immediately follows from the observation that for $u = 0, 2$ choosing $(\ell_1, \ldots, \ell_n)$ as in Lemma 31 we get that the first coordinate of $\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(q_j, r_j)$ is

$$\begin{aligned}(\psi_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(q_j, r_j))_1 &= \frac{1}{3^n}\left(q_j + N(\ell_1 \cdot 3^{n-1} + \cdots + \ell_n)\right) \\ &= q_i + u \cdot (3^{-1} + \cdots + 3^{m_0 \cdot N + 1 - n}) + 2 \cdot 3^{-n}\left(3^{m_0} + \cdots + 3^{km_0}\right),\end{aligned}$$

where $0 \leq k \leq N - 1$. □

Lemma 33. *Assume that $S_{a_1 \ldots a_n}$ is an n-good strip. For each $1 \leq i, j \leq K$ let $\mathbf{r}_i$ be the i-th row vector of the matrix $A_{a_1 \ldots a_n}$ and write $\mathbf{r}_i(j)$ for the j-th element of $\mathbf{r}_i$. Then*

$$\mathbf{r}_i \neq \mathbf{0} \text{ and } q_j \sim q_i \text{ imply } A_{a_1 \ldots a_n}(i, j) = \mathbf{r}_i(j) > 0. \tag{58}$$

Proof. Recall

$$\mathbf{r}_i(j) = \#\left\{(\ell_1, \ldots, \ell_n) \in \{0, 1, 2\}^n \mid Q_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(j) \subset Q_i \cap \widetilde{F}^n\right\}. \tag{59}$$

Assume that $\text{int}(Q_i \cap S_{a_1 \ldots a_n}) \neq \emptyset$. It is enough to prove that

(60)

$$j \leq K,\ q_i \sim q_j \Rightarrow \exists (\ell_1, \ldots, \ell_n) \in \{0, 1, 2\}^n \text{ s. t. } Q_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(j) \subset Q_i \cap \widetilde{F}^n.$$

Namely, by definition,

$$Q_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n}(j) \cap S_{a_1 \ldots a_n}^{\ell_1 \ldots \ell_n} \subset S_{a_1 \ldots a_n}.$$

To verify (60) we fix $j \leq K$ such that $q_i \sim q_j$. Let $u \in \{0,2\}$ be chosen such that assumption (**A2**) holds for $J_u^n(i)$. For this u, i, j and $(a_1, \dots, a_n)$ we choose an $(\ell_1, \dots, \ell_n) \in \{0,1,2\}^n$ which satisfies Corollary 32. This implies that we have $Q_{a_1 \dots a_n}^{\ell_1 \dots \ell_n}(j) \subset Q_i \cap \widetilde{F}^n$. Namely, $\left(\psi_{a_1 \dots a_n}^{\ell_1 \dots \ell_n}\right)_1 (q_j) \in J_k(i)$. $\square$

Corollary 34. *We assume that $S_{a_1 \dots a_n}$ is an n-good strip and we write $\mathbf{r}_i$ for the i-th row vector of the matrix $A_{a_1 \dots a_n}$. Assume that $\mathbf{r}_i \neq \mathbf{0}$.*

- **(a):** *If N is odd then all the elements of $\mathbf{r}_i$ are positive,*
- **(b):** *If N is even then for all j satisfying $q_j \sim q_i$, we have $\mathbf{r}_i(j) > 0$.*

Note that for a shape Q_i which is relevant for the n-good strip $S_{a_1 \dots a_n}$ we have $\mathbf{r}_i \neq \mathbf{0}$.

3.3. **The case when N is an even number.** The above argument shows that in the case when $3 \nmid N$ we can find $a_1, \dots, a_n$ such that all the rows of some matrices $A_{a_1 \dots a_n}$ are either all-positive or all-zero. Now we would like to add to Corollary 34 (b) and prove the same in the case when N is even.

In this section we always assume that N is even. We fix n which is large enough. (It will be specified later how large n has to be.) We always assume that $\mathbf{a} = (a_1, \dots, a_n) \in \{0,1,2\}^n$ is chosen in such a way that $a_n = 0$ and $S_{a_1 \dots a_n}$ is an n-good strip. For every $q \in \{0, \dots, N-1\}$ we can find a unique $i = i(q, \mathbf{a}) \in \{1, \dots, K\}$ such that

- **(a):** $q = q_i$
- **(b):** Q_i is a relevant shape for $S_{a_1 \dots a_n}$.
- **(c):** for $J(q) := \left[q + \frac{1}{3^n}, q + \frac{2}{3^n}\right]$ we have

(61)
$$J(q) \subset J_0(i) \subset (q, q+1) \cap \pi_1\left(e_1(i, a_1, \dots, a_n)\right) \cap \pi_1(e_2(i, a_1, \dots, a_n)),$$
where π_1 is the projection to the first axis.

Observe that

(62)
$$J(q) = q + \left[\sum_{k=1}^{n} u_k 3^{-k}, \sum_{k=1}^{n} u_k 3^{-k} + 3^{-n}\right] \text{ where } u_1 = \dots = u_{n-1} = 0, u_n = 1.$$

One of the motivations to consider the intervals $J(q)$ is as follows.

Fact 35. *For some $0 \leq q \leq N-1$ let $C = J(q) \times J'$, where*

$$J' := m + \left[\sum_{k=1}^{n} v_k 3^{-k}, \sum_{k=1}^{n} v_k 3^{-k} + \frac{1}{3^n}\right], \tag{63}$$

for some $0 \leq m \leq M-1$. *Then*

$$C \subset (q,m) + F^n \text{ if and only if } v_n \neq 1, \tag{64}$$

where F^n *is the* n*-th approximation of* F.

The proof of this Fact is an immediate corollary of Fact 21 and (62).

Given n and q define $\ell_1, \dots, \ell_n \in \{0,1,2\}^n$ as follows

$$\left\lfloor \frac{3^n q + 1}{N} \right\rfloor = \ell_1 \cdot 3^{n-1} + \cdots + \ell_{n-1} \cdot 3 + \ell_n.$$

Then we have

$$J(q) \subset \left[N \cdot \left(\frac{\ell_1}{3} + \cdots + \frac{\ell_n}{3^n} \right), N \cdot \left(\frac{\ell_1}{3} + \cdots + \frac{\ell_n}{3^n} + \frac{1}{3^n} \right) \right]. \tag{65}$$

Thus there is a unique $z_1 = z_1(q) \in \{0, \dots, N-1\}$ such that

$$z_1(q) := 3^n \cdot q + 1 - N \cdot 3^n \cdot \left(\frac{\ell_1}{3} + \cdots + \frac{\ell_n}{3^n} \right). \tag{66}$$

(Thus $z_1(q)$ is the number of the interval $J(q)$ when we count modulo N the horizontal intervals of width 3^{-n}.) It is easy to see that $q \to z_1(q)$ is a bijection on $\{0, \dots N-1\}$. Since N is even we obtain

$$\forall q \in \{0, \dots, N-1\}, \qquad q \not\sim z_1(q). \tag{67}$$

Further, since $3 \nmid N$, the map $z_1 : \{0, \dots, N-1\} \to \{0, \dots, N-1\}$ is a bijection. Now we define $z_2 := z_2(q) \in \{0, \dots, M-1\}$ as follows:

(68)

$$z_2(q) := \min \left\{ k \mid \left((z_1(q), k) + I^2 \right) \cap \operatorname{int}(S) \neq \emptyset,\ k + M \cdot \ell_n(q) \not\equiv 1 \bmod 3 \right\}.$$

We write

$$C(q, \mathbf{a}) := \psi_{a_1 \dots a_n}^{\ell_1(q) \dots \ell_n(q)}((z_1(q), z_2(q)) + I^2). \tag{69}$$

Using Fact 21 and $a_n = 0$, by (33) we obtain

$$C(q, \mathbf{a}) \subset Q_{i(q,\mathbf{a})} \cap \widetilde{F}^n \text{ and } C(q, \mathbf{a}) \cap S_{a_1 \dots a_n} \neq \emptyset. \tag{70}$$

Let $Q(q) := ((z_1(q), z_2(q)) + I^2) \bigcap S$. Note that we could choose $Q(q)$ independent of $\mathbf{a}$ only because we always assume here that we restrict our attention to those n-good strips $S_{a_1 \dots a_n}$ for which $a_n = 0$. Then

$$\psi_{a_1 \dots a_n}^{\ell_1 \dots \ell_n}(Q(q)) = C(q, \mathbf{a}) \cap S_{a_1 \dots a_n}. \tag{71}$$

Summarizing what we have proved above:

Lemma 36. *Assuming that* $S_{a_1 \dots a_n}$ *is an* n*-good strip and* $a_n = 0$*, for every* $m \in \{0, \dots, N-1\}$ *we can find* $1 \leq i, j \leq K$ *such that* $q_i \not\sim q_j$*,* $q_j = m$ *and* $A_{a_1 \dots a_n}(i,j) > 0$.

Proof. With $q := z_1^{-1}(m)$, $i := i(q, \mathbf{a})$ and $Q_j := Q(q)$ the assertion of the lemma follows. □

Lemma 37. *We can find n and $\mathbf{a} = (a_1, \ldots, a_n) \in \{0,1,2\}^n$ with $a_n = 0$ and $q' \not\sim q'' \in \{0, \ldots, N-1\}$ such that $S_{a_1,\ldots,a_n}$ is an n-good strip and both $Q(q')$ and $Q(q'')$ are relevant shapes for $S_{a_1,\ldots,a_n}$.*

Proof. First suppose that $M/N \geq 1$. Then $Q(z_1^{-1}(0))$ is either $Q_1 = I^2 \cap S$ or $Q_2 = ((0,1) + I^2) \cap S$. Each is relevant for *any* n-good strip $S_{a_1,\ldots,a_n}$. By Fact Fact 25 we can now choose any n-good strip $S_{a_1,\ldots,a_n}$ with $a_n = 0$ for which $Q(z_1^{-1}(1))$ is relevant and it is also relevant for $Q(z_1^{-1}(0))$. As $0 \not\sim 1$ we have $z_1^{-1}(0) \not\sim z_1^{-1}(1)$ as required.

Now suppose that $M/N < 1$. Then $Q(z_1^{-1}(0))$ is either Q_1 or Q_2. Clearly, if $Q(z_1^{-1}(0)) = Q_1$ then whichever way we choose $Q(z_1^{-1}(1)) \subset ([1,2] \times \mathbb{R}) \cap S$ we can find a narrow enough strip of slope M/N through this shape and Q_1. This would immediately give us by Fact 25 that $q' = z_1^{-1}(0),\ q'' = z_1^{-1}(1)$ satisfy the requirements of the lemma. So, we may assume that $Q(z_1^{-1}(0)) = Q_2$. By symmetry, for the same reason we can assume that $Q(z_1^{-1}(N-1)) = Q_{K-1} = ((N-1, M-1) + I^2) \cap S$. If $M/N > 1/2$ then $M/N < (M-1)/(N-1)$ and so we can find a narrow enough strip of slope M/N which traverses both Q_2 and Q_{K-1}. So, by Fact 25, we can choose $q' = z_1^{-1}(0),\ q'' = z_1^{-1}(N-1)$.

Now assume that $0 < M/N < 1/2$. First suppose that

$$Q(z_1^{-1}(1)) = Q_3 = ((1,0) + I^2) \cap S \text{ and } Q(z_1^{-1}(N-2)) = Q_{K-2} = ((N-2, M) + I^2) \cap S. \tag{72}$$

Then by elementary geometry one can find a narrow strip which intersects the interior of both Q_3 and Q_{K-2}. So, by Fact 25, we can choose $q' = z_1^{-1}(1),\ q'' = z_1^{-1}(N-2)$.

If (72) does not hold then either

$$Q(z_1^{-1}(1)) = Q_4 = ((1,1) + I^2) \cap S \tag{73}$$

or

$$Q(z_1^{-1}(N-2)) = Q_{K-3} = ((N-2, M-1) + I^2) \cap S. \tag{74}$$

If (73) holds we put $q' = z_1^{-1}(0),\ q'' = z_1^{-1}(1)$ and if (74) holds we put $q' = z_1^{-1}(N-2),\ q'' = z_1^{-1}(N-1)$. □

Now we are ready to prove Proposition 18.

Proof of Proposition 18. If N is odd then the assertion of the Proposition follows from Lemma 33. So, may assume that N is even. Let us fix n and $\widetilde{\mathbf{a}} = (\widetilde{a}_1, \ldots, \widetilde{a}_n)$, and q', q'' whose existence is guaranteed by Lemma 37. Using the notation of Lemma 36 we substitute $m = q'$ to define i', j' as in Lemma 36. Further, we also substitute $m = q''$ in

Lemma 36 to define i'', j''. As a shorthand notation we write in this proof

$$A := A_{\widetilde{a}_1,\dots,\widetilde{a}_n}.$$

Then by definition we have

$$A(i',j'), A(i'',j'') > 0. \tag{75}$$

Without loss of generality may assume that

$$q' = q_{j'} \text{ and } q_{i''} \text{ are even and } q'' = q_{j''} \text{ and } q_{i'} \text{ are odd.}$$

We will use the following observation: it follows from Corollary 34 **(b)** that for any $1 \leq k, \ell \leq K$ we have

$$A^2(k,\ell) \geq A(k,k) \cdot A(k,\ell) \geq A(k,\ell), \tag{76}$$

since all the non-zero entries of A are at least one. First we prove that

$$A^2(i',k) > 0 \text{ and } A^2(i'',k) > 0 \text{ for every } 1 \leq k \leq K. \tag{77}$$

To see that the first inequality holds observe that it follows from (75) and Corollary 34 **(b)** that whenever q_k is even we have

$$A^2(i',k) \geq A(i',j') \cdot A(j',k) > 0.$$

Similarly if q_k is an odd number then using (77) and Corollary 34 **(b)** we get

$$A^2(i',k) \geq A(i',k) > 0,$$

which completes the proof of the first half of (77). By symmetry, the second inequality in (77) can be proved in the same way.

Now let $u \in \{1,\dots,K\}$ be arbitrary such that the u-th row of A is a non-all-zero row. To prove Proposition 18 it is enough to show that for every $v \in \{1,\dots,K\}$ we have

$$A^3(u,v) > 0. \tag{78}$$

If q_u, q_v have the same parity then this follows from Corollary 34 **(b)** and (76). If q_u and q_v have different parity then without loss of generality we assume that q_u is odd. Since we assumed that $q_{i'}$ is also odd we get from Corollary 34 **(b)** that $A(u,i') > 0$. So, (77) yields that

$$A^3(u,v) \geq A(u,i') \cdot A^2(i',v) > 0. \tag{79}$$

Thus we have verified Proposition 18 with $n_0 := 3n$ and

$$(a_1,\dots,a_{n_0}) = (\widetilde{a}_1,\dots,\widetilde{a}_n,\widetilde{a}_1,\dots,\widetilde{a}_n,\widetilde{a}_1,\dots,\widetilde{a}_n).$$

□

3.4. **A corollary of the Perron-Frobenius theorem.** We fix n_0 and $A_{a_1\dots a_{n_0}}$ which satisfies Proposition 18. We consider the matrices

$$\mathcal{B} := \left\{A_{i_1\dots i_{n_0}}\right\}_{(i_1\dots i_{n_0})\in\{0,1,2\}^{n_0}}.$$

We write $T := 3^{n_0}$ and $\mathcal{B} = \{B_1,\dots,B_T\}$, where

$$B_1 := A_{a_1\dots a_{n_0}}.$$

Put

$$B_s := \sum_{k=1}^{T} B_k.$$

Since

$$B_s = \sum_{i_1\dots i_{n_0}} A_{i_1\dots i_{n_0}} = (A_0 + A_1 + A_2)^{n_0}.$$

using (18) one immediately gets that

$$B_s \text{ has each column sum } 8^{n_0}. \tag{80}$$

Further,

$$\forall 1 \le k \le T, \quad \text{each column sum of } B_k \ \in \left[2^{n_0}, 3^{n_0}\right]. \tag{81}$$

In particular all the matrices B_k are column allowable (every column contains a non-zero element) non-negative integer matrices.

We define the Lyapunov exponent for the random product of the matrices $\{B_i\}_{i=1}^{T}$, where for each i in every step we choose B_i independently with probability $1/T$. Using (20) the Lyapunov exponent

$$\gamma_B := \lim_{n\to\infty} \frac{1}{n} \log \|B_{i_1\dots i_n}\|_1, \text{ for a.a. } (i_1, i_2, \dots). \tag{82}$$

Then clearly we have

$$\gamma = \frac{1}{n_0} \cdot \gamma_B \tag{83}$$

where we recall that $T = 3^{n_0}$. Note that it follows from (21) that

$$\lim_{k\to\infty} \frac{1}{k} \cdot \sum_{j_1\dots j_k} \frac{1}{T^k} \cdot \log \|B_{j_1\dots j_k}\|_1 = \gamma_B. \tag{84}$$

Let $\widehat{B}_1$ be the matrix that we obtain from B_1 when we replace all the column vectors of B_1 that correspond to an all-zero row by all-zero columns. That is if the column and row vectors of B_1 are

$$B_1 = [\mathbf{c}_1,\dots,\mathbf{c}_K] \text{ and } B_1 = [\mathbf{r}_1,\dots,\mathbf{r}_K]$$

then the matrix $\widehat{B}_1$ is defined by its column vectors as follows:

$$\widehat{B}_1 = [\mathbf{c}_1^*,\dots,\mathbf{c}_K^*], \tag{85}$$

where

$$\mathbf{c}_i^* = \begin{cases} \mathbf{c}_i, & \text{if } \mathbf{r}_i > \mathbf{0}; \\ \mathbf{0}, & \text{if } \mathbf{r}_i = \mathbf{0}. \end{cases}$$

Note that for any k we have

$$B_1^{k+1} = \widehat{B}_1^k \cdot B_1. \tag{86}$$

Let

$$\ell := \#\left\{i : \mathbf{c}_i > \mathbf{0}\right\}. \tag{87}$$

We choose an orthogonal matrix Q that corresponds to a change of the order of the basis vectors in the natural basis such that

$$Q \cdot \widehat{B}_1 \cdot Q^T = \left[\begin{array}{c|c} C & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], \tag{88}$$

where C is a positive (all the elements are positive) $\ell \times \ell$ matrix. It follows from the Perron-Frobenius theorem [1, p. 185], [17, p. 9] that for the leading eigenvalue ρ and normalized left and right leading eigenvectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\ell$ of C we have

(89)

$$\mathbf{u}^T \cdot C = \rho \cdot \mathbf{u}^T,\ C \cdot \mathbf{v} = \rho \cdot \mathbf{v},\ \mathbf{u} > \mathbf{0},\ \mathbf{v} > \mathbf{0}, \sum_{i=1}^{\ell} u_i \cdot v_i = 1, \sum_{i=1}^{\ell} v_i = 1.$$

Furthermore, there exists $0 < \rho_0 < \rho$ and $\ell \times \ell$ matrices R^k such that for the $\ell \times \ell$ matrix

$$P := \mathbf{v} \cdot \mathbf{u}^T$$

we have

$$\forall k \qquad C^k = \rho^k \cdot P + R^k, \tag{90}$$

where for all $1 \le i, j \le K$ the (i,j)-th element $r_{i,j}^{(k)}$ of the matrix R^k satisfies

$$|r_{i,j}^{(k)}| < c_1 \cdot \rho_0^k, \tag{91}$$

for some constant $c_1 > 0$. Sometimes we will have to extend vectors defined in $\mathbb{R}^\ell$ to $\mathbb{R}^K$. We will do this in two different ways: namely, for $\mathbf{a} \in \mathbb{R}^\ell$ we write

$$\mathbf{a}_* := \left[\begin{array}{c} \mathbf{a} \\ \hline \mathbf{0} \end{array}\right] \text{ and } \mathbf{a}^* := Q^{-1}\mathbf{a}_*. \tag{92}$$

The meaning of $\mathbf{a}^*$ is as follows: to provide that all the ℓ positive rows of B_1 become the first ℓ rows we needed to use a permutation of the basis vectors of the natural basis. This permutation was provided by multiplying by the orthogonal matrix Q on the left. To permute all the coordinates to the original order we have to multiply by Q^{-1} on the left.

Remark 38. **(a):** *In the following definition we use the fact that by the definition of Q the last $K-\ell$ rows of the matrix $Q \cdot B_1 \cdot Q^T$ are all-zero rows and the first ℓ rows are all-positive rows. That is there exists an $\ell \times K$ matrix D such that*

$$Q \cdot B_1 \cdot Q^T = \left[\frac{D}{\mathbf{0}}\right] \text{ and } D = [C|E] \text{ with } E > \mathbf{0}. \tag{93}$$

(b): *By the definition of Q we have*

(94)
$$B_1 Q^{-1} \mathbf{v}_* = Q^{-1}((C \cdot \mathbf{v})_*) = Q^{-1} \rho \mathbf{v}_* = Q^{-1} \left[\frac{C|E}{\mathbf{0}}\right] \mathbf{v}_* = \rho Q^{-1} \mathbf{v}_*.$$

Note that Q is an orthogonal matrix that corresponds to some change of the order in the natural basis. Hence $Q \geq \mathbf{0}$. In this way we see that ρ is an eigenvalue of the matrix B_1. Furthermore, $\mathbf{v}^* = Q^{-1}\mathbf{v}_*$ is a non-negative eigenvector of the eigenvalue ρ.

Definition 39. *For every n and for every $(b_1, \dots, b_n) \in \{0,1,2\}^n$:*

(a): *We write $E_{b_1 \dots b_n}$ for the $\ell \times K$ matrix which satisfies*

$$\left[\frac{E_{b_1 \dots b_n}}{\mathbf{0}}\right] = Q \cdot B_1 \cdot Q^T \cdot Q \cdot B_{b_1 \dots b_n} \cdot Q^T = Q \cdot B_1 \cdot B_{b_1 \dots b_n} \cdot Q^T.$$

(b): *We define the positive vector of K components*

$$\mathbf{u}^T_{b_1 \dots b_n} := \mathbf{u}^T \cdot E_{b_1 \dots b_n}.$$

(c): *For $\varepsilon \in \mathbb{R}$ we define the (column) vector $\boldsymbol{\varepsilon} \in \mathbb{R}^\ell$ as a vector with all components ε.*

Lemma 40. *Let $0 < \varepsilon_1 < \min_{1 \leq i \leq \ell} v_i$. Then there exists k_0 such that for every n and for every $(b_1, \dots, b_n) \in \{0,1,2\}^n$ we have*

(95)
$$\rho^{k_0} \cdot \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon_1}\right) \cdot \mathbf{u}^T_{b_1 \dots b_n} < C^{k_0} \cdot E_{b_1 \dots b_n} < \rho^{k_0} \cdot \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon_1}\right) \cdot \mathbf{u}^T_{b_1 \dots b_n}.$$

Proof. The proof follows from (91). Namely,

(96)
$$k_0 := k_0(\varepsilon_1) \text{ is defined as } k_0 := \min \left\{ k : c_1 \cdot \left\{\frac{\rho_0}{\rho}\right\}^k < \frac{\varepsilon_1}{10} \cdot \min_{1 \leq i \leq \ell} u_i \right\}.$$

Then by (91) we have $\frac{1}{\rho^{k_0}} \cdot |r^{(k_0)}_{i,j}| < \frac{\varepsilon_1}{10} \cdot \min u_i$. Hence

$$\rho^{k_0}\left(\mathbf{v} \cdot \mathbf{u}^T - \frac{1}{10}\,\boldsymbol{\varepsilon_1} \cdot \mathbf{u}^T\right) < C^{k_0} < \rho^{k_0}\left(\mathbf{v} \cdot \mathbf{u}^T + \frac{1}{10}\,\boldsymbol{\varepsilon_1} \cdot \mathbf{u}^T\right).$$

This completes the proof of the lemma because all the elements of all the matrices and vectors in the last inequality as well as in the assertion of the lemma are nonnegative. □

Lemma 41. *There is* $(i_1, \ldots, i_{n_0}) \in \{0,1,2\}^{n_0}$ *such that*

$$\|A_{a_1\ldots a_{n_0}} \cdot \mathbf{v}^*\|_1 \neq \|A_{i_1\ldots i_{n_0}} \cdot \mathbf{v}^*\|_1, \tag{97}$$

where we recall that $\mathbf{v}^*$ *was defined by the convention introduced in (92).*

Proof. To get a contradiction we assume that

$$\forall (i_1, \ldots, i_{n_0}) \in \{0,1,2\}^{n_0}, \ \|A_{a_1\ldots a_{n_0}} \cdot \mathbf{v}^*\|_1 = \|A_{i_1\ldots i_{n_0}} \cdot \mathbf{v}^*\|_1. \tag{98}$$

Then

$$\begin{aligned} 3^{n_0} \cdot \rho &= 3^{n_0} \cdot \|\rho \cdot \mathbf{v}^*\|_1 = 3^{n_0} \cdot \|B_1 \cdot \mathbf{v}^*\|_1 \\ &= \sum_{(i_1,\ldots,i_{n_0})} \|A_{i_1\ldots i_{n_0}} \cdot \mathbf{v}^*\|_1 = \left\| \sum_{(i_1,\ldots,i_{n_0})} A_{i_1\ldots i_{n_0}} \cdot \mathbf{v}^* \right\|_1 \\ &= \|A_s^{n_0} \cdot \mathbf{v}^*\|_1 = 8^{n_0}\|\mathbf{v}^*\|_1 = 8^{n_0} \end{aligned}$$

so $\rho = 8^{n_0}/3^{n_0}$. However this is impossible since $8^{n_0}/3^{n_0}$ cannot be a root of the characteristic polynomial of B_1 which is a matrix of integer coefficients. □

We assume from now on that we numbered the elements of $\mathcal{B}$ such that

$$\|B_1 \cdot \mathbf{v}\| \neq \|B_2 \cdot \mathbf{v}\|. \tag{99}$$

Without loss of generality we may assume that $\|B_1 \cdot \mathbf{v}\| < \|B_2 \cdot \mathbf{v}\|$.

Definition 42. *Let us define* $0 < \varepsilon_1 < \frac{1}{20} \cdot \min v_i$ *and* $c_0 > 0$ *such that*

$$\frac{\|B_1 \cdot (\mathbf{v} + 10\ \boldsymbol{\varepsilon_1})^*\|_1}{\|(\mathbf{v} - 10\ \boldsymbol{\varepsilon_1})\|_1} + c_0 < \frac{\|B_2 \cdot (\mathbf{v} - 10\ \boldsymbol{\varepsilon_1})^*\|_1}{\|(\mathbf{v} + 10\ \boldsymbol{\varepsilon_1})\|_1}. \tag{100}$$

Lemma 43. *Here we use the notation of Lemma 40. For an arbitrary* $(b_1, \ldots, b_n) \in \{0,1,2\}^n$ *we define*

$$F_{b_1\ldots b_n} := B_1^{k_0+1} \cdot B_{b_1\ldots b_n}.$$

Then for an arbitrary n *and for arbitrary* $(b_1, \ldots, b_n) \in \{0,1,2\}^n$ *we have*

$$\frac{\|B_1 \cdot F_{b_1\ldots b_n}\|_1}{\|F_{b_1\ldots b_n}\|_1} + c_0 < \frac{\|B_2 \cdot F_{b_1\ldots b_n}\|_1}{\|F_{b_1\ldots b_n}\|_1}, \tag{101}$$

where $\|A\|_1$ *means the sum of the modulus of the elements of the matrix.*

Proof. First we start with a simple observation that we will use at the end of this argument. Let $\mathbf{a}, \mathbf{b} \geq 0$ be vectors of K components.

$$\text{If } A = \mathbf{a} \cdot \mathbf{b}^T \text{ then } \|A\|_1 = \|\mathbf{a}\|_1 \cdot \|\mathbf{b}^T\|_1. \tag{102}$$

Because QAQ^T is obtained from A by permuting the rows and columns

$$\|A\|_1 = \|Q \cdot A \cdot Q^T\|_1.$$

Thus in order to verify (101) it is enough to estimate the ratio of the norms

$$\|Q \cdot B_j \cdot F_{b_1\dots b_n} \cdot Q^T\|_1 \text{ for } j=1,2 \text{ and } \|Q \cdot F_{b_1\dots b_n} \cdot Q^T\|_1.$$

Using (88), (86) and Definition 39 **(a)** we obtain

$$Q \cdot F_{b_1\dots b_n} \cdot Q^T = \left[\begin{array}{c|c} C^{k_0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \cdot \left[\begin{array}{c} E_{b_1\dots b_n} \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c} C^{k_0} \cdot E_{b_1\dots b_n} \\ \hline \mathbf{0} \end{array}\right]. \tag{103}$$

Lemma 40 asserts that

$$\rho^{k_0} \cdot \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right) \cdot \mathbf{u}^T_{b_1\dots b_n} < C^{k_0} \cdot E_{b_1\dots b_n} < \rho^{k_0} \cdot \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right) \cdot \mathbf{u}^T_{b_1\dots b_n},$$

where we recall that $\mathbf{u}^T_{b_1\dots b_n} = \mathbf{u}^T \cdot E_{b_1\dots b_n}$. Using (102) this implies that

$$\rho^{k_0} \cdot \left\| \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right) \right\|_1 \cdot \left\|\mathbf{u}^T_{b_1\dots b_n}\right\|_1 \le \|F_{b_1\dots b_n}\|_1 \le \rho^{k_0} \cdot \left\| \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right) \right\|_1 \cdot \left\|\mathbf{u}^T_{b_1\dots b_n}\right\|_1. \tag{104}$$

Now we estimate

$$\|Q \cdot B_j \cdot F_{b_1\dots b_n} \cdot Q^T\|_1 = \|Q \cdot B_j \cdot Q^T \cdot Q \cdot F_{b_1\dots b_n} \cdot Q^T\|_1 \qquad j=1,2.$$

Also from the assertion of Lemma 40 we obtain that

$$\rho^{k_0} \cdot \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^* \cdot \mathbf{u}^T_{b_1\dots b_n} \le Q^T \cdot Q \cdot F_{b_1\dots b_n} \cdot Q^T \le \rho^{k_0} \cdot \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^* \cdot \mathbf{u}^T_{b_1\dots b_n}. \tag{105}$$

Using (102) again we get on the one hand

$$\|Q \cdot B_j \cdot F_{b_1\dots b_n} \cdot Q^T\|_1 \le \rho^{k_0} \cdot \left\| B_j \cdot \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^* \right\|_1 \cdot \left\|\mathbf{u}^T_{b_1\dots b_n}\right\|_1 \tag{106}$$

and on the other hand

$$\|Q \cdot B_j \cdot F_{b_1\dots b_n} \cdot Q^T\|_1 \ge \rho^{k_0} \cdot \left\| B_j \cdot \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^* \right\|_1 \cdot \left\|\mathbf{u}^T_{b_1\dots b_n}\right\|_1. \tag{107}$$

Putting together these last two inequalities with (104) the assertion of the Lemma immediately follows from (100). Namely, for $j=1,2$

$$\frac{\left\|B_j \cdot \left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^*\right\|_1}{\left\|\left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)\right\|_1} \le \frac{\|B_j \cdot F_{b_1\dots b_n}\|_1}{\|F_{b_1\dots b_n}\|_1} \le \frac{\left\|B_j \cdot \left(\mathbf{v} + \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)^*\right\|_1}{\left\|\left(\mathbf{v} - \frac{1}{10}\,\boldsymbol{\varepsilon}_\mathbf{1}\right)\right\|_1}. \tag{108}$$

□

Lemma 44. *For every m and $b_1,\dots b_m$ and $i=1,\dots,T$ we have*

$$\frac{\|B_i \cdot B_{b_1\dots b_m}\|_1}{\|B_{b_1\dots b_m}\|_1} \in [2^{n_0}, 3^{n_0}]. \tag{109}$$

Proof. Using the notation and the assertion of Fact 16 and the fact that, for all i, every column sum of B_i is between 2^{n_0} and 3^{n_0} we obtain

$$\sum_{i=1}^{K} 2^{n_0}\|\mathbf{r}_{B_{b_1\dots b_n}}(i)\|_1 \leq \|B_i\cdot B_{b_1\dots b_m}\|_1 \leq \sum_{i=1}^{K} 3^{n_0}\|\mathbf{r}_{B_{b_1\dots b_n}}(i)\|_1.$$

The simple observation that $\|B_{b_1\dots b_n}\|_1 = \sum_{i=1}^{K}\|\mathbf{r}_{B_{b_1\dots b_n}}(i)\|_1$ completes the proof of the lemma. □

Definition 45. **(a):** *Let*

$$R := \left\{(x_1,\dots,x_T)\in[2^{n_0},3^{n_0}]^T : \exists i,j \text{ such that } |x_i-x_j|\geq c_0\right\},$$

where $c_0 > 0$ is the constant defined in Lemma 43. Using the well known inequality between the arithmetic and geometry means we obtain that the continuous function

$$f(x_1,\dots,x_T) := \log\frac{\sum_{j=1}^{T} x_j}{T} - \log\sqrt[T]{x_1\cdots x_T}$$

takes only positive values on the compact set R. We define

$$\delta_0 := \min f|_R > 0. \tag{110}$$

(b): *Let $m > k_0+1$. Put*

$\mathcal{I}_m := \{1,\dots,T\}^m$, $\mathcal{I}'_m := \{\mathbf{i}\in\mathcal{I}_m : i_1=\dots=i_{k_0+1}=1\}$ *and* $\mathcal{I}''_m := \mathcal{I}_m\backslash\mathcal{I}'_m$.

Fact 46. *For every $\mathbf{i}\in\mathcal{I}'_m$ we have*

$$\frac{1}{T}\cdot\log\prod_{j=1}^{T}\frac{\|B_j\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}+\delta_0 \leq \log\frac{1}{T}\sum_{j=1}^{T}\frac{\|B_j\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}. \tag{111}$$

Proof. It follows from Lemma 43 that for every $\mathbf{i}\in\mathcal{I}'_m$

$$\frac{\|B_1\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}+c_0 < \frac{\|B_2\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}.$$

This and Lemma 44 imply that for every $\mathbf{i}\in\mathcal{I}'_m$

$$\left(\frac{\|B_1\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1},\dots,\frac{\|B_T\cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}\right)\in R.$$

The assertion of the Fact immediately follows from (110). □

Observe that for every $\mathbf{i} \in \mathcal{I}_m$ we have

$$\sum_{j=1}^{T} \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} = \frac{\|\sum_{j=1}^{T} B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} = \frac{\|B_s \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} = \frac{8^{n_0}\|B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} = 8^{n_0}, \tag{112}$$

where in the last but one step we used (25) and (80). This and Fact 46 imply that

$$\forall \mathbf{i} \in \mathcal{I}'_m : \ \frac{1}{T} \cdot \log \prod_{j=1}^{T} \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \leq n_0 \cdot \log \frac{8}{3} - \delta_0. \tag{113}$$

Lemma 47. *For every $m \geq k_0 + 1$ we have*

$$\frac{1}{T^m} \cdot \sum_{\mathbf{i} \in \mathcal{I}_m} \frac{1}{T} \log \prod_{j=1}^{T} \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \leq n_0 \cdot \log \frac{8}{3} - T^{-(k_0+1)} \cdot \delta_0. \tag{114}$$

Proof.

$$\begin{aligned}
\frac{1}{T^m} \cdot \sum_{\mathbf{i} \in \mathcal{I}_m} \frac{1}{T} \log \prod_{j=1}^{T} \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} &= \frac{1}{T^m} \cdot \sum_{\mathbf{i} \in \mathcal{I}_m} \frac{1}{T} \cdot \sum_{j=1}^{T} \log \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \\
&= \frac{1}{T^m} \cdot \sum_{\mathbf{i} \in \mathcal{I}'_m} \frac{1}{T} \cdot \sum_{j=1}^{T} \log \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \\
&+ \frac{1}{T^m} \cdot \sum_{\mathbf{i} \in \mathcal{I}''_m} \frac{1}{T} \cdot \sum_{j=1}^{T} \log \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \\
&\leq \frac{\#\mathcal{I}'_m}{T^m} \cdot n_0 \cdot \log \frac{8}{3} - \frac{\#\mathcal{I}'_m}{T^m} \cdot \delta_0 \\
&+ \frac{\#\mathcal{I}''_m}{T^m} \cdot \log \frac{1}{T} \sum_{j=1}^{T} \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1} \\
&= n_0 \cdot \log \frac{8}{3} - T^{-(k_0+1)} \cdot \delta_0,
\end{aligned}$$

where in the inequality we used (113). □

3.5. **The proof of Theorem 9.** It follows from Corollary 17 that to prove our Theorem 9 it is enough to check that

$$\gamma < \log \frac{8}{3}, \tag{115}$$

where the Lyapunov exponent γ was defined in (20). To do so we need to use a theorem of Furstenberg about the integral representation of the Lyapunov exponent. Now we use the definitions from [17, Chapter 3].

Definition 48. *We recall that a non-negative $K \times K$ matrix is called column allowable if every column contains a non-zero element. Since we will use this theory for matrices which are the product of A_0, A_1, A_2 in the sequel we will always be working with column allowable matrices. We write $\mathcal{CA}$ for the set of $K \times K$ non-negative, column allowable matrices. Further let*

(116)

$$\mathcal{CA}_p := \{A \in \mathcal{CA} : \text{ if } \mathbf{r} \text{ is a row vector of } A \text{ then either } \mathbf{r} = \mathbf{0} \text{ or } \mathbf{r} > \mathbf{0}\}.$$

For the vectors with all elements positive $\mathbf{x} = (x_1, \dots, x_K) > \mathbf{0}$ and $\mathbf{y} = (y_1, \dots, y_K) > \mathbf{0}$ we define the pseudo-metric

$$d(\mathbf{x}, \mathbf{y}) := \log \left[\frac{\max_i (x_i/y_i)}{\min_j (x_j/y_j)} \right].$$

This is not exactly a metric because $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \lambda \mathbf{y}$ for some real number λ, but d defines a metric on

$$\Delta := \left\{ \mathbf{x} = (x_1, \dots, x_K) \in \mathbb{R}^K : x_i > 0 \text{ and } \sum_{i=1}^{K} x_i = 1 \right\}.$$

We call it projective distance. For all $A \in \mathcal{CA}$ we define

$$\widetilde{A} : \Delta \to \Delta \qquad \widetilde{A}(\mathbf{x}) := \frac{\mathbf{x}^T \cdot A}{\|\mathbf{x}^T \cdot A\|_1}.$$

Finally for any $A \in \mathcal{CA}$ the Birkhoff contraction coefficient $\tau_B(A)$ is defined as the Lipschitz constant for $\widetilde{A}$. That is

$$\tau(A) := \sup_{\mathbf{x}, \mathbf{y} \in \Delta,\ \mathbf{x} \neq \mathbf{y}} \frac{d(\mathbf{x}^T \cdot A, \mathbf{y}^T \cdot A)}{d(\mathbf{x}, \mathbf{y})}.$$

Lemma 49 ([17])**.** **(a):** *For every $i = 1, \dots, T$ we have $\tau(B_i) \leq 1$.*

(b): *The map B_1 is a strict contraction in the projective distance. That is*

$$h := \tau(B_1) < 1.$$

Proof. Part [**a**]: Since $B_i \in \mathcal{CA}$ the statement can be checked easily. Also it appears as formula [17, (3.1)].

Part [**b**]: Note that $B_1 \in \mathcal{CA}_p$. Therefore we can use [17, Theorem 3.10] and [17, Theorem 3.12] which yields the assertion. □

We will need the following notation from [9]:

Definition 50. *On the complete metric space* (Δ, d) *we write* $M(\Delta)$ *for the set of all probability measures on* Δ *for which* $\mu(\phi) < \infty$ *holds for all real valued Lipschitz functions* ϕ *defined on* (Δ, d). *After Hutchinson we define the distance of* $\mu, \nu \in M(\Delta)$ *by*

$$L(\mu, \nu) := \sup \left\{ \mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1 \right\},$$

We will consider the metric space $(M(\Delta), L)$. Using that (Δ, d) is a complete metric space, the main result of [9] implies that

Proposition 51. *[**Kravchenko** [9, Theorem 4.2]] The metric space* $(M(\Delta), L)$ *is complete.*

We introduce the operator $\mathcal{F} : M(\Delta) \to M(\Delta)$

$$\mathcal{F}\nu(H) := \frac{1}{T} \cdot \sum_{i=1}^{T} \nu \left(\widetilde{B}_i^{-1}(H) \right).$$

for a Borel set $H \subset \Delta$. Using $\nu \in M(\Delta)$, for every Lipschitz function ϕ we have

$$\mathcal{F}\nu(\phi) = \frac{1}{T} \cdot \sum_{i=1}^{T} \nu(\phi \circ \widetilde{B}_i). \tag{117}$$

Lemma 52. **(a):** *$\mathcal{F}$ is a contraction on the metric space* $(M(\Delta), L)$.
(b): *There is a unique fixed point* $\nu \in M(\Delta)$ *of* $\mathcal{F}$ *and for all* $\mu \in \mathcal{F}$ *we have* $L(\nu, \mathcal{F}^n \mu) \to 0$.

Proof. **Proof of (a):** Note that our IFS $\left\{ \widetilde{B}_i \right\}_{i=1}^{T}$ is not contracting, only contracting on average (that is the arithmetic mean of the Lipschitz constants of the functions in the IFS is less than one). Let $z := T/(h + (T-1)) > 1$. Fix an arbitrary $\phi \in \mathrm{Lip}(1)$. We write

$$\psi := \frac{z}{T} \cdot \left(\phi \circ \widetilde{B}_1 + \cdots + \phi \circ \widetilde{B}_T \right).$$

Using Lemma 49 we obtain that also $\psi \in \mathrm{Lip}(1)$. Then for an arbitrary $\nu, \mu \in M(\Delta)$ we have

$$z \cdot [\mathcal{F}\nu(\phi) - \mathcal{F}\mu(\phi)] = \nu(\psi) - \mu(\psi) \leq L(\nu, \mu). \tag{118}$$

So, we obtained that

$$\forall \mu, \nu \in M(\Delta), \qquad L(\mathcal{F}\nu, \mathcal{F}\mu) \leq \frac{1}{z} \cdot L(\nu, \mu), \tag{119}$$

which completes the proof of Part (a).

Proof of (b): This follows from Proposition 51 and the Banach Fixed Point Theorem. (A contraction on a complete metric space has a unique fixed point.) □

From now on we always write $\nu \in M(\Delta)$ for the unique fixed point of the operator $\mathcal{F}$ on $M(\Delta)$. That is

$$\nu(\phi) = \frac{1}{T^n} \cdot \sum_{i_1 \dots i_n} \nu(\phi \circ \widetilde{B}_{i_1 \dots i_n}). \tag{120}$$

holds for all Lipschitz functions ϕ and $n \geq 1$. Following the idea of Furstenberg [5], it is a key point of our argument that we would like to give an integral representation of the Lyapunov exponent γ_B as an integral of a function φ to be introduced below against the measure ν.

Definition 53. *Let $\varphi : \Delta \to \mathbb{R}$ be defined by*

$$\varphi(\mathbf{x}) := \frac{1}{T} \cdot \sum_{k=1}^{T} \log \|\mathbf{x} \cdot B_k\|_1, \qquad \mathbf{x} \in \Delta. \tag{121}$$

Lemma 54. *We have* $\mathrm{Lip}(\varphi) \leq 1$ *on the metric space* (Δ, d).

Proof. Let $\mathbf{x}, \mathbf{y} \in \Delta$. We assume that $\mathbf{x} \neq \mathbf{y}$. Then

$$\min_j \frac{x_j}{y_j} < 1 \text{ and } \max_i \frac{x_i}{y_i} > 1, \tag{122}$$

since $\sum x_i = \sum y_i = 1, x_i, y_i > 0$. We fix k with $1 \leq k \leq T$. It is enough to prove that the function

$$\psi_k : \mathbf{x} \mapsto \log \|\mathbf{x} \cdot B_k\|_1$$

is in Lip(1) because φ is the arithmetic mean of the functions $\{\psi_k\}_{k=1}^{T}$. Now fix $\mathbf{x}, \mathbf{y} \in \Delta$, $\mathbf{x} \neq \mathbf{y}$.

$$\psi_k(\mathbf{x}) - \psi_k(\mathbf{y}) = \log \frac{\|\mathbf{x} \cdot B_k\|_1}{\|\mathbf{y} \cdot B_k\|_1} \tag{123}$$

Note that for arbitrary $\mathbf{a} \in \Delta$ we have

$$\|\mathbf{a} \cdot B_k\|_1 = a_1 \cdot \|\mathbf{r}_1\|_1 + \cdots + a_K \cdot \|\mathbf{r}_K\|_1, \tag{124}$$

where $\mathbf{r}_1, \dots, \mathbf{r}_K$ are the row vectors of the matrix B_k. This is so because the matrix B_k is non-negative. Hence from (123) we obtain

$$\psi_k(\mathbf{x}) - \psi_k(\mathbf{y}) = \log \frac{\sum_{i=1}^{K} x_i \cdot \|\mathbf{r}_i\|_1}{\sum_{i=1}^{K} y_i \cdot \|\mathbf{r}_i\|_1}. \tag{125}$$

Further, for all $1 \leq i \leq K$, we have

$$\min_\ell \frac{x_\ell}{y_\ell} \cdot y_i \cdot \|\mathbf{r}_i\|_1 \leq x_i \cdot \|\mathbf{r}_i\|_1 \leq \max_\ell \frac{x_\ell}{y_\ell} \cdot y_i \cdot \|\mathbf{r}_i\|_1. \tag{126}$$

Thus, using $y_i > 0$ for all $1 \le i \le K$, we obtain

$$\min_\ell \frac{x_\ell}{y_\ell} \le \frac{\sum_{i=1}^{K} x_i \cdot \|\mathbf{r}_i\|_1}{\sum_{i=1}^{K} y_i \cdot \|\mathbf{r}_i\|_1} \le \max_\ell \frac{x_\ell}{y_\ell}.$$

Putting this and (125) together yields that

$$\log \min_\ell \frac{x_\ell}{y_\ell} \le \psi_k(\mathbf{x}) - \psi_k(\mathbf{y}) \le \log \max_\ell \frac{x_\ell}{y_\ell}. \tag{127}$$

From this and (122) we get

$$|\psi_k(\mathbf{x}) - \psi_k(\mathbf{y})| \le \log \max_\ell \frac{x_\ell}{y_\ell} - \log \min_\ell \frac{x_\ell}{y_\ell} = \log \frac{\max_\ell \frac{x_\ell}{y_\ell}}{\min_\ell \frac{x_\ell}{y_\ell}} = d(\mathbf{x}, \mathbf{y}).$$

This implies that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \quad \le \quad \frac{1}{T} \sum_{k=1}^{T} |\psi_k(\mathbf{x}) - \psi_k(\mathbf{y})| \le d(\mathbf{x}, \mathbf{y})$$

which completes the proof of the lemma. □

Proof of Theorem 9. Using (83) and (115) the only thing left to prove is that

$$\gamma_B < n_0 \cdot \log \frac{8}{3}. \tag{128}$$

Let

$$\mathbf{w} := \frac{1}{K} \cdot \mathbf{e} \text{ where } \mathbf{e} := (1, \ldots, 1) \in \mathbb{R}^K.$$

We define the sequence of measures $\nu_n \in \mathcal{M}^1$ by

$$\nu_0 := \delta_{\mathbf{w}} \text{ and } \nu_n(H) := (\mathcal{F}^n \nu_0)(H) = \frac{1}{T^n} \cdot \sum_{i_1 \ldots i_n} \nu_0(\widetilde{B}^{-1}_{i_1 \ldots i_n}(H)),$$

where $H \subset \Delta$ is a Borel set. Observe that

$$\begin{aligned}
\int_\Delta \varphi(\mathbf{x}) d\nu_n(\mathbf{x}) &= \frac{1}{T^n} \sum_{i_1 \dots i_n} \int_\Delta \varphi(\mathbf{x}) d\nu_0 \left(\widetilde{B}^{-1}_{i_n \dots i_1}(\mathbf{x}) \right) \\
(129) \quad &= \frac{1}{T^n} \sum_{i_1 \dots i_n} \varphi \left(\widetilde{B}_{i_n \dots i_1}(\mathbf{w}) \right) \\
&= \frac{1}{T} \sum_{k=1}^{T} \frac{1}{T^n} \sum_{i_1 \dots i_n} \log \| \widetilde{B}_{i_n \dots i_1}(\mathbf{w}) \cdot B_k \|_1 \\
&= \frac{1}{T} \sum_{k=1}^{T} \frac{1}{T^n} \sum_{i_1 \dots i_n} \log \frac{\| \mathbf{w} \cdot B_{i_1 \dots i_n} \cdot B_k \|_1}{\| \mathbf{w} \cdot B_{i_1 \dots i_n} \|_1} \\
&= \frac{1}{T^{n+1}} \sum_{i_1 \dots i_{n+1}} \log \| \mathbf{w} \cdot B_{i_1 \dots i_{n+1}} \|_1 - \frac{1}{T^n} \sum_{i_1 \dots i_n} \log \| \mathbf{w} \cdot B_{i_1 \dots i_n} \|_1 \\
&= \frac{1}{T^n} \cdot \sum_{i_1 \dots i_n} \frac{1}{T} \log \prod_{k=1}^{T} \frac{\| \mathbf{w} \cdot B_k \cdot B_{i_1 \dots i_n} \|_1}{\| \mathbf{w} \cdot B_{i_1 \dots i_n} \|_1} \\
&= \frac{1}{T^n} \cdot \sum_{i_1 \dots i_n} \frac{1}{T} \log \prod_{k=1}^{T} \frac{\| B_k \cdot B_{i_1 \dots i_n} \|_1}{\| B_{i_1 \dots i_n} \|_1}.
\end{aligned}$$

Using Lemma 47 we obtain that for all $n \geq k_0 + 1$

$$\int_\Delta \varphi(\mathbf{x}) d\nu_n(\mathbf{x}) \leq n_0 \cdot \log \frac{8}{3} - T^{-(k_0+1)} \cdot \delta_0. \tag{130}$$

In the rest of the proof we will verify that the limit of the left hand side is equal to γ_B from which it immediately follows that (128) holds. Namely, results of Furstenberg [5] indicate that the Lyapunov exponent γ_B (defined in (82)) has the following integral representation

$$\gamma_B = \int_\Delta \varphi(\mathbf{x}) d\nu(\mathbf{x}) = \lim_{n \to \infty} \int_\Delta \varphi(\mathbf{x}) d\nu_n(\mathbf{x}), \tag{131}$$

where the second equality follows from Lemma 52. Although, we believe that this follows from [5], for the convenience of the reader we give a short proof of (131) here.

We start with an easy observation: in (129) we put together the left hand side of the first line and the right side of the fifth line. Then we apply for both $\ell = n + 1$ and $\ell = n$ that

$$\frac{1}{T^\ell} \sum_{i_1 \dots i_\ell} \log \| \mathbf{w} \cdot B_{i_1 \dots i_\ell} \|_1 = \log 1/K + \frac{1}{T^\ell} \sum_{i_1 \dots i_\ell} \log \| B_{i_1 \dots i_\ell} \|_1$$

to obtain

(132)
$$\int_\Delta \varphi(\mathbf{x}) d\nu_n(\mathbf{x}) = \frac{1}{T^{n+1}} \sum_{i_1 \dots i_{n+1}} \log \|B_{i_1 \dots i_{n+1}}\|_1 - \frac{1}{T^n} \sum_{i_1 \dots i_n} \log \|B_{i_1 \dots i_n}\|_1.$$

Let us define the measure

$$\eta_\ell := \frac{1}{\ell} \sum_{n=1}^{\ell} \nu_n.$$

Clearly $\eta_\ell \in M(\Delta)$ holds for all ℓ and it follows from Lemma 52 that for all Lipschitz functions ϕ we have

$$\lim_{\ell \to \infty} \eta_\ell(\phi) = \lim_{n \to \infty} \nu_n(\phi) = \nu(\phi). \tag{133}$$

Note that by (132) the integral $\eta_\ell(\varphi)$ is given by a telescopic sum. This yields

$$\eta_\ell(\varphi) = \frac{1}{\ell} \sum_{n=1}^{\ell} \nu_n(\varphi) = \frac{1}{\ell} \left(\frac{1}{T^{\ell+1}} \sum_{i_1 \dots i_{\ell+1}} \log \|B_{i_1 \dots i_{\ell+1}}\|_1 - \frac{1}{T} \cdot \sum_{i=1}^{T} \log \|B_i\|_1 \right).$$

Now the assertion of (131) immediately follows from (84). As was noted above, (128) follows from (130) and (131) which completes the proof of Theorem 9. $\square$

4. The remaining proofs

The angle θ projection from S to the y-axis is denoted by PROJ^θ. We are using notation defined in §3.1. The proj^θ projection of the IFS $\mathcal{G} := \left\{ g_i(x,y) = \frac{1}{3}(x,y) + \frac{1}{3}\mathbf{t}_i \right\}_{i=1}^{8}$ is the IFS

$$\Phi := \left\{ \varphi_i^\theta(t) = \frac{1}{3} \cdot t + \frac{1}{3} \cdot \mathrm{proj}^\theta(\mathbf{t}_i) \right\}_{i=1}^{8}.$$

That is for every $i = 1, \dots, 8$ the following diagram is commutative:

$$\begin{array}{ccc} F & \xrightarrow{g_i} & F \\ \downarrow{\scriptstyle \mathrm{proj}^\theta} & & \downarrow{\scriptstyle \mathrm{proj}^\theta} \\ I^\theta & \xrightarrow[\varphi_i^\theta]{} & I^\theta \end{array}$$

4.1. **The measures ν^θ and $\widetilde{\nu}^\theta$.** We recall that, for $\theta \in [0, \pi/2)$ and $a \in I^\theta$, $\mathrm{Line}_{\theta,a}$ denotes the line segment in $[0,1]^2$ of angle θ through the point $(0,a)$. Further, for any $a \in [0,1]$, let $\mathrm{LINE}_{\theta,a}$ be the line segment in S of angle θ through the point $(0,a)$. Using the notation of Definition 19 we define the measure $\widetilde{\nu}$ on the Borel subsets of $\widetilde{F}$ as follows:

$$\widetilde{\nu} := \frac{1}{N} \sum_{i=1}^{K} \left((q_i, r_i) + \nu\right)|_S.$$

The angle θ projection of $\widetilde{\nu}$ to the y-axis is called $\widetilde{\nu}^\theta$. When $\pi^\theta := \mathrm{proj}^\theta \circ \Pi$, this can be summarized by

the following diagram is commutative.

$$\begin{array}{ccc} (\Sigma_8, \mu_8) & & \\ \downarrow \Pi_* & \searrow \pi^\theta_* & \\ (F, \nu) & \xrightarrow[\mathrm{proj}^\theta_*]{} & (I^\theta, \nu^\theta) \end{array} \qquad \text{and} \qquad (\widetilde{F}, \widetilde{\nu}) \xrightarrow{\mathrm{PROJ}^\theta_*} ([0,1], \widetilde{\nu}^\theta).$$

It is immediate from the definitions and from the symmetries of the Sierpinski carpet that for Borel sets $H \subset [-\tan\theta, 0]$ and $P \subset [0,1]$ we have
(134)
$\widetilde{\nu}^\theta(P) = \nu^\theta(H) = 0$ implies that $\nu^\theta(P) = 0$ and $\nu^\theta(1 - \tan\theta - H) = 0.$

On the other hand, for $H \subset [-\tan\theta, 0]$, $\nu^\theta(H) = 0$ does not imply that $\nu^\theta(1 + H) = 0$. This makes our investigation more complicated than the analogous proof for the Lebesgue measure. We will use the following observation.

Lemma 55. *The measure $\widetilde{\nu}^\theta$ is invariant under the map $y \mapsto 3y$ mod 1.*

Proof. For $(i_1, \dots, i_n) \in \{0,1,2\}^n$ we write

$$I_{i_1 \dots i_n} := \left[\sum_{k=1}^{n} i_k \cdot 3^{-k}, \sum_{k=1}^{n} i_k \cdot 3^{-k} + 3^{-n}\right].$$

Since the ν measure of every level n shape j is $8^{-n}\nu(Q_j)$ we obtain

$$\widetilde{\nu}^\theta(I_{i_1 \dots i_n}) = \sum_{j=1}^{K} \#_{i_1 \dots i_n}(j) \cdot 8^{-n} \nu(Q_j), \tag{135}$$

where $\#_{i_1 \dots i_n}(j)$ denotes the number of level n shapes j contained in $\widetilde{F}^n \cap S_{i_1 \dots i_n}$. That is

$$\#_{i_1 \dots i_n}(j) = \sum_{k=1}^{K} A_{i_1 \dots i_n}(k, j). \tag{136}$$

Since from (18) all the column sums of the matrix $A_0 + A_1 + A_2$ are equal to 8, this readily implies that

$$\sum_{i_0=0}^{2} \#_{i_0 i_1 \dots i_n}(j) = 8 \cdot \#_{i_1 \dots i_n}(j). \tag{137}$$

Since the inverse image by $y \mapsto 3y \bmod 1$ of $I_{i_1 \dots i_n}$ is $\cup_{i_0=0}^{2} I_{i_0 i_1 \dots i_n}$ formulae (135) and (137) together imply the assertion of the lemma. □

4.2. **The dimension of the intersection is a constant ν^θ-almost surely.**

Lemma 56. *For every $\theta \in [0, \pi)$ the following functions are μ_8-almost everywhere constant:*

$$\mathbf{i} \mapsto \dim_{\mathrm{H}}(E_{\theta,\pi^\theta(\mathbf{i})}),\ \mathbf{i} \mapsto \overline{\dim}_{\mathrm{B}}(E_{\theta,\pi^\theta(\mathbf{i})}),\ \mathbf{i} \mapsto \underline{\dim}_{\mathrm{B}}(E_{\theta,\pi^\theta(\mathbf{i})}). \tag{138}$$

Proof. The fact that these functions are Borel measurable can be proved in exactly the same way as the analogous statement proved in [8, Lemma 2.4]. Now, for every $\mathbf{i} \in \Sigma_8$ and $k \in \{1, \dots, 8\}$, we have $g_k\left(\mathrm{Line}_{\theta,\pi^\theta(\mathbf{i})}\right) \subset \mathrm{Line}_{\theta,\pi^\theta(k\mathbf{i})}$, which implies that these three functions have the property

$$\forall \mathbf{j} \in \Sigma, \qquad f(\sigma \mathbf{j}) \leq f(\mathbf{j}). \tag{139}$$

Such a function is μ_8-almost everywhere constant since (σ, μ_8) is ergodic and, for each t, $\sigma\{\mathbf{i} : f(\mathbf{i}) \leq t\} \subset \{\mathbf{i} : f(\mathbf{i}) \leq t\}$. □

Clearly, for every θ, a

$$\text{if } \Pi(\mathbf{i}), \Pi(\mathbf{j}) \in E_{\theta,a} \text{ then } \dim(E_{\theta,\pi^\theta(\mathbf{i})}) = \dim(E_{\theta,\pi^\theta(\mathbf{j})}) \tag{140}$$

if dim means any one of Hausdorff dimension, lower box dimension or upper box dimension.

Proof of Proposition 3. The assertion of Proposition 3 follows immediately from (140) and Lemma 56. □

Before proving Proposition 8 we remark that it is analogous to Theorem 7. In this case however, the IFS Φ satisfies the so called Weak Separation Property (WSP). Thus, due to the results of Lau, Ngai and Rao, (see [10]) the measures ν_θ and $\mathcal{L}$eb are singular. Hence, neither of the previous two assertions implies the other one. In fact, as follows from our main result,

$$\text{if } \tan\theta \in \mathbb{Q}, \quad d^\theta(\mathcal{L}\text{eb}) < s - 1 \text{ and } d^\theta(\nu^\theta) \geq s - 1. \tag{141}$$

The proof of Theorem 7 is based on a method of Kenyon-Peres which uses a theorem due to Ledrappier (see [8, Proposition 2.6]). We use their method but we have to tackle an additional problem: unlike the Lebesgue measure, our measure ν^θ is not translation invariant which makes this case much more difficult.

Proof of Proposition 8. In this proof when we write dim it means any of the following dimensions: $\dim_{\mathrm{H}}, \underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$. By Lemma 55 we can apply [8, Proposition 2.6] exactly as in the proof of [11, Theorem 1.1 (3)] to obtain

$$\text{for } \widetilde{\nu}^\theta\text{-almost all } b \in [0,1], \dim_{\mathrm{H}}(\mathrm{LINE}_{\theta,b} \cap \widetilde{F}) = \dim_{\mathrm{B}}(\mathrm{LINE}_{\theta,b} \cap \widetilde{F}) \tag{142}$$

and the box dimension exists.

We fix $\mathbf{a} := (a_1, \dots, a_{n_0}) \in \{0,1,2\}^{n_0}$ as constructed in Proposition 18. We use the definitions of Section 3.2.1 (see figure 4). For every $j \in \{1, \dots, K\}$, in $S_{\mathbf{a}}$ there are altogether 3^{n_0} level n shapes j. These were denoted (see (35)) by

$$Q_{\mathbf{a}}^{\overline{\ell}}(j) : \overline{\ell} = (\ell_1, \dots, \ell_{n_0}) \in \{0,1,2\}^{n_0}.$$

When we consider $\widetilde{F}^{n_0} \cap S_{\mathbf{a}}$ some of these shapes are completely deleted by the construction of the n_0 approximation $\widetilde{F}^{n_0}$ of the translated copies of the Sierpinski carpet while some other level n_0 shapes are preserved completely by this construction. Thus for every $j \in \{1, \dots, K\}$ we can define the set of indices $\mathcal{I}_1(j) \subset \mathcal{I}(j) \subset \{0,1,2\}^{n_0}$ such that

$$\widetilde{F}^{n_0} \cap S_{\mathbf{a}} = \bigcup_{j=1}^{K} \bigcup_{\overline{\ell} \in \mathcal{I}(j)} Q_{\mathbf{a}}^{\overline{\ell}}(j) \text{ and } \widetilde{F}^{n_0} \cap S_{\mathbf{a}} \cap Q_1 = \bigcup_{j=1}^{K} \bigcup_{\overline{\ell} \in \mathcal{I}_1(j)} Q_{\mathbf{a}}^{\overline{\ell}}(j). \tag{143}$$

It follows from the special choice of $\mathbf{a}$ that we have

$$\mathcal{I}_1(j) \neq \emptyset. \tag{144}$$

By self-similarity for every $b \in [0,1]$ and $1 \le j \le K$

$$\text{if } Q_{\mathbf{a}}^{\overline{\ell}}(j) \subset \widetilde{F}^n \text{ then } \dim\left(\mathrm{LINE}_{\theta,b} \cap \widetilde{F} \cap Q_{\mathbf{a}}^{\overline{\ell}}(j)\right) = \dim(\mathrm{LINE}_{\theta,3^{n_0}b} \cap Q_j \cap \widetilde{F}), \tag{145}$$

where $3^{n_0}b$ is written for $3^{n_0}b \bmod 1$. Using (144) and (145) we see that

for every $b \in I_{\mathbf{a}} := \left[\sum_{k=1}^{n_0} a_k 3^{-k}, \sum_{k=1}^{n_0} a_k 3^{-k} + 3^{-n_0}\right]$ we have

$$\dim(E_{\theta,b}) = \max_j \dim(\mathrm{LINE}_{\theta,3^{n_0}b} \cap Q_j) = \dim(\mathrm{LINE}_{\theta,b} \cap \widetilde{F}). \tag{146}$$

Using this, (142) and (134) we obtain that for ν^θ-almost all $b \in I_{\mathbf{a}}$ we have

$$\dim_{\mathrm{H}}(E_{\theta,b}) = \dim_{\mathrm{B}}(E_{\theta,b})$$

and the box dimension exists. Clearly, $\nu^\theta(I_{\mathbf{a}}) > 0$. This and Proposition 3 together imply that the conclusion of Proposition 8 holds. □

4.3. **The local dimension of ν^θ and the box dimension of the slices.** Let $\theta \in [0,\pi/2)$ be arbitrary. For $a \in I^\theta$ let $N_{\theta,a}(n)$ be the number of level n squares of F^n that intersect $E_{\theta,a}$. By definition

(147)
$$\underline{\dim}_{\mathrm{B}}(E_{\theta,a}) = \liminf_{r\to 0} \frac{\log N_{\theta,a}(n)}{n\log 3} \text{ and } \overline{\dim}_{\mathrm{B}}(E_{\theta,a}) = \limsup_{r\to 0} \frac{\log N_{\theta,a}(n)}{n\log 3}.$$

We write $\mathrm{Line}_{\theta,a}(3^{-n})$ for the 3^{-n} vertical neighborhood of $\mathrm{Line}_{\theta,a}$. That is

$$\mathrm{Line}_{\theta,a}(3^{-n}) := \left\{(x,y) : |y-(x\cdot\tan\theta + a)| < 3^{-n},\ x\in[0,1]\right\}.$$

For arbitrary n we shall estimate the ν measure of $\mathrm{Line}_{\theta,a}(3^{-n})$ by the ν measure of those squares of level n or close to n that intersect $\mathrm{Line}_{\theta,a}$. First we fix k_0 such that $1+\tan\theta < 3^{k_0}$ and observe that if a level $n+k_0$ square from F^{n+k_0} intersects $E_{\theta,a}$ then this square is contained in $\mathrm{Line}_{\theta,a}(3^{-n})$. Since the ν measure of such a square is equal to $8^{-(n+k_0)}$ we see that

$$(148)\qquad \forall\theta,a: \qquad 8^{-(n+k_0)}\cdot N_{\theta,a}(n+k_0) \le \nu\left(\mathrm{Line}_{\theta,a}(3^{-n})\right).$$

Now we prove an opposite inequality.

Lemma 57. *Let $\theta\in[0,\pi/2)$ and $a\in I^\theta$ be arbitrary. We have*

$$(149)\qquad \nu\left(\mathrm{Line}_{\theta,a}(3^{-n})\right) < 1000\cdot\left(\left\lfloor\frac{1}{\tan\theta}\right\rfloor+1\right)\cdot 8^{-n}\cdot N_{\theta,a}(n).$$

Proof. In this proof we write $G_{\theta,a}(n)$ for the collection of those "good" level n squares from F^n that intersect $E_{\theta,a}$. Further we write $B_{\theta,a}(n)$ for the collection of those "bad" level n squares from F^n that do not intersect $E_{\theta,a}$ but do intersect $\mathrm{Line}_{\theta,a}(3^{-n})$. Since all the level n squares from F^n have ν-measure 8^{-n}, to prove this Lemma it is enough to show that

$$(150)\qquad \#B_{\theta,a}(n) < 1000\cdot\left(\left\lfloor\frac{1}{\tan\theta}\right\rfloor+1\right)\cdot\#G_{\theta,a}(n).$$

Choose $U\in B_{\theta,a}(n)$. Without loss of generality we may assume that U is situated below the line $\mathrm{Line}_{\theta,a}$. Write

$$(151)\qquad U^W, U^{W^2},\dots,U^{W^k}$$

for the sequence of consecutive level n squares on the left of U with k chosen so that U^{W^k} is the first that intersects $\mathrm{Line}_{\theta,a}$. Then $k\le \left\lfloor\frac{1}{\tan\theta}\right\rfloor+1$. If we write U^N for the level n square above U and

$$(152)\qquad U^N, U^{NW},\dots,U^{NW^k}$$

for the consecutive level n squares to the left of U^N, all which meet line $\mathrm{Line}_{\theta,a}$, then either one of them, V say, is in F^n and we will say that V is associated with U or none of them is in F^n. This is possible only if there is $\ell < n$ such that $3^{-\ell} > k \cdot 3^{-n}$ and a level ℓ square, let us call it X, which is situated in such a way that its bottom line contains the bottom lines of all the squares from (152). Then the level ℓ square X^S below X is in F^ℓ and the level n squares in (151) are in the top row in X^S and lie in F^n. Then U^{W^k} belongs to $G_{\theta,a}(n)$ and we say that it is associated to U. In this way we have seen that for every "bad" square U there is an associated "good" square V within distance $\left(\left\lfloor \frac{1}{\tan\theta} \right\rfloor + 1\right) \cdot 3^{-n}$. It follows that all the elements of $G_{\theta,a}(n)$ can be associated with less than $1000 \cdot \left(\left\lfloor \frac{1}{\tan\theta} \right\rfloor + 1\right)$ squares from $B_{\theta,a}(n)$ which completes the proof of the lemma. □

Proof of Proposition 4. Putting formulas (148) and (149) together we obtain that for all $\theta \in [0, \pi/2)$ we have

$$8^{k_0} N_{\theta,a}(n + k_0) \leq 8^n \nu^\theta([a - 3^{-n}, a + 3^{-n}]) \leq k_1 N_{\theta,a}(n), \tag{153}$$

where

$$k_0 = \left\lfloor \frac{\log(1 + \tan\theta)}{\log 3} \right\rfloor + 1, \qquad k_1 = 1000 \left(\left\lfloor \frac{1}{\tan\theta} \right\rfloor + 1 \right). \tag{154}$$

Taking logarithms and dividing by $\log 3^n$ we obtain

$$s - o(n) \leq \frac{\log N_{\theta,a}(n)}{\log 3^n} + \frac{\log \nu^\theta([a - 3^{-n}, a + 3^{-n}])}{\log 3^{-n}} \leq s + o(n). \tag{155}$$

□

Proof of Theorem 5. The assertion immediately follows from (3) and Proposition 4 since $\dim_{\mathrm{H}}(\nu^\theta) \leq 1$. □

Proof of Theorem 11. This is immediate from Theorem 5 and Proposition 8. □

Proof of Corollary 10. This follows from Theorem 9 and Proposition 4. □

Proof of Theorem 6. We only need to prove part (a). $\nu^\theta \sim \mathcal{L}\mathrm{eb}$ follows from [16, Proposition 3.1], which states that every self-similar measure is either equivalent to the Lebesgue measure or singular. Using the argument from [12, p.140] we see that for $\mathcal{L}\mathrm{eb}$ almost all $a \in I^\theta$ the conditional measure

$$\nu_{\theta,a}(B) := \lim_{\delta \to 0} \frac{\nu(B(\delta))}{2\delta}, \qquad B \subset E_{\theta,a} \text{ Borel}$$

exists and is finite and positive from [12, (10.2)]. Now (8) follows from (153). □

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