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Hausdorff and packing measure for solenoids

MICHAŁ RAMS† and KÁROLY SIMON‡

† *Michał Rams Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland*
(e-mail: m.rams@impan.gov.pl)
‡ *Institute of Mathematics, Technical University of Budapest, H-1529 PO box 91, Hungary*
(e-mail: simonk@math.bme.hu)

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Abstract. We prove that the solenoid with two different contraction coefficients has zero Hausdorff and positive packing measure in its own dimension and the SBR measure is equivalent to the packing measure on the attractor. Further, we prove similar statements for Slanting Baker maps with intersecting cylinders (in $\mathbb{R}^2$).

1. *Introduction*

The SBR (natural) measure carries the most important information about strange attractors. For many conformal hyperbolic attractors it is equivalent to the Hausdorff measure on the attractor. However, it happens that the appropriate dimensional Hausdorff measure of the attractor is zero while the packing measure is positive and finite and equivalent to the SBR measure. In such a case it is the packing measure which is dynamically relevant. Such a phenomenon has been previously observed by Sullivan in the context of parabolic Kleinian groups, then by Denker and Urbański [**5**] in the context of parabolic rational functions, then by Urbański [**20**] for non-recurrent rational functions. Also, the same phenomenon was observed for infinite iterated function systems [**9**], parabolic dynamical systems [**10**] and finite iterated function systems with overlapping cylinders [**13**]. In this paper we point out that such a situation (when the packing measure is the dynamically relevant measure) may occur even for the simplest axiom A diffeomorphisms. Based on this we believe the following.

CONJECTURE 1. *Let* Λ *be the attractor of an axiom A diffeomorphism. Then 'typically' its SBR measure is equivalent to the appropriate dimensional packing measure restricted to* Λ.

The examples we will consider include the solenoid, which is a most natural non-conformal hyperbolic attractor, and the Slanting Baker maps. The Smale–Williams solenoid (see [**15**] and [**16**] for an illustration and more details) has an expanding and two contracting directions. As an example, let $\bar{\Lambda}$ be the attractor of the map

$$(t, x, z) \to (2t \ (\mathrm{mod}\ 1), \lambda_1 x + \varepsilon \cos(2\pi t), \lambda_2 z + \varepsilon \sin(2\pi t)), \tag{1.1}$$

the map being defined on the torus $S^1 \times D$, where D is the unit disk. We assume that the contraction ratios are different so that the map is non-conformal. In order to assure the map to be injective, we also have to assume that the greater contraction ratio λ_1 is smaller than $\frac{1}{2}$.

It follows from a recent result due to Hasselblatt and Schmeling [**7**], that all angular sections of $\bar{\Lambda}$ share the same Hausdorff and packing dimensions. On the other hand, it was proved in [**17**] that the Hausdorff and the upper box dimensions (hence the packing dimension as well) of $\bar{\Lambda}$ are $1 + s$, where $s = \log 2/{-\log \lambda_1}$. Combining these theorems, one can see that the Hausdorff and packing dimensions of all angular sections of $\bar{\Lambda}$ are equal to s. We prove that the $(1 + s)$-dimensional Hausdorff measure is zero and the $(1 + s)$-dimensional packing measure is positive and finite. Similarly, almost every section has zero s-dimensional Hausdorff measure and positive s-dimensional packing measure. Most importantly, the SBR measure for the solenoid (1.1) is equivalent to the $(1 + s)$-dimensional packing measure.

We also consider plane maps, so called Slanting Baker maps. They were first studied by Falconer in [**6**]. These are maps of the rectangle $[0, 1] \times [-1, 1]$ into itself. An example of these is given by the formula

$$(t, x) \to (2t \ \mathrm{mod}\ 1, \lambda(x - \Phi(t))), \tag{1.2}$$

where $\Phi(t) = 1 - |2t - 1|$ is the tent map, investigated previously by Carter and Mauldin in [**3**]. Our results imply in this particular case that the attractor Λ has zero Hausdorff measure but positive and finite packing measure in the dimension $1 + s$, where $s = \log 2/{-\log \lambda}$, whenever $\lambda < \frac{1}{2}$. Also in this case the SBR measure is equivalent to the $(1 + s)$-dimensional packing measure.

Both in the case of solenoid and of Slanting Baker maps we need some linearity assumptions to prove results on the packing measure. We can prove the Hausdorff measure results in much greater generality.

2. *Results*

Following Bothe [**1**] we consider more general solenoid maps than that in (1.1). Namely, let D be the unit disk in R^2 centered at the origin. We consider a map $\bar{f}$ defined on $[0, 1] \times D \subset R^3$ by the formula

$$\bar{f}(t, x, z) := (g(t), \lambda_1(t, x), \lambda_2(t, z)) \tag{2.1}$$

where the component functions $g : [0, 1] \to [0, 1]$ and $\lambda_1, \lambda_2 : [0, 1] \times [-1, 1] \to [-1, 1]$ satisfy the following assumptions:

(a) we can partition $[0, 1]$ into closed intervals $I_1, \ldots, I_m$ with disjoint interiors;

(b) for every $1 \leq k \leq m$, $g : \mathrm{int}(I_k) \to (0, 1)$ holds and is an onto and $\mathcal{C}^2$ map with $|g'(x)| > c > 1$ for $x \in \mathrm{int}(I_k)$;

(c) the second and third component functions λ_1, λ_2 are $\mathcal{C}^2$ maps with partial derivatives satisfying $0 < q_1 \le (\partial/\partial x)\lambda_1, (\partial/\partial z)\lambda_2 \le q_2 < 1$.

Furthermore, we say that $\bar{f}$ is *linear* in the special case when $g(t)$ and $\lambda_i(t,x)$ are of the form

$$g(t) = mt \bmod 1 \quad \text{and} \quad \lambda_i(t,x) = \lambda_i x + r_i(t), \quad i = 1, 2 \tag{2.2}$$

where $m \in \mathbb{N}$, $0 < \lambda_i < 1$, $i = 1, 2$ are constants. Notice that here we do not require the linearity of $r_i(t)$ $i = 1, 2$.

Put

$$\bar{\varphi}_1(t,x,z) := \log \frac{\partial}{\partial x}\lambda_1(t,x), \quad \bar{\varphi}_2(t,x,z) := \log \frac{\partial}{\partial z}\lambda_2(t,z). \tag{2.3}$$

Let $P = P_{\bar{f}^{-1}}$ be the topological pressure for the transformation $\bar{f}^{-1}$ and let s_1 and s_2 be the solutions of the pressure formulas:

$$P(s_1\bar{\varphi}_1) = 0 \quad \text{and} \quad P(s_2\bar{\varphi}_2) = 0. \tag{2.4}$$

We assume that

$$s_2 < s_1 < 1 \tag{2.5}$$

holds. Because of the symmetry between the second and third component functions, without loss of generality we may always require that (2.5) holds if the solutions of the two pressure formulas are different. This means that the contraction in the direction of the z-axis is stronger than in the direction of the x-axis.

Observe that the first two component functions of $\bar{f}$ depend only on the first two variables. So we may consider the projection of $\bar{f}$ to the first two coordinates. In this way we obtain

$$f(t,x) := (g(t), \lambda_1(t,x)). \tag{2.6}$$

The attractors of $\bar{f}$ and f are $\bar{\Lambda}$ and Λ, respectively. That is

$$\bar{\Lambda} := \bigcap_{n=0}^{\infty} \bar{f}^n([0,1] \times D) \quad \text{and} \quad \Lambda := \bigcap_{n=0}^{\infty} f^n([0,1] \times [-1,1]).$$

It is important that $\bar{f}$ is a one-to-one map, but f is not. Thus the unstable lines of Λ may intersect each other.

Over each object in $\mathbb{R}^3$ we use a bar (like $\bar{f}$ and $\bar{\Lambda}$). The same notation without the bar means the projection of that object in space to the coordinate plane of the t and x axes (like f and Λ).

It was proved in [**1**] and [**17**] that

$$\dim_H \bar{\Lambda} = \dim_P \bar{\Lambda} = 1 + s_1 \quad \text{and} \quad \dim_H \Lambda = \dim_P \Lambda = 1 + s_1 \tag{2.7}$$

hold if all intersections between unstable lines of the attractor Λ are transversal. This *transversality* condition was checked for the solenoid (1.1) with two different constant coefficients in [**17**]. The main results of this paper are as follows.

THEOREM 1. *If there are at least two intersecting unstable lines of Λ then both $\mathcal{H}^{1+s_1}(\bar{\Lambda}) = 0$ and $\mathcal{H}^{1+s_1}(\Lambda) = 0$.*

THEOREM 2. *If all intersections between the unstable lines of Λ are transversal and $\bar{f}$ is linear (see (2.2)), then the $(1+s_1)$-dimensional packing measures of both of the attractors Λ and $\bar{\Lambda}$ are positive and finite. That is*

$$0 < \mathcal{P}^{1+s}(\Lambda) \leq \mathcal{P}^{1+s}(\bar{\Lambda}) < \infty. \tag{2.8}$$

Moreover, the SBR measures both for f and $\bar{f}$ are equivalent to the $(1+s_1)$-dimensional packing measures on Λ and $\bar{\Lambda}$, respectively.

Remark 1. The maps (1.1) and (1.2) mentioned in §1 are linear in our sense and the transversality condition mentioned in Theorem 2 holds, so all the results above apply to them.

Remark 2. In Theorem 1 we do not require that the intersections between unstable lines of Λ be transversal. However, we can guarantee that the Hausdorff dimension is $1 + s_1$ only if this transversality condition holds. Bothe proved in [**1**] that if we assume that the contractions are strong enough, then this transversality condition holds on a residual subset of endomorphisms f with intersecting unstable lines. So, the transversality condition typically holds in some sense in the case of strong contractions.

Remark 3. If there are no intersections between the unstable lines of Λ then the Manning McCluskey Theorem applies for Λ. It follows that the $(1 + s_1)$-dimensional Hausdorff measures of both Λ and $\bar{\Lambda}$ are positive and finite. Then obviously both Λ and $\bar{\Lambda}$ have positive and finite $(1+s_1)$-dimensional packing measures as well. Therefore, without loss of generality we may assume in the rest of the paper the following.

PRINCIPAL ASSUMPTION. *There are two unstable lines of the attractor Λ intersecting each other.*

3. *Notation*

For any $H \subset [0,1] \times D$ and $t \in [0,1]$ we write $H(t)$ for the t-angular section of H. That is $H(t) := H \cap (\{t\} \times D)$.

To construct a symbolic dynamic, we introduce the symbolic space $\Sigma := \{1, \ldots, m\}^{\mathbb{Z}}$. As usual we write σ for the left shift. For technical reasons the right shift σ^{-1} on Σ will commute with $\bar{f}$ on $\bar{\Lambda}$ via the natural projection $\bar{\Pi} : \Sigma \to \bar{\Lambda}$. To define this natural projection we write $I_{i_1\ldots i_n} := \bigcap_{k=1}^{n} g^{-(k-1)}(I_k)$; further, put $\bar{\Delta}_{i_1\ldots i_n} := I_{i_1\ldots i_n} \times D$ and $\Delta_{i_1\ldots i_n} := I_{i_1\ldots i_n} \times [-1,1]$. For an $\mathbf{i} \in \Sigma$ we define

$$\bar{\Pi}(\mathbf{i}) := \lim_{n\to\infty} (\bar{\Delta}_{i_0\ldots i_{-(n-1)}} \cap \bar{f}^n(\bar{\Delta}_{i_n\ldots i_1})). \tag{3.1}$$

We call a set $\bar{\Delta}_{i_0\ldots i_{-(n-1)}}$ the *vertical n-cylinder* and $\bar{S}_{i_1\ldots i_n} := \bar{f}^n(\bar{\Delta}_{i_n\ldots i_1})$ is called the *horizontal n-cylinder*, while the set $\bar{C}^{i_1\ldots i_n}_{i_0\ldots i_{-(n-1)}} := \bar{\Delta}_{i_0\ldots i_{-(n-1)}} \cap \bar{f}^n(\bar{\Delta}_{i_n\ldots i_1})$ is called the *n-cylinder*. Note that a horizontal n-cylinder $\bar{S}_{i_1\ldots i_n}$ is a tube from the wall $\{0\} \times D$ to the wall $\{1\} \times D$ in $[0,1] \times D$. In the special case when our map is of the form (1.1), all its angular sections $\bar{S}_{i_1\ldots i_n}(t)$, for $t \in [0,1]$, are ellipses with half axes λ_1^n, λ_2^n. Note the first (t) coordinate of the point $\bar{\Pi}(\mathbf{i}) \in [0,1] \times D$ is $\bigcap_{n=0}^{\infty} \bar{\Delta}_{i_0\ldots i_{-(n-1)}}$. This is determined by the non-positive coordinates of $\mathbf{i}$. So, we may introduce $\bar{\Pi}^{-}(\mathbf{i}) := \bigcap_{n=0}^{\infty} \bar{\Delta}_{i_0 \ldots i_{-(n-1)}}$. In this way $\bar{\Pi}^{-}$ defines a map both from Σ and from

$$\Sigma^{-} := \{(i_0, i_{-1}, \ldots) | i_k \in \{1, \ldots m\} \text{ for } k \leq 0\}$$

into [0, 1].

We also see for any $\mathbf{i} \in \Sigma$ that the intersection of tubes $\bigcap_{n=0}^{\infty} \bar{S}_{i_1 \ldots i_n}$ is a curve called $\bar{\Lambda}_{i_1 i_2 \ldots}(t)$ for $t \in [0, 1]$. Clearly $\bar{\Lambda} = \bigcup_{\mathbf{i} \in \Sigma} \bar{\Lambda}_{i_1 i_2 \ldots}$. Let us call $\overline{\mathcal{C}urves}$ the set of all curves $\bar{\Lambda}_{i_1 i_2 \ldots}$. That is, $\overline{\mathcal{C}urves} := \{\bar{\Lambda}_{i_1 i_2 \ldots} | \mathbf{i} \in \Sigma^{+}\}$, where

$$\Sigma^{+} := \{(i_1, i_2, \ldots) | i_k \in \{1, \ldots m\} \text{ for } k \geq 1\}.$$

Then $\bar{\Pi}^{+}(\mathbf{i}) := \bar{\Lambda}_{i_1 i_2} \ldots$ defines a map both from Σ and Σ^{+} into $\overline{\mathcal{C}urves}$. Furthermore, we define

$$\rho_0(t, x, z) := t, \quad \rho_1(t, x, z) := (t, x), \quad \rho_2(t, x, z) := (t, z).$$

Changing from $\mathbb{R}^3$ to the (t, x) coordinate plane we repeat all the above definitions, using the same notation without bars over the symbols. Obviously,

$$\Pi = \rho_1 \circ \bar{\Pi}, \quad \Lambda_{i_1 i_2 \ldots}(t) = \rho_1 \circ \bar{\Lambda}_{i_1 i_2 \ldots}(t), \quad \mathcal{C}urves = \rho_1 \circ \overline{\mathcal{C}urves}.$$

LEMMA 1. *There is a uniform bound K for the derivative of the $\mathcal{C}^2$ curves $t \mapsto \Lambda_{i_1 i_2 \ldots}(t)$. That is, there exists a K such that for every $i_1 i_2 \cdots$ the curve $|(d/dt)\Lambda_{i_1 i_2 \ldots}(t)| \leq K$ holds for every $i_1 i_2 \cdots$ and for every $t \in [0, 1]$.*

Proof. Using that

$$\Lambda_{i_1 i_2 \ldots}(t) = [t, \lambda_1(g_{i_1}^{-1}(t), \lambda_1(g_{i_2}^{-1} \circ g_{i_1}^{-1}(t), \lambda_1(\ldots)))] \tag{3.2}$$

we immediately get the statement of the lemma. □

Let μ be a measure on $\mathbb{R}^n$ and X be a point in the support of μ. Then the upper and lower s-dimensional density of the measure μ is defined by

$$\underline{D}(\mu, X, s) = \liminf_{r \to 0} \frac{\mu(B(X, r))}{r^s}, \quad \overline{D}(\mu, X, s) = \limsup_{r \to 0} \frac{\mu(B(X, r))}{r^s}.$$

The following lemma is well known (see [**8**]).

LEMMA 2. *If $\overline{D}(\mu, X, s) \geq c$ for all $X \in E$, then $\mathcal{H}^s(E) \leq$ constant $\cdot (\mu(E)/c)$; further, if $\underline{D}(\mu, X, s) \leq c$ for all $X \in E$, then $\mathcal{P}^s(E) \geq$ constant $\cdot (\mu(E)/c)$.*

4. *The proof of Theorem 2*

Our aim in this section is to prove Theorem 2. Therefore, we always assume in this section that all intersections between unstable lines of Λ are transversal. We start with an easy lemma we will use later.

LEMMA 3. *Let η be a measure, l_n a family of real-valued functions, each satisfying the inequality*

$$\eta(\{x \mid l_n(x) > M\}) \leq h(M)$$

for a function $h : \mathbb{R} \to \mathbb{R}$. Denote $l(x) = \underline{\lim}_{n \to \infty} l_n(x)$. Then

$$\eta(\{x \mid l(x) > M\}) \leq h(M).$$

Proof. We can write

$$\{x \mid l(x) > M\} = \bigcup_N \bigcap_{n>N} \{x \mid l_n(x) > M\}.$$

This is a union of an increasing family of sets, hence the measure of the union equals to the supremum of measures of these sets. However,

$$\eta\left(\bigcap_{n>N} \{x \mid l_n(x) > M\}\right) \le \eta(\{x \mid l_{N+1}(x) > M\}) \le h(M). \qquad \square$$

We work with the linear Slanting Baker map

$$f(t,x) = (mt \bmod 1, \lambda x + r(t)). \tag{4.1}$$

As we assume transversality, there exists Q such that any two $\omega, \tau \in \Sigma^+$ with $\omega_1 \ne \tau_1$ the lines $\Lambda_\omega, \Lambda_\tau$ intersect each other in at most Q points. The mapping $mt \bmod 1$ will be denoted by g. In our case $s_1 = \log m / -\log \lambda$, hence $m\lambda < 1$. Let us define the functions $\bar{L}_t : \Sigma^+ \to \bar{\Lambda}$ and $L_t : \Sigma^+ \to \Lambda$ by

$$\bar{L}_t(\omega) = \bar{\Lambda}_\omega(t) \quad \text{and} \quad L_t = \rho_1 \circ \bar{L}_t. \tag{4.2}$$

Let $\tilde{\mu}$ be the Bernoulli measure on Σ^+ given by the probability vector $(1/m, \ldots, 1/m)$. Denote by μ_t its projection under L_t. We denote

$$\mu = \int \mu_t \, dt.$$

Note that $\mu = \Pi_* \tilde{\mu}$ and that μ is the SBR measure for the Slanting Baker map (4.1). Similarly, $\bar{\mu} = \bar{\Pi}_* \tilde{\mu}$ is the SBR measure for the solenoid. When we write ω^n, τ^n or $\mathbf{i}^n$ we always mean that they are elements of $\{1, \ldots, m\}^n$.

Let U be an open interval. We will denote by $U_{\omega^n}(t_0)$ the intersection of the line $t = t_0$ with the image of the strip $\{(t,x) \mid t \in \Delta_{\omega_n \ldots \omega_1}, x \in U\}$ under f^n. We write

$$J(\omega^n, \tau^n) = \{t \mid U_{\omega^n}(t) \cap U_{\tau^n}(t) \ne \emptyset\}.$$

We can find Q (independent of ω, τ) such that for every n this set is a union of at most Q intervals $J^{(i)}(\omega^n, \tau^n)$. We assumed that all intersections between unstable lines of Λ are transversal. It follows that there exists $0 < c < C$ such that if $\omega_1 \ne \tau_1$ then each of these intervals (except possibly those containing 0 or 1 which may be shorter) has length

$$c\lambda^n \le |J^{(i)}(\omega^n, \tau^n)| \le C\lambda^n, \tag{4.3}$$

since $|U_{\omega^n}| = |U_{\tau^n}| = |U|\lambda^n$.

For a $t \in [0,1]$, $k > 0$ and ω^n, τ^n, let us denote the number of those $\mathbf{i}^k$ for which $U_{\mathbf{i}^k \omega^n}(t) \cap U_{\mathbf{i}^k \tau^n}(t) \ne \emptyset$ by $J_k(\omega^n, \tau^n)(t)$. That is,

$$J_k(\omega^n, \tau^n)(t) := \sharp\{\mathbf{i}^k \mid t \in J(\mathbf{i}^k \omega^n, \mathbf{i}^k \tau^n)\}. \tag{4.4}$$

Therefore,

$$J_k(\omega^n, \tau^n)(t) = \sum_{\mathbf{i}^k} \mathbf{1}_{J(\omega^n, \tau^n)}(g_{i^k}^{-k}(t)) \tag{4.5}$$

One can see this as the characteristic function of the set $g^k(J(\omega^n, \tau^n))$, counted with multiplicities.

LEMMA 4. *For every n big enough, every ω^n, τ^n and every t_1, t_2, we have*

$$|J_k(\omega^n, \tau^n)(t_1) - J_k(\omega^n, \tau^n)(t_2)| \le Q. \tag{4.6}$$

Moreover, if $\omega_1 \neq \tau_1$ then for every n:

$$\int J_k(\omega^n, \tau^n)(t)\, dt \le CQ\lambda^n m^k. \tag{4.7}$$

Proof. Denote $G(t) = mt$, acting on the real line. This mapping is semiconjugated to g, hence

$$J_k(\omega^n, \tau^n)(t) = \sharp\{d \in \mathbb{Z} \mid t + d \in G^k(J(\omega^n, \tau^n))\}.$$

The sets $J^{(i)}(\omega^n, \tau^n)$ are intervals, hence sets $G^k(J^{(i)}(\omega^n, \tau^n))$ are also intervals, only m^k times longer. They are disjoint, hence we may write

$$J_k(\omega^n, \tau^n)(t) = \sum_i \sharp\{d \in \mathbb{Z} \mid t + d \in G^k(J^{(i)}(\omega^n, \tau^n))\}.$$

Now every one of the summands on the right-hand side of this equality may differ at most by 1 when we change t and that implies that (4.6) holds.

The second part of the assertion immediately follows from (4.3) and from the definition of $J_k(\omega^n, \tau^n)(t)$. □

We denote

$$\tilde{A}_n(t) = \sharp\{(\omega^n, \tau^n) \mid U_{\omega^n}(t) \cap U_{\tau^n}(t) \neq \emptyset\}.$$

Similarly,

$$A_n(t) = \sharp\{(\omega^n, \tau^n) \mid \omega_1 \neq \tau_1, U_{\omega^n}(t) \cap U_{\tau^n}(t) \neq \emptyset\}.$$

We use the convention $\tilde{A}_0 \equiv 1$, $A_0 \equiv 0$. As A_n is the sum of characteristic functions of all possible sets $J(\omega^n, \tau^n)$ with $\omega_1 \neq \tau_1$ (there are $m^{2n-1}(m-1)$ of them), we can use (4.3) to obtain the following estimation:

$$\int A_n(t)\, dt \le C' Q\lambda^n m^{2n-1}(m-1), \tag{4.8}$$

which is again true for all n.

The following proposition is crucial for the rest of the section.

PROPOSITION 1. *There exists $K > 0$ such that for almost every t (with respect to the Lebesgue measure),* $\limsup_{n\to\infty} m^{-n}\tilde{A}_n(t) \le K$.

Proof. If $\omega_1 = \tau_1$ then the sets $U_{\omega^n}(t)$ and $U_{\tau^n}(t)$ intersect each other if and only if the sets $U_{\omega_2\ldots\omega_n}(g_{\omega_1}^{-1}(t))$ and $U_{\tau_2\ldots\tau_n}(g_{\omega_1}^{-1}(t))$ do, because the mapping f restricted to Δ_{ω_1} is bijective. Hence

$$\tilde{A}_n(t) = A_n(t) + \sum_{i=1}^{m} \tilde{A}_{n-1}(g_i^{-1}(t)).$$

Recursively, the formula follows

$$\tilde{A}_n(t) = m^n + \sum_{k=0}^{n} \sum_{\tilde{t}\in g^{-k}(t)} A_{n-k}(\tilde{t}).$$

We can write

$$m^{-n}\tilde{A}_n(t) = m^{-n}\sum_{k=0}^{K(n)}\sum_{\tilde{t}\in g^{-k}(t)} A_{n-k}(\tilde{t}) + \left(1 + m^{-n}\sum_{k=K(n)}^{n}\sum_{\tilde{t}\in g^{-k}(t)} A_{n-k}(\tilde{t})\right)$$

where $K(n) = ((-\log c - n\log\lambda + \log Q)/(\log m - \log\lambda))$. The two summands will be denoted by $B_{1,n}(t)$ and $B_{2,n}(t)$.

Let us start our estimations from the second one. If $k > K(n)$, then $c\lambda^{n-k}m^k > Q$. Then for any t_1 and t_2 and for any ω^{n-k}, τ^{n-k} such that $\omega_1 \neq \tau_1$, Lemma 4 gives us

$$\frac{1}{Q+1} \leq \frac{J_k(\omega^{n-k}, \tau^{n-k})(t_1)}{J_k(\omega^{n-k}, \tau^{n-k})(t_2)} \leq Q + 1.$$

One has only to notice that

$$\sum_{\tilde{t}\in g^{-k}(t)} A_{n-k}(\tilde{t}) = \sum_{\omega^{n-k},\tau^{n-k};\omega_1\neq\tau_1} J_k(\omega^{n-k}, \tau^{n-k})(t)$$

to get a similar statement for $B_{2,n}$: for all t_1, t_2,

$$\frac{1}{Q+1} \leq \frac{B_{2,n}(t_1)}{B_{2,n}(t_2)} \leq Q + 1. \tag{4.9}$$

We will now estimate the average values of $B_{1,n}$ and $B_{2,n}$. Using (4.8) we get

$$\int B_{1,n}(t)\,dt \leq m^{-n}\sum_{k=0}^{K(n)} m^k CQ\lambda^{n-k}m^{2n-2k-1}(m-1) \approx (m\lambda)^{n-K(n)}$$

and as $m\lambda < 1$ and $n - K(n)$ is asymptotically a linear function of n, we get

$$\sum_{n=1}^{\infty}\int B_{1,n}(t)\,dt < \infty,$$

so that $B_{1,n}(t)$ goes to zero for almost every t. Similarly,

$$\int B_{2,n}(t)\,dt \leq m^{-n}\left(m^n + \sum_{k=K(n)}^{n} m^k C'Q\lambda^{n-k}m^{2n-2k-1}(m-1)\right) \approx 1$$

hence (by (4.9)) $B_{2,n}(t)$ is universally bounded. The assertion follows. □

Until now, we did not need any assumptions about the set U. Now we will assume it is so large that

$$\Lambda \subset [0,1] \times U.$$

Choose t and let $(t, x) = \Lambda_\omega(t)$. The measure μ_t is defined as the Π^+ projection of the $\{1/m, \ldots, 1/m\}$ distributed Bernoulli measure on the symbolic space Σ^+. We can choose constants $0 < c_1, c_2$ (dependent on U) such that for any $x \in \Lambda_{\omega^n}(t)$, if $U_{\omega^n}(t) \cap U_{\tau^n}(t) \neq \emptyset$, then the ball $B_{c_1\lambda^n}(x)$ will contain $U_{\tau^n}(t)$ (hence $\Lambda_{\tau^n}(t)$ as well); while if $U_{\omega^n}(t) \cap U_{\tau^n}(t) = \emptyset$, then the ball $B_{c_2\lambda^n}(x)$ will not intersect $\Lambda_{\tau^n}(t)$. The constants c_1, c_2 depend only on U (but not on t, ω or n). We obtained

$$\begin{aligned} \mu_t(B_{c_1\lambda^n}(x)) &\geq m^{-n}\tilde{A}_t(\omega^n) \\ \mu_t(B_{c_2\lambda^n}(x)) &\leq m^{-n}\tilde{A}_t(\omega^n) \end{aligned} \tag{4.10}$$

where $\tilde{A}_t(\omega^n) = \sharp\{\tau^n \mid U_{\omega^n}(t) \cap U_{\tau^n}(t) \neq \emptyset\}$.

Note that

$$\sum_{\omega^n} \tilde{A}_t(\omega^n) = \tilde{A}_n(t),$$

hence

$$\sharp\{\tau^n \mid \tilde{A}_t(\tau^n) \geq M\} \leq \frac{\tilde{A}_n(t)}{M}. \tag{4.11}$$

We may now prove Theorem 2.

Proof. First we prove that the packing measure of the attractor is positive. Note that it is enough to prove this for the two-dimensional map $\bar{f}$ (4.1) as the corresponding attractor Λ is the projection of $\bar{\Lambda}$.

We introduce a new open interval V, containing U. All the statements we proved for U in this section also remain true for V (except perhaps the constants may change). We will use the notation $\tilde{A}_n(t; U)$ or $\tilde{A}_n(t; V)$ (and similarly for other functions) to distinguish the functions defined above for U from analogical ones we define for V.

Given ω, $\Lambda_\omega(t)$ is a Lipschitz function. Hence we can choose V in such a way that

$$U_{\omega^n}(t) \cap U_{\tau^n}(t) \neq \emptyset \Longrightarrow \text{for all } \tilde{t} \in (t - \lambda^n, t + \lambda^n), \quad V_{\omega^n}(\tilde{t}) \cap V_{\tau^n}(\tilde{t}) \neq \emptyset.$$

We have

$$\tilde{A}_t(\tau^n; V) \geq \sup_{\tilde{t} \in (t-\lambda^n, t+\lambda^n)} \tilde{A}_{\tilde{t}}(\tau^n; U),$$

hence (by (4.11))

$$\sharp\{\tau^n \mid \exists \tilde{t} \in (t - \lambda^n, t + \lambda^n),\ \tilde{A}_{\tilde{t}}(\tau^n; U) \geq M\} \leq \frac{\tilde{A}_n(t; V)}{M}. \tag{4.12}$$

We want to prove that the $(1 + s_1)$-dimensional packing measure of Λ is positive. In order to do this we only need to prove that the lower $(1 + s_1)$-dimensional density of the measure μ is finite for μ-almost all (t, x). We may assume that our t is chosen such that the assertion of Proposition 1 is satisfied for $\tilde{A}_n(t; V)$. For almost all x we have

$$\underline{D}(\mu, (t, x), 1 + s) \leq C'' \underline{\lim}_{n\to\infty} \lambda^{-n(1+s_1)} \mu((t - \lambda^n, t + \lambda^n) \times B_{c_2\lambda^n}(\Lambda_\omega(\tilde{t}))).$$

We recall that $\lambda^{-s_1} = m$. Using (4.10) we get

$$\mu((t - \lambda^n, t + \lambda^n) \times B_{c_2\lambda^n}(\Lambda_\omega(\tilde{t}))) \leq 2\lambda^n m^{-n} \sup_{\tilde{t} \in (t-\lambda^n, t+\lambda^n)} \tilde{A}_{\tilde{t}}(\omega^n; U).$$

Hence, we are interested in the lower limit (when n goes to the infinity) of $\sup_{\tilde{t} \in (t-\lambda^n, t+\lambda^n)} \tilde{A}_{\tilde{t}}(\omega^n; U)$. We can now use Lemma 3 for $\eta = \mu_t$, $h(M) = (K + 1)/M$, $l_n(\omega) = \sup_{\tilde{t} \in (t-\lambda^n, t+\lambda^n)} \tilde{A}_{\tilde{t}}(\omega^n; U)$ from (4.12) and Proposition 1; for Lebesgue-almost all t we get that

$$\mu_t(\{x \mid \underline{D}(\mu, (t, x), 1 + s) \leq M\}) \geq 1 - \frac{2(K + 1)}{C''M}.$$

Using (4.2) we obtain a little bit more than the positivity of the packing measure, namely

$$\frac{d\mathcal{P}^{1+s_1}}{d\mu} > 0 \quad \mu\text{-a.e.} \tag{4.13}$$

If our Slanting Baker map is the two-dimensional projection of the solenoid map, the same result must be true for the solenoid map as well (for the measure $\bar{\mu}$).

Now we prove our remaining assertions, both for the (two-dimensional) Slanting Baker map f and for the (three-dimensional) solenoid map $\bar{f}$. In the case of the Slanting Baker map, consider another Slanting Baker map $\tilde{f}$ with the same g and λ, but bijective on its attractor $\tilde{\Lambda}$; in the case of the solenoid map let $\tilde{f}$ be a linear solenoid with same g, λ_1, with $\lambda_2 = \lambda_1$ and let $\tilde{\Lambda}$ be its attractor. It is well known in both cases that such a $\tilde{\Lambda}$ has positive and finite $(1+s_1)$-dimensional Hausdorff and packing measures and the density $d\mathcal{P}^{1+s_1}/d\tilde{\mu}$ (where the packing measure is restricted to the attractor) is uniformly bounded from both below and above, see [**11**]. We then have the semiconjugacy (for Slanting Baker maps) or conjugacy (for solenoid maps) acting from $\tilde{\Lambda}$ onto Λ (for Slanting Baker maps) or $\bar{\Lambda}$ (for solenoid maps), given by

$$h_t = L_t \circ \tilde{L}_t^{-1}, \quad h_t = \bar{L}_t \circ \tilde{L}_t^{-1}$$

for the Slanting Baker transformation and solenoid, respectively. It is well defined, because the projection $\tilde{L}_t$ from Σ^+ onto $\{(\tilde{t}, \dots) \in \tilde{\Lambda} \mid \tilde{t} = t\}$ is one to one.

For both Slanting Baker maps and solenoid maps this (semi)conjugacy is Lipschitz. Then the densities $d\mathcal{P}^{1+s_1}/d\mu$ (for Slanting Baker maps) or $d\mathcal{P}^{1+s_1}/d\bar{\mu}$ (for solenoid maps) may be greater than $d\mathcal{P}^{1+s_1}/d\tilde{\mu}$ by at most a multiplicative constant, hence they are uniformly bounded from above. This proves the finiteness of the $(1+s_1)$-dimensional packing measure of Λ and (together with (4.13)) equivalence between $\mathcal{P}^{1+s_1}$ and μ $(\bar{\mu})$. □

5. *The proof of Theorem 1*

In this section we consider the nonlinear case. We will define a measure supported on $\bar{\Lambda}$ which is not invariant but has full Hausdorff dimension.

5.1. *The measures we need*

We define the potentials $\Psi_1, \Psi_2 : \Sigma \to \mathbb{R}$ by $\Psi_1(\mathbf{j}) := \log(\partial/\partial x)\lambda_1(\rho_1(\bar{\Pi}(\sigma\mathbf{j})))$ and $\Psi_2(\mathbf{j}) := \log(\partial/\partial z)\lambda_2(\rho_2(\bar{\Pi}(\sigma\mathbf{j})))$. Observe that $\psi_k(\mathbf{j}) = \bar{\varphi}_k(\bar{\Pi}(\sigma^{-1}\mathbf{j}))$, $k = 1, 2$.

Then using that $\bar{\Pi} : \Sigma \to \bar{\Lambda}$ is a homeomorphism, it follows from [**2**, Proposition 2.13] that

$$P(s_1\Psi_1) = 0 \quad \text{and} \quad P(s_2\Psi_2) = 0 \tag{5.1}$$

hold. Denote the Gibbs measures of the potential $s_l\Psi_l$ by ν_l $(l = 1, 2)$. Then there exists a $d > 0$ such that

$$\nu_l(i_1 \dots i_n) \in [d^{-1}, d] \exp\left(s_l \sum_{k=0}^{n-1} \Psi_l(\sigma^k(\mathbf{i}))\right) \tag{5.2}$$

holds $(l = 1, 2)$ (see [**2**, p. 10]). Consider Σ as a product space of $\Sigma^- \times \Sigma^+$ and we write ν_l^-, ν_l^+ for the induced measures on Σ^- and Σ^+, respectively. Then it follows from the σ-invariance of the measure ν_l and from (5.2) that

$$\nu_l \sim \nu_l^- \times \nu_l^+ \quad l = 1, 2. \tag{5.3}$$

Note that $\bar{\Pi}_*\nu_l^-$ is a measure on $[0, 1]$ and $\bar{\Pi}_*\nu_l^+$ is a measure on $\overline{\mathcal{C}urves}$. In the linear case, the measure $\bar{\Pi}_*\nu_l^-$ is the Lebesgue measure. However, in general $\bar{\Pi}_*\nu_l^-$ is singular to the Lebesgue measure. Let η be the absolutely continuous invariant measure for g. Then $\Pi_*^{-1}(\eta)$ is a measure on Σ^-. We define a measure supported on $\bar{\Lambda}$ which is not invariant in the nonlinear case, but has full Hausdorff dimension. We also call it μ since in the linear case it is invariant and even coincides with the SBR measure. In general, there is no invariant measure of full Hausdorff dimension see [**11**].

$$\mu := \Pi_*((\Pi^-)_*^{-1}(\eta) \times \nu_1^+). \tag{5.4}$$

LEMMA 5. *For ν_1-almost every $\mathbf{j} \in \Sigma$ and for any $\varepsilon > 0$ there exists an $L = L(\mathbf{j})$ such that, if $n > L$, then*

$$\frac{\sum_{k=0}^{n-1} \Psi_1(\sigma^k \mathbf{j})}{\sum_{k=0}^{n-1} \Psi_2(\sigma^k \mathbf{j})} \le \frac{s_2}{s_1} + \varepsilon. \tag{5.5}$$

Proof. Using the definition of ν_1 and ν_2 and the variational principle (see [**2**]) twice, we get that

$$0 = P(s_1\Psi_1) = h_{\nu_1}(\sigma) + s_1 \int \Psi_1 \, d\nu_1,$$

and

$$0 = P(s_2\Psi_2) = h_{\nu_2}(\sigma) + s_2 \int \Psi_2 \, d\nu_2 \ge h_{\nu_1}(\sigma) + s_2 \int \Psi_2 \, d\nu_1.$$

Since $\Psi_i < 0$, $i = 1, 2$, we obtain that $\int \Psi_1 \, d\nu_1 \big/ \int \Psi_2 \, d\nu_1 \le s_2/s_1$.

From the ergodicity of ν_1 we have that for ν_1-a.e. $\mathbf{j} \in \Sigma$, $(1/n)\sum_{k=0}^{n-1} \Psi_1(\sigma^k \mathbf{j}) \to \int \Psi_1(\mathbf{j})\, d\nu_1(\mathbf{j})$ and $(1/n)\sum_{k=0}^{n-1} \Psi_2(\sigma^k \mathbf{j}) \to \int \Psi_2(\mathbf{j})\, d\nu_1(\mathbf{j})$ which immediately follows the statement of the lemma. $\square$

5.2. *Bounded distortion lemmas.* We recall that f, $\Delta_{i_1 \ldots i_n}$, Π, $S_{i_1 \ldots i_n}$, $\mathcal{C}^{i_1 \ldots i_n}_{i_0 \ldots i_{-(n-1)}}$ and Λ were defined as the ρ_1 projection of $\bar{f}$, $\bar{\Delta}_{i_1 \ldots i_n}$, $\bar{\Pi}$, $\bar{S}_{i_1 \ldots i_n}$, $\bar{\mathcal{C}}^{i_1 \ldots i_n}_{i_0 \ldots i_{-(n-1)}}$ and $\bar{\Lambda}$, respectively. In this section, we work mainly in the (t, x) coordinate plane. Therefore, to simplify the formulas, we write λ instead of λ_1. Because of the symmetry, all the results remain valid if we apply the projection ρ_2 instead of ρ_1.

Let us denote the set of n-cylinders $\mathcal{C}^{i_1 \ldots i_n}_{i_0 \ldots i_{-(n-1)}}$ on the (t, x) coordinate plane by $\mathcal{C}_n$. It follows from the hyperbolicity of the map f that there is a constant $c_1 > 0$ and $0 < p < 1$ such that for every n-cylinder, $\operatorname{diam}(\mathcal{C}^{i_1 \ldots i_n}_{i_0 \ldots i_{-(n-1)}}) < c_1 p^n$. From this, we immediately obtain the following lemma.

LEMMA 6. *There exists a constant $c_2 > 0$ such that for any $k \in \mathbb{N}$ and any $C \in \mathcal{C}_k$, we have $|\log(\partial/\partial x)\lambda(z_1) - \log(\partial/\partial x)\lambda(z_2)| < c_2 p^k$ for any $z_1, z_2 \in C$.*

In the following two lemmas, we frequently use the second component function f_2^n of the function f^n. That is, $f^n(t, x) = (g^n(t), f_2^n(t, x))$. Obviously $f_2^n(t, x) = \lambda(g^{n-1}(t), f_2^{n-1}(t, x))$. Thus, using the chain rule,

$$\frac{\partial}{\partial x} f_2^n(P) = \prod_{k=0}^{n-1} \frac{\partial}{\partial x} \lambda(P_k), \tag{5.6}$$

where $P = P_0 = (t, x)$ and $P_k := f^k(P)$.

LEMMA 7. *There is a constant $c_3 > 0$, such that if $P^1, P^2 \in C \in \mathcal{C}_n$ for some n, then*

$$c_3^{-1} < \frac{(\partial/\partial x) f_2^n(P^1)}{(\partial/\partial x) f_2^n(P^2)} < c_3 \tag{5.7}$$

holds.

Proof. Note that $P_k^1 := f^k(P^1)$ and $P_k^2 := f^k(P^2)$ are in the same $n-k$ cylinder. Thus, it follows from Lemma 6 that $|\log(\partial/\partial x)\lambda(P_k^1) - \log(\partial/\partial x)\lambda(P_k^2)| < c_2 p^{n-k}$. Therefore, we have $\left|\sum_{k=0}^{n-1} \log(\partial/\partial x)\lambda(P_k^1) - \sum_{k=0}^{n-1} \log(\partial/\partial x)\lambda(P_k^2)\right| < c_2/(1-p)$ for every n. Using (5.6) we obtain that

$$\left|\log \frac{(\partial/\partial x) f_2^n(P^1)}{(\partial/\partial x) f_2^n(P^2)}\right| < \frac{c_2}{1-p}.$$

This completes the proof with $c_3 := \exp(c_2/(1-p))$. □

LEMMA 8. *There exists a $c_4 > 0$ such that, for all $t \in [0,1]$ and for every $x_1, x_2 \in [-1,1]$, we have*

$$c_4^{-1} < \frac{(\partial/\partial x) f_2^n(P^1)}{(\partial/\partial x) f_2^n(P^2)} < c_4 \tag{5.8}$$

for $P^1 = (t, x_1)$ and $P^2 = (t, x_2)$.

Proof. The same was proved in [**19**, Lemma 3.1]. □

Putting together the last two lemmas, we obtain that for $t_1, t_2 \in I_{j_0 \dots j_{-(n-1)}}$ for some $j_0 \dots j_{-(n-1)}$, and then for arbitrary $x_1, x_2 \in [-1,1]$,

$$c_5^{-1} < \frac{(\partial/\partial x) f_2^n(t_1, x_1)}{(\partial/\partial x) f_2^n(t_2, x_2)} < c_5 \tag{5.9}$$

holds with $c_5 = c_3 c_4^2$. Namely, let $(j_1, \dots, j_n)$ be arbitrary. We choose points $P_1, P_2 \in \mathcal{C}^{j_1 \dots j_n}_{j_0 \dots j_{-(n-1)}}$ such that their first coordinates are t_1, t_2, respectively. Then it follows from Lemmas 7 and 8 that

$$c_4^{-1} < \frac{g(t_1, x_1)}{g(P_1)} < c_4, \quad c_3^{-1} < \frac{g(P_1)}{g(P_2)} < c_3, \quad c_4^{-1} < \frac{g(P_2)}{g(t_2, x_2)} < c_4.$$

It immediately follows that (5.9) holds.

We are going to use the lemmas above as follows. Let $t \in I_{j_n \dots j_1}$ and let $\theta = g^n(t)$. Then $f_2^n(t, \cdot) : (\{t\} \times [-1,1]) \to (\{\theta\} \times [-1,1])$ and $S_{j_1 \dots j_n} = f^n(I_{j_n \dots j_1} \times [-1,1])$; further,

$$|S_{j_1 \dots j_n}(\theta)| = 2 \frac{\partial}{\partial x} f_2^n(t, x) \tag{5.10}$$

holds for an $x \in [-1,1]$.

Vice versa, for any $\theta \in [0,1]$ and $j_1 \dots j_n$ we can find a (t, x) such that $t \in I_{j_n \dots j_1}$, $\theta = g^n(t)$ and (5.10) holds. In this way (5.9) implies that for any $j_1 \dots j_n$ and $\theta_1, \theta_2 \in [0,1]$

$$c_5^{-1} < \frac{|S_{j_1 \dots j_n}(\theta_1)|}{|S_{j_1 \dots j_n}(\theta_2)|} < c_5. \tag{5.11}$$

This means that the ratios of the width of $S_{j_1\ldots j_n}$ for different θ are uniformly bounded. We need one more bounded distortion lemma. Let us denote the set of finite words in the alphabet $\{1,\ldots,m\}$ by Σ^*. That is, $\Sigma^* := \bigcup_{k=1}^{\infty}\{1,\ldots,m\}^k$. Usually we write $\mathbf{i},\mathbf{j},\boldsymbol{\tau},\boldsymbol{\omega}$ for the elements of Σ^*.

LEMMA 9. *For any* $\theta_1,\theta_2,\theta_3,\theta_4 \in [0,1]$ *and any* $\mathbf{i},\mathbf{j},\boldsymbol{\tau} \in \Sigma^*$, *if*

$$\frac{|S_{\mathbf{i}}(\theta_1)|}{|S_{\mathbf{j}}(\theta_2)|} \in (e_1, e_2)$$

then

$$\frac{|S_{\boldsymbol{\tau}\mathbf{i}}(\theta_3)|}{|S_{\boldsymbol{\tau}\mathbf{j}}(\theta_4)|} \in (c_5^{-3}e_1, c_5^3 e_2). \tag{5.12}$$

Proof. Using that

$$f(S_{\mathbf{i}} \cap \Delta_{\tau_k\ldots\tau_1}) = S_{\tau_k\mathbf{i}} \cap \Delta_{\tau_{k-1}\ldots\tau_1} \tag{5.13}$$

we obtain that $f^k(S_{\mathbf{i}} \cap \Delta_{\tau_k\ldots\tau_1}) = S_{\boldsymbol{\tau}\mathbf{i}}$ and $f^k(S_{\mathbf{j}} \cap \Delta_{\tau_k\ldots\tau_1}) = S_{\boldsymbol{\tau}\mathbf{j}}$. Thus, there exist $t_1, t_2 \in \Delta_{\tau_k\ldots\tau_1}$ and $x_1, x_2 \in [-1,1]$ such that

$$|S_{\mathbf{i}}(t_1)|\left|\frac{\partial}{\partial x} f_2^k(t_1,x_1)\right| = |S_{\boldsymbol{\tau}\mathbf{i}}(\theta_3)| \quad \text{and} \quad |S_{\mathbf{j}}(t_2)|\left|\frac{\partial}{\partial x} f_2^k(t_2,x_2)\right| = |S_{\boldsymbol{\tau}\mathbf{j}}(\theta_4)|. \tag{5.14}$$

Hence,

$$\frac{|S_{\boldsymbol{\tau}\mathbf{i}}(\theta_3)|}{|S_{\boldsymbol{\tau}\mathbf{j}}(\theta_4)|} = \frac{|S_{\mathbf{i}}(t_1)||(\partial/\partial x) f_2^k(t_1,x_1)|}{|S_{\mathbf{j}}(t_2)||(\partial/\partial x) f_2^k(t_2,x_2)|} \in (c_5^{-3}e_1, c_5^3 e_2)$$

follows from the assumption of the lemma and (5.9). □

5.3. *Lemmas about intersecting horizontal cylinders.* In this section we prove that there are many horizontal cylinders (as many as we wish) of approximately the same size lying close (in comparison to their size) to each other.

LEMMA 10. *For an arbitrary* $N \in \mathbb{N}$ *we can find* $\mathbf{i}^1,\ldots,\mathbf{i}^N$, *such that:*

(a) $S_{\mathbf{i}^k}(\theta) \subset S_{\mathbf{i}^1}(\theta)$ *and* $|S_{\mathbf{z}\mathbf{i}^k}(\theta)| \geq q_1 c_5^{-3}|S_{\mathbf{z}\mathbf{i}^1}(\theta)|$ *for every* $\mathbf{z} \in \Sigma^*$ *and* $\theta \in [0,1]$;

(b) $|S_{\mathbf{i}^k}(\theta)|/|S_{\mathbf{i}^l}(\theta)| \in [c_5^{-3}q_1, c_5^3/q_1]$ *for all* $1 \leq k,l \leq N$ *and* $\theta \in [0,1]$; *we recall that* q_1 *was defined as the minimum of* $(\partial/\partial x)\lambda$.

First observe that (b) immediately follows from (a).

Proof. We use mathematical induction to prove (a). For $N = 1$ the statement is trivial. For $N > 1$ assume that we have already constructed $\mathbf{i}^1,\ldots,\mathbf{i}^{N-1}$ satisfying (a). It follows from our principal assumption that there exist $\omega, \tau \in \Sigma^+$ such that the curves $\Lambda_\omega(t)$ and $\Lambda_\tau(t)$ intersect each other at a certain $t_0 \in [0,1]$ and $\tau_1 \neq \omega_1$. Then there exist L such that for each $k, n \geq L$ the horizontal cylinders $S_{\tau_1\ldots\tau_k}$ and $S_{\omega_1\ldots\omega_n}$ also cross each other. Let $\boldsymbol{\tau} := (\tau_1,\ldots,\tau_L)$. Then

$$f^L(S_{\mathbf{i}^p} \cap \Delta_{\tau_L\ldots\tau_1}) = S_{\boldsymbol{\tau}\mathbf{i}^p} \tag{5.15}$$

holds for $1 \leq p \leq N-1$.

Therefore, from the assumption we obtain that $S_{\boldsymbol{\tau}\mathbf{i}^p} \subset S_{\boldsymbol{\tau}\mathbf{i}^1}$ for $2 \le p \le N-1$. We can choose $k > N$ and $t', t'' \in [0,1]$ such that, for $\boldsymbol{\omega} = \omega_1, \dots, \omega_k$ and $\theta \in [t', t'']$, we have

$$S_{\boldsymbol{\omega}}(\theta) \subset S_{\boldsymbol{\tau}\mathbf{i}^1}(\theta) \quad \text{and} \quad q_1|S_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)| \le |S_{\omega}(\theta)|. \tag{5.16}$$

Choose $\mathbf{j} := j_1, \dots, j_q$ such that $\Delta_{j_q \dots j_1} \subset [t', t'']$. Using (5.16) and Lemma 9 we obtain that $S_{\mathbf{j}\omega}(t) \subset S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^1}(t)$ and $c_5^{-3}q_1|S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^1}(t)| \le |S_{\mathbf{j}\omega}(t)|$ holds for all $t \in [0,1]$. Also, it follows from the assumption that $S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^p}(t) \subset S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^1}(t)$ and $c_5^{-3}q_1|S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^1}(t)| \le |S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^p}(t)|$ holds for $2 \le p \le N-1$ and all $t \in [0,1]$. Thus the N different horizontal cylinders $S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^1}, S_{\mathbf{j}\boldsymbol{\omega}}, S_{\mathbf{j}\boldsymbol{\tau}\mathbf{i}^p}$ for $2 \le p \le N-1$ satisfy the assumption. □

Fix N and $\mathbf{i}^1, \dots, \mathbf{i}^N$ constructed above. Let Q be the subset of Σ^+ covered infinitely many times by $\bigcup_{n \ge 0} \sigma^{-n}\mathbf{i}^1$. That is,

$$Q = \{\mathbf{j} \in \Sigma^+ \mid \exists \text{ infinitely many } k \in \mathbb{N} \text{ such that } \sigma^k \mathbf{j} \subset \mathbf{i}^1\} \tag{5.17}$$

Remark 4. It follows from the ergodicity of the measure ν_1^+ that $\nu_1^+(Q) = 1$. Furthermore, it follows from the definition of Q that there are infinitely many k such that for some $\boldsymbol{\tau}^k = (\tau_1, \dots, \tau_k)$, $\Lambda_{\mathbf{j}} \subset S_{\boldsymbol{\tau}^k\mathbf{i}^1}$ holds. This implies that, for every $\theta \in [0,1]$, the $r = |S_{\boldsymbol{\tau}^k\mathbf{i}^1}(\theta)|$ neighborhood in $[-1,1]$ of the second coordinate of the point $\Lambda_{\mathbf{j}}(\theta)$ contains N intervals $S_{\boldsymbol{\tau}^k\mathbf{i}^1}(\theta), \dots, S_{\boldsymbol{\tau}^k\mathbf{i}^N}(\theta)$ of approximately the same size. So, the upper t-density of the measure $\mathcal{L}eb \times \nu^+$ is infinite almost everywhere. This follows that the t-dimensional Hausdorff measure is zero.

In fact we prove more than this, namely an analogue statement in space.

5.4. *The axes of the ellipses.* Consider $\bar{S}_{\mathbf{j}}(t)$ for an arbitrary $t \in [0,1]$. This is an ellipse in the very special case when $\bar{f}$ is defined by (1.1). In general, $\bar{S}_{\mathbf{j}}(t)$ is not an ellipse but contained in the rectangle with vertices

$$\{\bar{f}^n(\theta,-1,0),\ \bar{f}^n(\theta,1,0),\ \bar{f}^n(\theta,0,-1),\ \bar{f}^n(\theta,0,1)\}$$

where $\theta \in I_{j_n \dots j_1}$ and $g^n(\theta) = t$. Let $A_{\mathbf{j}}^1(t)$, $A_{\mathbf{j}}^2(t)$ be the length of the horizontal and vertical sides of this rectangle above. That is, $A_{\mathbf{j}}^1(t)$ is the distance of the first two vertices and $A_{\mathbf{j}}^2(t)$ is the distance of the last two vertices of the rectangle above. Then we can express $A_{\mathbf{j}}^k(t)$ with ψ_k.

LEMMA 11. *Let* $\mathbf{j} = (j_1 \dots j_n)$ *be arbitrary. For any* $t \in [0,1]$, $\boldsymbol{\omega} \in (j_1 \dots j_n)$, *that is* $\omega_l = j_l$ *for* $1 \le l \le n$, *the following holds:*

$$\frac{A_{\mathbf{j}}^k(t)}{\exp\left(\sum_{l=0}^{n-1} \psi_k(\sigma^l \boldsymbol{\omega})\right)} \in \left(\frac{2}{c_5}, 2c_5\right) \quad k = 1, 2. \tag{5.18}$$

Proof. Fix $t \in [0,1]$. We can find θ such that $g^n(\theta) = t$ and $\theta \in I_{j_n \dots j_1}$. It is enough to prove the lemma for $k = 1$. Using (5.10) and the chain rule (we may use it because this is essentially a one-dimensional computation) we obtain that there exists an $x_0 \in [-1,1]$ such that

$$A_{\mathbf{j}}^1(t) = 2\prod_{k=0}^{n-1}\left|\frac{\partial}{\partial x} f_2^n(\theta, x_0)\right| = 2\prod_{k=0}^{n-1}\frac{\partial}{\partial x}\lambda_1(P_k)$$

where $P_0 := (\theta, x_0)$ and $P_k := f^k(P_0)$. Choose an arbitrary $\omega \in \Sigma \cap (j_1 \dots j_n)$. That is, $\omega_1 = j_1, \dots, \omega_n = j_n$. Then $P_0' := \Pi(\sigma^n \boldsymbol{\omega}) \in \Delta_{j_n \dots j_1}$. Using that $P_0 \in \Delta_{j_n \dots j_1}$ also holds, we obtain from (5.9) that

$$A^1_{\mathbf{j}}(t) \in (2c_{5^{-1}}, 2c_5) \prod_{k=0}^{n-1} \frac{\partial}{\partial x} \lambda_1(\Pi(\sigma^{n-k}\boldsymbol{\omega})).$$

In the last step we use that f commutes with σ^{-1}, which implies that

$$\frac{A^1_{\mathbf{j}}(t)}{\exp\big(\sum_{k=1}^{n} \log(\partial/\partial x)\lambda_1(\Pi(\sigma^k\boldsymbol{\omega}))\big)} \in (2c_5^{-1}, 2c_5).$$

Using that $\rho_2(\bar{\Pi}(\sigma^k\boldsymbol{\omega})) = \Pi(\sigma^k\boldsymbol{\omega})$, the summand in the denominator is just $\psi_1(\bar{\Pi}(\sigma^{k-1}\boldsymbol{\omega}))$ which completes the proof. □

Let m' be the maximum length of the words $\mathbf{i}^1, \dots, \mathbf{i}^N \in \Sigma^*$. We are going to prove that the horizontal axis $A^1_{\mathbf{j}}(t)$ is longer than the vertical axis $A^2_{\mathbf{j}}(t)$ for almost every $t \in [0, 1]$. Actually, we prove a little more than this.

LEMMA 12. *Let $K = \frac{1}{2}(s_2/s_1 + 1)$. Then there exist T such that if $n > T$ then for every $\mathbf{j} \in \Sigma$ we have*

$$K \sum_{k=0}^{n-1} \psi_2(\sigma^k\mathbf{j}) > \sum_{k=0}^{n-m'-1} \psi_2(\sigma^k\mathbf{j}).$$

Proof. If n is big enough,

$$\frac{\sum_{k=0}^{n-m'} \psi_2(\sigma^k\mathbf{j}) + \sum_{k=n-m'}^{n-1} \psi_2(\sigma^k\mathbf{j})}{\sum_{k=0}^{n-m'-1} \psi_2(\sigma^k\mathbf{j})} \le 1 + \frac{-m' \log q_1}{-(n-m')\log q_2} < \frac{1}{K}$$

holds.

This implies the statement of the lemma, since in the left-hand side the denominator is negative. □

Now we are ready to prove the main lemma of this section.

LEMMA 13. *For ν_1^+-almost every $\mathbf{j} \in \Sigma^+$, there exists $M = M(\mathbf{j})$, such that, for all $n \ge M$ and $\theta \in [0, 1]$,*

$$\frac{A^1_{j_1 \dots j_n}(\theta)}{A^2_{j_1 \dots j_{n-m'}}(\theta)} \ge 4c_5.$$

Proof. Fix a T which satisfies Lemma 12. Let $K' := \frac{1}{2}(s_2/s_1 + K)$. It follows from Lemma 5 that there is an $M = M(\mathbf{j})$ such that, if $n > M$, then

$$\sum_{k=0}^{n-1} \psi_1(\sigma^k\mathbf{j}) \ge K' \sum_{k=0}^{n-1} \psi_2(\sigma^k\mathbf{j}) > K \sum_{k=0}^{n-1} \psi_2(\sigma^k\mathbf{j}) > \sum_{k=0}^{n-m'-1} \psi_2(\sigma^k\mathbf{j}).$$

Thus from (5.18) and Lemma 12

$$\begin{aligned}
\frac{A^1_{j_1\cdots j_n}(\theta)}{A^1_{j_1\cdots j_{n-m'}}(\theta)} &= \frac{A^1_{j_1\cdots j_n}(\theta)}{\exp\big(\sum_{k=0}^{n-1}\psi_1(\sigma^k\mathbf{j})\big)} \frac{\exp\big(\sum_{k=0}^{n-1}\psi_1(\sigma^k\mathbf{j})\big)}{\exp\big(\sum_{k=0}^{n-m'-1}\psi_2(\sigma^k\mathbf{j})\big)} \frac{\exp\big(\sum_{k=0}^{n-m'-1}\psi_2(\sigma^k\mathbf{j})\big)}{A_{j_1\cdots j_{n-m'}}(\theta)} \\
&\geq \frac{2}{c_5}\frac{\exp\big(K'\sum_{k=0}^{n-1}\psi_2(\sigma^k\mathbf{j})\big)}{\exp\big(K\sum_{k=0}^{n-1}\psi_2(\sigma^k\mathbf{j})\big)}\frac{1}{2c_5} \geq \frac{2}{c_5}\exp((K-K')n\log q_2^{-1})\frac{1}{2c_5} \\
&= \frac{1}{c_5^2}\left(\frac{1}{q_2}\right)^{n(K-K')} > 4c_5
\end{aligned}$$

if n is big enough. □

Let $K_l := \{\mathbf{j} \in \Sigma \mid M(\mathbf{j}) \leq l\}$. Let $\mathbf{j}$ be an element of the set $K_l \cap Q$. (Q was defined in (5.17).) Choose a $\boldsymbol{\tau}$ whose length is greater than l such that $\mathbf{j}$ starts with $\boldsymbol{\tau}\mathbf{i}^1$. Then, for every $\theta \in [0, 1]$,

$$\frac{A^1_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)}{A^2_{\boldsymbol{\tau}}(\theta)} \geq 4c_5 \tag{5.19}$$

holds. This is because the length of $\boldsymbol{\tau}$ is greater than $M(\mathbf{j})$ and the length of $\mathbf{i}^1$ is less than m', so we can apply Lemma 13.

5.5. *Density lemmas*

LEMMA 14. *Let* $\mathbf{j} \in \bigcup_{l\geq 1}(\mathbf{Q}\cap\mathbf{K}_l)$. *Then we can find infinitely many* $\boldsymbol{\tau}$ *such that* $\mathbf{j}$ *starts with* $\boldsymbol{\tau}\mathbf{i}^1$ *and for every* $\theta \in [0, 1]$ *and for all* $1 \leq u, v \leq N$ *we have* $\mathrm{Dist}(\bar{S}_{\boldsymbol{\tau}\mathbf{i}^u}(\theta), \bar{S}_{\boldsymbol{\tau}\mathbf{i}^v}(\theta)) < 2A^1_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)$ *where* Dist *means the Hausdorff distance.*

Proof. Fix an l such that $\mathbf{j} \in \mathbf{K}_l$. We know that for $1 \leq u, v \leq N$, $S_{\boldsymbol{\tau}\mathbf{i}^u}(\theta), S_{\boldsymbol{\tau}\mathbf{i}^v}(\theta) \subset S_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)$, thus $\mathrm{Dist}(S_{\boldsymbol{\tau}\mathbf{i}^u}(\theta), S_{\boldsymbol{\tau}\mathbf{i}^v}(\theta)) < A^1_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)$. On the other hand, since $\bar{S}_{\boldsymbol{\tau}\mathbf{i}^u} \subset \bar{S}_{\boldsymbol{\tau}}$ holds for all $u \leq N$, the Hausdorff distance between the projections to the z-axis of $\bar{S}_{\boldsymbol{\tau}\mathbf{i}^u}(\theta)$ and $\bar{S}_{\boldsymbol{\tau}\mathbf{i}^v}(\theta)$ is less than $A^2_{\boldsymbol{\tau}}(\theta) < A^1_{\boldsymbol{\tau}\mathbf{i}^1}(\theta)$. This completes the proof. □

For a $\mathbf{j} \in \Sigma^+$ and $\theta \in (0, 1)$, and for an $r > 0$, we define

$$\mathrm{Cyl}(\theta, \mathbf{j}, r) := \{(t, Y) \in [0, 1] \times D \mid |t - \theta| < r,\ \mathrm{dist}((t, Y), \bar{\Lambda}_{\mathbf{j}}(t)) < r\}, \tag{5.20}$$

where dist is the Euclidean distance.

Obviously, there exist constants c_7, c_8 such that

$$B(\bar{\Lambda}_{\mathbf{j}}(\theta), c_7 r) \subset \mathrm{Cyl}(\theta, \mathbf{j}, r) \subset B(\bar{\Lambda}_{\mathbf{j}}(\theta), c_8 r). \tag{5.21}$$

PROPOSITION 2. *For* μ*-almost every* $X \in \bar{\Lambda}$ *the upper* $(1 + s_1)$*-density of the measure* μ *is infinite. That is,*

$$\bar{D}(\mu, X, 1 + s_1) := \limsup_{r\to 0}\frac{\mu(B(X, r))}{r^{1+s_1}} = \infty.$$

Proof. It is enough to prove that for a constant c_{10} and μ-almost every $X \in \bar{\Lambda}$, we have $\bar{D}(\mu, X, 1 + s_1) > c_{10}N$ since N was arbitrary. We may assume that $X = \Lambda_{\mathbf{j}}(\theta)$ for a

$\mathbf{j} \in \bigcup_{l \geq 1}(\mathbf{Q}_N \cap \mathbf{K}_l)$. Then it follows from Lemma 14 and (5.11) that for $r = 4c_5 A^1_{\bar{\boldsymbol{\tau}}\mathbf{i}^1}(\theta)$ and for all $|t - \theta| < r$, $\bar{S}_{\boldsymbol{\tau}\mathbf{i}^u}(t) \subset \mathrm{Cyl}(\theta, \mathbf{j}, r)$ holds for $1 \leq u \leq N$. Thus,

$$\frac{\mu(\mathrm{Cyl}(\theta, \mathbf{j}, r))}{r^{1+s_1}} \geq \frac{2r \sum_{k=1}^N \nu_1^+(\boldsymbol{\tau}\mathbf{i}^k)}{r^{1+s_1}} \geq \frac{N c_9 (A^1_{\bar{\boldsymbol{\tau}}\mathbf{i}^1}(\theta))^{s_1}}{(4c_5)^{s_1}(A^1_{\bar{\boldsymbol{\tau}}\mathbf{i}^1}(\theta))^{s_1}} \geq N c_{10}.$$

In the second inequality we used that $\nu_1^+(\boldsymbol{\tau}\mathbf{i}^k) \geq d^{-1}(2^{s_1} c_5^{s_1})^{-1}(A^1_{\bar{\boldsymbol{\tau}}\mathbf{i}^1}(\theta))^{s_1}$ what immediately follows from (5.2) and (5.18). This completes the proof. □

5.6. *The Absolute Continuity Lemma.* The proof of the theorem is based on the previous proposition and the next lemma.

LEMMA 15. *The* $(1 + s_1)$*-dimensional Hausdorff measure* $\mathcal{H}^{1+s_1}$ *(restricted to* $\bar{\Lambda}$*) is absolutely continuous with respect to* μ.

Proof. We are going to write $\mathcal{H}^{1+s_1}$ for the restriction of the $(1+s_1)$-dimensional Hausdorff measure to $\bar{\Lambda}$ and for brevity we write η instead of $(\Pi^-)^{-1}_*(\eta)$ when we are on Σ^-. If $\mu(A) = 0$ for an $A \subset \bar{\Lambda}$, then $\eta \times \nu_1^+(\bar{\Pi}^{-1}(A)) = 0$. Fix an $\varepsilon > 0$. The set $\bar{\Pi}^{-1}(A) \subset \Sigma$ can be covered by a countable system of cylinders $\{C_i\}$ of the form $C_i = (\omega^i_{-m_i}, \ldots, \omega^i_0, \ldots, \omega^i_{n_i})$, such that $\sum_{i \geq 1} \mu(\bar{\Pi}(C_i)) < \varepsilon$. We may assume about the length (in Σ) and shape of these cylinders that for a constant $c_{11} > 0$ and for all $\theta \in [0, 1]$

$$\frac{|I_{\omega^i_{-m_i}\cdots\omega^i_0}|}{\mathrm{diam}(\bar{S}_{\omega^i_1\cdots\omega^i_{n_i}}(\theta))} \in (c_{11}^{-1}, c_{11}) \quad \text{and} \quad \sum_{j_1\cdots j_{n_i} \in \{1,\ldots,m\}^{n_i}} (A^2_{j_1\cdots j_{n_i}}(\theta))^{s_1} < \varepsilon. \tag{5.22}$$

Namely, by subdividing the cylinders the first requirement is easy to fulfill (see (5.11)). Considering the second part of (5.22), it follows from (5.2) that $\nu_2(j_1, \ldots, j_l) \approx (A^2_{j_1\cdots j_l}(\theta))^{s_2}$. Thus $\sum_{j_1\cdots j_l}(A^2_{j_1\cdots j_l}(\theta))^{s_2} <$ constant. From the definitions $A^2_{j_1\cdots j_l}(\theta) < q_2^l$. So,

$$\sum_{j_1\cdots j_l}(A^2_{j_1\cdots j_l}(\theta))^{s_1} < q_2^{l(s_1-s_2)} \sum_{j_1\cdots j_l}(A^2_{j_1\cdots j_l}(\theta))^{s_2} < q_2^{l(s_1-s_2)}\text{constant}.$$

Hence, there exists an l_0 such that for $l \geq l_0$

$$\sum_{j_1\cdots j_l}(A^2_{j_1\cdots j_l}(\theta))^{s_1} < \varepsilon. \tag{5.23}$$

The second part of (5.22) requires that the length of the positive part (in Σ) of all cylinders C_i are at least l_0. That is, $n_i \geq l_0$. By subdivisions, if necessary we can construct such a cover of Σ.

We divide the index set $\mathbb{N}$ into two parts. Let us write $A^k_{\mathbf{j}}$ for $A^k_{\mathbf{j}}(0)$ $(k = 1, 2)$. Let $J' := \{i \mid A^1_{\omega^i_1,\ldots,\omega^i_{n_i}} \geq A^2_{\omega^i_1,\ldots,\omega^i_{n_i}}\}$ and $J'' := \{i \mid A^2_{\omega^i_1,\ldots,\omega^i_{n_i}} > A^1_{\omega^i_1,\ldots,\omega^i_{n_i}}\}$. For an $i \in J'$ we have $\mathrm{diam}(\bar{\Pi}(C_i)) \in (c_{12}^{-1}, c_{12}) A^1_{\omega^i_1,\ldots,\omega^i_{n_i}}$, where $c_{12} = c_5 c_{11} e$, where the constant e is defined by

$$\frac{|I_{\omega^i_{-m_i}\cdots\omega^i_0}|}{\eta(I_{\omega^i_{-m_i}\cdots\omega^i_0})} \in (e^{-1}, e).$$

The existence of such an e follows from the Folklore Theorem [**12**, p. 352]. On the other hand, using (5.2) and (5.18) we obtain that $\nu_1^+(\omega_1^i,\dots,\omega_{n_i}^i) \in (c_{13}^{-1}, c_{13})(A^1_{\omega_1^i,\dots,\omega_{n_i}^i})^{s_1}$, where $c_{13} = d(2c_5)^{s_1}$. In this way

$$\mu(\bar{\Pi}(C_i)) \geq \eta(\omega_{-m_i}^i,\dots,\omega_0^i)\nu_1^+(\omega_1^i,\dots,\omega_{n_i}^i) > c_{14}A^1_{\omega_1^i,\dots,\omega_{n_i}^i}(A^1_{\omega_1^i,\dots,\omega_{n_i}^i})^{s_1}$$

thus $\mu(\bar{\Pi}(C_i)) > c_{14}(A^1_{\omega_1^i,\dots,\omega_{n_i}^i})^{1+s_1}$. This follows from

$$\sum_{i\in J'}(\operatorname{diam}(\bar{\Pi}(C_i)))^{1+s_1} < c_{12}^{1+s_1}\sum_{i\in J'}(A^1_{\omega_1^i,\dots,\omega_{n_i}^i})^{1+s_1} < \varepsilon c_{12}c_{14}, \tag{5.24}$$

since $\Sigma\mu(\bar{\Pi}(C_i)) < \epsilon$. In the rest of the proof, we give a similar estimate for the index set J''. We may assume that $\{C_i\}$ is ordered in such a way that $\{n_i\}$ is a non-decreasing sequence. We define a sequence of finite words $\{\mathbf{j}^k\}$ as follows: Let $\{\mathbf{j}^1\} := (\omega_1^1,\dots,\omega_{n_1}^1)$. If we have already defined $\mathbf{j}^1,\dots,\mathbf{j}^{k-1}$, then we define $\mathbf{j}^k$ in the following way. Let

$$l := \min\{i \mid (\omega_1^i,\dots,\omega_{n_i}^i)\cap \mathbf{j}^p = \emptyset \text{ for all } 1\leq p\leq k-1\}.$$

Then $\mathbf{j}^k := (\omega_1^l,\dots,\omega_{n_l}^l)$. Using that any two cylinders of Σ^+ are either disjoint or one of them contains the other, we obtain that $\bigcup_i \bar{\Pi}(C_i) \subset \bigcup_{k\geq 1}\bar{S}_{\mathbf{j}^k}$ and $\bar{S}_{\mathbf{j}^k}\cap\bar{S}_{\mathbf{j}^l} = \emptyset$ for any two different k, l. Further, from the definition of J'', $A^2_{\mathbf{j}^k} > A^1_{\mathbf{j}^k}$ holds. For each k we partition the interval $[0,1]$ into $n_k := [1/A^2_{\mathbf{j}^k}]$ sub-intervals called $\{J_l^k\}_{l=1}^{n_k}$ with equal length. Let $\bar{E}_l^k := (J_l^k\times D)\cap\bar{S}_{\mathbf{j}_k}$ for $1\leq k$ and $1\leq l\leq n_k$. So,

$$\bigcup_i \bar{\Pi}(C_i) \subset \bigcup_{k\geq 1}\bigcup_{l=1}^{n_k}\bar{E}_l^k. \tag{5.25}$$

Therefore

$$\sum_{k\geq 1}\sum_{l=1}^{n_k}|\bar{E}_l^k|^{1+s_1} < 2\sum_{k\geq 1}\sum_{l=1}^{n_k}|J_l^k||A^2_{\mathbf{j}^k}|^{s_1} < \sum_{k\geq 1}n_k\frac{1}{n_k}|A^2_{\mathbf{j}^k}|^{s_1} \tag{5.26}$$

$$= \sum_{k\geq 1}(A^2_{\mathbf{j}^k})^{s_1} < \text{constant}\sum_{i_1\dots i_{l_0}}(A^2_{i_1\dots i_{l_0}})^{s_1} < \varepsilon\cdot\text{constant}. \tag{5.27}$$

The last but one inequality can be proved as follows. We partition the cylinder $(i_1,\dots,i_{l_0})$ into cylinders $\{\boldsymbol{\omega}^k\}_{k=1}^{\infty}$ arbitrarily. Then

$$(A^2_{i_1\dots i_{l_0}})^{s_1} = ((A^2_{i_1\dots i_{l_0}})^{s_2})^{s_1/s_2} \approx (\nu_2^+(i_1\dots i_{l_0}))^{s_1/s_2} \geq \sum_{k\geq 1}(\nu_2^+(\boldsymbol{\omega}^k))^{s_1/s_2} \approx \sum_{k\geq 1}(A^2_{\boldsymbol{\omega}^k})^{s_1}$$

since $s_1/s_2 > 1$. From (5.24) and (5.26) we obtain that the $(1+s_1)$-dimensional Hausdorff measure of $\bar{\Lambda}$ is less than or equal to constant $\cdot\,\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $\mathcal{H}^{1+s_1}(\bar{\Lambda}) = 0$ which completes the proof. □

5.7. *The proof.* Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let E be the set of $X\in\bar{\Lambda}$ for which $\bar{D}(\mu, X, 1+s_1) = \infty$. Using that μ is a finite Borel measure it follows from [**4**, Proposition 2.2(b)] that $\mathcal{H}^{1+s_1}(E) = 0$. Then Proposition 2 implies that $\mu(\bar{\Lambda} - E) = 0$. From Lemma 15 we obtain that $\mathcal{H}^{1+s_1}(\bar{\Lambda} - E) = 0$. In this way we have proved that $\mathcal{H}^{1+s_1}(\bar{\Lambda}) = 0$. □

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REFERENCES

[1] H. G. Bothe. The Hausdorff dimension of certain solenoids. *Ergod. Th. & Dynam. Sys.* **15** (1995), 449–474.

[2] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Springer Lecture Notes in Mathematics, 470)*. Springer, Berlin, 1979.

[3] P. H. Carter and R. D. Mauldin. Attractors for cylinder maps. *Preprint*, 1990.

[4] K. Falconer. *Techniques in Fractal Geometry*. Wiley, New York, 1997.

[5] M. Denker and M. Urbański. Measures for parabolic rational maps. *Ergod. Th. & Dynam. Sys.* **12** (1992), 53–66.

[6] K. Falconer. Hausdorff dimension of some fractals. *J. Stat. Phys.* **47** (1987), 123–132.

[7] B. Hasselblatt and J. Schmeling. Oral communication, 2000.

[8] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, Cambridge, 1995.

[9] D. Mauldin and M. Urbański. Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.* **73** (1996), 105–154.

[10] D. Mauldin and M. Urbański. Parabolic iterated function systems. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1423–1447.

[11] H. McCluskey and A. K. Manning. Hausdorff dimension for horseshoes. *Ergod. Th. & Dynam. Sys.* **3** (1983), 251–260.

[12] W. de Melo and S. van Strien. *One-dimensional Dynamics*. Springer, Berlin, 1993.

[13] Y. Peres, K. Simon and B. Solomyak. Self-similar sets of zero Hausdorff and positive Packing measure. *Israel J. Math.* **117** (2000), 353–379.

[14] Y. Peres, M. Rams, K. Simon and B. Solomyak. Equivalence of positive Hausdorff measure and the open set condition for non-conformal sets. *Proc. Amer. Math. Soc.* **129** (2001), 2688–2699.

[15] Y. Pesin. *Dimension Theory in Dynamical Systems (Chicago Lectures in Mathematics)*. The University of Chicago Press, 1997.

[16] M. Shub. *Global Stability of Dynamical Systems*. Springer, Berlin, 1987.

[17] K. Simon. The Hausdorff dimension of the Smale–Williams solenoid with different contraction coefficients. *Proc. Amer. Math. Soc.* **125**(4) (1997), 1221–1228.

[18] K. Simon. Hausdorff dimension for non-invertible maps. *Ergod. Th. & Dynam. Sys.* **13** (1993), 199–212.

[19] K. Simon and B. Solomyak. Hausdorff dimension for horseshoes in $\mathbb{R}^3$. *Ergod. Th. & Dynam. Sys.* **19** (1999), 1343–1364.

[20] M. Urbański. Rational functions with no recurrent critical points. *Ergod. Th. & Dynam. Sys.* **14** (1994), 391–414.