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EQUIVALENCE OF POSITIVE HAUSDORFF MEASURE AND THE OPEN SET CONDITION FOR SELF-CONFORMAL SETS

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ABSTRACT. A compact set K is *self-conformal* if it is a finite union of its images by conformal contractions. It is well known that if the conformal contractions satisfy the "open set condition" (OSC), then K has positive s-dimensional Hausdorff measure, where s is the solution of Bowen's pressure equation. We prove that the OSC, the strong OSC, and positivity of the s-dimensional Hausdorff measure are equivalent for conformal contractions; this answers a question of R. D. Mauldin. In the self-similar case, when the contractions are linear, this equivalence was proved by Schief (1994), who used a result of Bandt and Graf (1992), but the proofs in these papers do not extend to the nonlinear setting.

1. INTRODUCTION

Let $V \subset \mathbb{R}^d$. Recall that a map $S: V \to V$ is **contracting** if there exists $0 < \gamma(S) < 1$ such that $|S(x) - S(y)| \le \gamma(S) \cdot |x - y|$ for all $x, y \in V$; if equality holds here for all $x, y \in V$, then S is a **contracting similitude**. Let $\{S_i\}_{i=1}^m$ be a collection of contracting maps on an open set $V \subset \mathbb{R}^d$ and suppose that for some closed set $X \subset V$ we have $S_i(X) \subset X$ for all $i \le m$. By [6], there is a unique non-empty compact set $\mathcal{K} \subset X$ such that

$$\mathcal{K} = \bigcup_{i=1}^{m} S_i \mathcal{K}\,. \tag{1.1}$$

If all S_i are similitudes, then $\mathcal{K}$ satisfying (1.1) is called **self-similar**.

The contracting maps $\{S_i\}_{i=1}^m$ of V are said to satisfy the **Open Set Condition** (OSC) if there is a non-empty open set $U \subset V$ such that $S_i U \subset U$ for all i, and $S_i U \cap S_j U = \emptyset$ for $i \ne j$. The **strong Open Set Condition** holds if the set U in the definition of the OSC can be chosen with $U \cap \mathcal{K} \ne \emptyset$, where $\mathcal{K}$ is a compact set satisfying (1.1).

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Next, consider a collection of contracting similitudes $\{S_i\}_{i=1}^m$ and let $\mathcal{K}$ be the corresponding self-similar set. The **similarity dimension** for this collection is defined as the unique positive solution s of the equation $\sum_{i=1}^m \gamma(S_i)^s = 1$. It is immediate that the Hausdorff measure $\mathcal{H}^s(\mathcal{K})$ is finite. Hutchinson [6] proved that if the OSC holds, then $\mathcal{H}^s(\mathcal{K})$ is positive and hence the Hausdorff dimension of $\mathcal{K}$ equals s.

Bandt and Graf [1] gave a very useful characterization of self-similar sets with positive Hausdorff measure in the similarity dimension. Let $\mathcal{A}^*$ be the set of finite "words" in the alphabet $\mathcal{A} = \{1, \dots, m\}$ and denote $S_u = S_{u_1} \circ \dots \circ S_{u_n}$ for $u = u_1 \dots u_n \in \mathcal{A}^*$. For $u \in \mathcal{A}^*$ let $\mathcal{K}_u = S_u(\mathcal{K})$. We say that two maps S_u and S_v are **ε-relatively close** if

$$|S_u(x) - S_v(x)| \le \varepsilon \min\{\mathrm{diam}(\mathcal{K}_u), \mathrm{diam}(\mathcal{K}_v)\} \quad \text{for all} \;\; x \in \mathcal{K}. \tag{1.2}$$

Bandt and Graf [1] proved that $\mathcal{H}^s(\mathcal{K}) > 0$ if and only if there exists $\varepsilon > 0$ such that for distinct u, v in $\mathcal{A}^*$, the maps S_u and S_v are not ε-relatively close. Building on [1], Schief [11] proved that $\mathcal{H}^s(\mathcal{K}) > 0$ is equivalent to the OSC and also to the strong OSC.

Much of the theory has been extended from self-similar to self-conformal sets (see, *e.g.*, [10, 2]). Let $V \subset \mathbb{R}^d$ be an open set. A $\mathcal{C}^1$-map $S: V \to \mathbb{R}^d$ is **conformal** if the differential $S'(x): \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|S'(x)y| = |S'(x)| \cdot |y| \neq 0$ for all $x \in V$ and $y \in \mathbb{R}^d$, $y \neq 0$. We say that $\{S_i : X \to X\}_{i \le m}$ is a **conformal iterated function system** on a compact set $X \subset \mathbb{R}^d$ if each S_i extends to an injective conformal map $S_i : V \to V$ on an open connected set $V \supset X$ and $\sup\{|S_i'(x)| : x \in V\} < 1$. We assume Hölder continuity of the differentials, that is, there exists $\alpha > 0$ such that for all $i \le m$,

$$||S_i'(x)| - |S_i'(y)|| \le \mathrm{const} \cdot |x - y|^\alpha \quad \text{for all} \;\; x, y \in V. \tag{1.3}$$

We should note that for $d \ge 2$ Hölder continuity (and, in fact, real analyticity) of $|S_i'(\cdot)|$ follows from conformality and injectivity.

Under these assumptions the unique non-empty compact set $\mathcal{K} \subset X$ satisfying (1.1) is called **self-conformal**. The role of similarity dimension is played by the unique solution s of the Bowen equation $P(s) = 0$, where the pressure $P(t)$ is defined by

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in \mathcal{K}} \sum_{u \in \mathcal{A}^n} |S_u'(x)|^t, \quad \text{for} \;\; t > 0. \tag{1.4}$$

It is well-known that $\mathcal{H}^s(\mathcal{K}) < \infty$. The definitions of ε-relatively close maps (1.2) and of the compositions S_u extend to this setting.

We say that the **Bandt-Graf condition** holds if there exists $\varepsilon > 0$ such that for distinct u, v in $\mathcal{A}^*$, the maps S_u and S_v are not ε-relatively close. Our main result is the complete equivalence theorem for self-conformal sets.

Theorem 1.1. *For a conformal i.f.s. $\{S_i\}_{i \le m}$, satisfying the Hölder condition, and the associated self-conformal set $\mathcal{K}$, the following are equivalent:*
- **(a)** *the OSC;*
- **(b)** $\mathcal{H}^s(\mathcal{K}) > 0$ *where* $s > 0$ *is such that* $P(s) = 0$*;*
- **(c)** *the Bandt-Graf condition;*
- **(d)** *the strong OSC.*

The implication (a) $\Rightarrow$ (b) is standard (see, *e.g.*, [4, p. 89]), so we just need to prove that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Perhaps surprisingly, the existing proofs of these implications in the self-similar case do not extend to the nonlinear setting. The elegant method of Bandt and Graf [1] for the proof of (b) $\Longleftrightarrow$ (c) is very much dependent on the set $\mathcal{K}$ being precisely self-similar. In several places of [1] it was crucial that $\sum_j |S_j'(x)|^s = 1$ for all x. We have to use a more "robust" method to allow for distortion.

The implication (a) $\Rightarrow$ (d) answers a question of R. D. Mauldin (see [7, Question 9.1]). This implication was stated by Fan and Lau in [5, Lemma 2.6]. Although their approach is very promising, unfortunately, the proof in [5] contains a gap, as was pointed out by N. Patzschke (personal communication). A more detailed comment on this is given at the end of the paper.

We also obtain the following corollary, which extends Schief's result [11, Cor. 2.3]:

Corollary 1.2. *If $\mathcal{K} \subset \mathbb{R}^d$ is self-conformal and the solution of the pressure equation s equals d, then $\mathcal{H}^d(\mathcal{K}) > 0$ implies that $\mathcal{K}$ is the closure of its interior.*

2. Generalizing the Bandt-Graf theorem

After some preliminaries, which will be needed in Section 3 as well, we prove the implication (b) $\Rightarrow$ (c) in Theorem 1.1, generalizing the result of Bandt and Graf [1].

We consider a conformal contracting i.f.s. $\{S_i\}_{i=1}^m$ satisfying the Hölder condition (1.3) on an open set V, such that $S_i(X) \subset X$ for a compact set $X \subset V$. Let $\mathcal{A} = \{1, \ldots, m\}$ and equip the sequence space $\mathcal{A}^{\mathbb{N}}$ with the product topology. We write $\mathcal{A}^* = \bigcup_{n \geq 1} \mathcal{A}^n$ for the set of finite "words" in the alphabet $\mathcal{A}$. The symbol σ denotes the left shift on $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^*$. The map $\Pi : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}^d$ defined by

$$\Pi(\omega) = \lim_{n \to \infty} S_{\omega_1 \ldots \omega_n}(x), \quad x \in V,$$

is called the **natural projection** map (clearly, it does not depend on x). The self-conformal set associated with the i.f.s. is $\mathcal{K} = \Pi(\mathcal{A}^{\mathbb{N}})$. Let

$$\mathcal{O}(F, r) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, F) < r\}$$

denote the r-neighborhood of a compact set $F \subset \mathbb{R}^d$. The closed ball of radius r centered at $x \in \mathbb{R}^d$ is denoted by $B(x, r)$. We write $[x, y]$ to denote the line segment connecting x and y in $\mathbb{R}^d$.

Fix $\delta_0 > 0$ so that $\mathcal{O}(X, 3\delta_0) \subset V$ and let

$$V' = \mathcal{O}(X, \delta_0), \qquad V'' = \mathcal{O}(X, 2\delta_0).$$

Since $S_i X \subset X$ and $|S_i'(x)| < 1$ for all $x \in V$, we also have $S_i V' \subset V'$ and $S_i V'' \subset V''$ for all i.

Next we recall the standard bounded distortion property of conformal i.f.s. satisfying the Hölder condition (see, *e.g.*, [8, Lemma 2.1]): there exists $C_1 \geq 1$ such that for all $u \in \mathcal{A}^*$,

$$|S_u'(x)| \leq C_1 |S_u'(y)| \quad \text{for all } x, y \in V''. \tag{2.1}$$

Denote

$$\|S_u'\| = \sup_{x \in V''} |S_u'(x)|.$$

The property (2.1) yields (see, *e.g.*, [8, Lemma 2.2]) that there exists $C_2 \geq 1$ such that for all $u \in \mathcal{A}^*$,

$$C_2^{-1}\|S_u'\| \cdot |x-y| \leq |S_u(x) - S_u(y)| \leq C_2\|S_u'\| \cdot |x-y| \quad \text{for all} \ \ x, y \in V'. \tag{2.2}$$

This implies

$$B(x,r) \subset V' \ \Rightarrow \ S_u B(x,r) \supset B(S_u(x), C_2^{-1}\|S_u'\|r) \quad \text{for all} \ \ u \in \mathcal{A}^* \tag{2.3}$$

(see, *e.g.*, [8, Cor. 2.3]). Denote $d_u = \text{diam}(\mathcal{K}_u)$ for $u \in \mathcal{A}^*$. By (2.2), there exists $C_3 \geq 1$ such that

$$C_3^{-1}\|S_u'\| \leq d_u \leq C_3\|S_u'\| \quad \text{for all} \ \ u \in \mathcal{A}^*. \tag{2.4}$$

By (2.1) and (2.4), there exists $C_4 \geq 1$ such that for all $u, v \in \mathcal{A}^*$,

$$C_4^{-1} \max\{\|S_u'\|d_v, \|S_v'\|d_u\} \leq d_{uv} \leq C_4 \min\{\|S_u'\|d_v, \|S_v'\|d_u\}. \tag{2.5}$$

Let $\omega \wedge \tau$ denote the common initial block (possibly empty) of two sequences $\omega, \tau \in \mathcal{A}^{\mathbb{N}}$. We equip the space $\mathcal{A}^{\mathbb{N}}$ with a metric

$$\rho(\omega, \tau) = d_{\omega\wedge\tau} \quad \text{for} \ \ \omega \neq \tau. \tag{2.6}$$

It follows from the bounded distortion properties that the product topology on $\mathcal{A}^{\mathbb{N}}$ coincides with the one defined by ρ. Clearly, the natural projection map $\Pi : (\mathcal{A}^{\mathbb{N}}, \rho) \to \mathbb{R}^d$ is Lipschitz.

The reader is referred to [3, 4] for the background on thermodynamic formalism. Define a Hölder continuous function on $\mathcal{A}^{\mathbb{N}}$ by $\phi(\omega) = \log|S_{\omega_1}'(\Pi(\sigma\omega))|$. The pressure function $P(t)$ of $t\phi$ with respect to the shift σ can be expressed by (1.4). There is a unique value s such that $P(s) = 0$. Let μ be the Gibbs measure on $\mathcal{A}^{\mathbb{N}}$ for the potential $s\phi$. Denoting by $[u]$ the cylinder set corresponding to $u \in \mathcal{A}^*$, we have by the definition of the Gibbs measure and the bounded distortion principle (2.1) that there exists $C_5 \geq 1$ such that

$$C_5^{-1}\|S_u'\|^s \leq \mu[u] \leq C_5\|S_u'\|^s \quad \text{for all} \ \ u \in \mathcal{A}^*. \tag{2.7}$$

Lemma 2.1. **(i)** *The measure μ is equivalent to the s-dimensional Hausdorff measure on $\mathcal{A}^{\mathbb{N}}$ with the metric ρ.*

(ii) *The restriction of the Hausdorff measure $\mathcal{H}^s|_{\mathcal{K}}$ is absolutely continuous with respect to the measure $\nu = \mu \circ \Pi^{-1}$ on $\mathcal{K}$.*

Proof. (i) A ball in the metric ρ is a cylinder $[u]$ for some $u \in \mathcal{A}^*$. Any collection of cylinders in $\mathcal{A}^{\mathbb{N}}$ contains a disjoint subcollection with the same union. Now the claim is immediate by comparing (2.4), (2.6) and (2.7).

(ii) Suppose that $\nu(B) = 0$ for some Borel set $B \subset \mathcal{K}$. Then $\mu(\Pi^{-1}B) = 0$; hence the s-dimensional Hausdorff measure of $\Pi^{-1}B \subset \mathcal{A}^{\mathbb{N}}$ is zero by part (i) of this lemma. It follows that $\mathcal{H}^s(B) = 0$ since Π is Lipshitz. □

Proof of (b) ⇒ (c) in Theorem 1.1. We are going to prove that if for any $\varepsilon > 0$ there exist $u \neq v$ such that S_u and S_v are ε-relatively close, then $\mathcal{H}^s(\mathcal{K}) = 0$. First we make a few useful observations concerning ε-relatively close maps.

Claim 1. *If S_u, S_v are ε-relatively close, then S_{wu} and S_{wv} are $C_2C_4\varepsilon$-relatively close for every $w \in \mathcal{A}^*$.* Indeed, we have by (2.2), (1.2) and (2.5) for $x \in \mathcal{K}$:

$$\begin{aligned} |S_{wu}(x) - S_{wv}(x)| &\leq C_2\|S_w'\| \cdot |S_u(x) - S_v(x)| \\ &\leq C_2\|S_w'\| \cdot \varepsilon \min\{d_u, d_v\} \\ &\leq C_2C_4\varepsilon \min\{d_{wu}, d_{wv}\}. \end{aligned}$$

CLAIM 2. *If S_{w_1}, S_{w_2} are ε-relatively close, then S_{w_1u} and S_{w_2u} are $C_4\|S_u'\|^{-1}\varepsilon$-relatively close for every $u \in \mathcal{A}^*$.* Indeed, in view of (1.2) and (2.5),

$$\begin{aligned} |S_{w_1u}(x) - S_{w_2u}(x)| &= |S_{w_1}(S_u(x)) - S_{w_2}(S_u(x))| \\ &\leq \varepsilon \min\{d_{w_1}, d_{w_2}\} \\ &\leq C_4\|S_u'\|^{-1}\varepsilon \cdot \min\{d_{w_1u}, d_{w_2u}\}. \end{aligned}$$

CLAIM 3. *If S_u, S_v are ε-relatively close, then*

$$d_v \leq (1+2\varepsilon)\cdot d_u.$$

This is immediate from the definition (1.2).

CLAIM 4. *If S_u, S_v are δ-relatively close and S_v, S_w are δ-relatively close, then S_u, S_w are $2\delta(1+2\delta)$-relatively close.* Indeed, by (1.2) and Claim 3, $\min\{d_u, d_v\} \leq (1+2\delta)\min\{d_u, d_w\}$ and $\min\{d_v, d_w\} \leq (1+2\delta)\min\{d_u, d_w\}$. The rest is immediate.

Lemma 2.2. *Suppose that for any $\varepsilon > 0$ there exist $u \neq v$ such that S_u and S_v are ε-relatively close. Then for any $N \in \mathbb{N}$ and any $\varepsilon > 0$ there exist distinct $u_1, \ldots, u_N$ such that S_{u_i}, S_{u_j} are ε-relatively close for all $1 \leq i < j \leq N$.*

Proof. It is enough to show that if the statement holds for N, then it holds for $2N$. Assuming it holds for N, find distinct $u_1, \ldots, u_N$ such that $S_{u_1}, \ldots, S_{u_N}$ are pairwise δ_1-relatively close where $\delta_1 = \frac{1}{4}(C_2C_4)^{-1}\varepsilon$. Next let

$$\delta_2 = (1/4)C_4^{-1}\min_{j\leq N}\|S_{u_j}'\|\cdot\varepsilon$$

and find $w_1 \neq w_2$ such that S_{w_1}, S_{w_2} are δ_2-relatively close. Then the $2N$ words w_ku_j, $k = 1, 2$, $1 \leq j \leq N$, are all distinct, and we claim that the maps $\{S_{w_ku_j} : k = 1, 2; 1 \leq j \leq N\}$ are pairwise ε-relatively close. Indeed, $S_{w_1u_i}, S_{w_1u_j}$ are $\frac{\varepsilon}{4}$-close by Claim 1 and $S_{w_1u_j}, S_{w_2u_j}$ are $\frac{\varepsilon}{4}$-close by Claim 2. Now Claim 4 implies that $S_{w_1u_i}, S_{w_2u_j}$ are δ_3-close, with $\delta_3 = \frac{\varepsilon}{2}(1+\frac{\varepsilon}{2})$. We have $\delta_3 \leq \varepsilon$ for $\varepsilon \leq 2$, which we can certainly assume, and the lemma is proved. □

Now we resume the proof of (b) ⇒ (c) in Theorem 1.1. Fix $N \in \mathbb{N}$ and find distinct $u_1, \ldots, u_N$ such that $S_{u_1}, \ldots, S_{u_N}$ are pairwise 1-relatively close. Recall that $\nu = \mu \circ \Pi^{-1}$ is the push-down measure on $\mathcal{K}$. We claim that for ν-a.e. x,

$$\limsup_{r\to 0}\frac{\nu B(x,r)}{r^s} \geq c\cdot N, \tag{2.8}$$

with a constant $c > 0$ independent of N.

It is well-known (see [3] or [4, Cor. 5.6]) that the Gibbs measure μ is an ergodic invariant measure for the shift σ on $\mathcal{A}^{\mathbb{N}}$. Since $\mu[u_1] > 0$, the block u_1 occurs infinitely often in μ-a.e. sequence ω by the Ergodic Theorem. Let $\Omega \subset \mathcal{A}^{\mathbb{N}}$ be the set of all such ω. Fix $\omega \in \Omega$. We know that there exist arbitrarily large n such that $\sigma^n\omega \in [u_1]$. Fix such n, let $w = \omega_1\ldots\omega_n$, and consider the words $v_j = wu_j$ for $j = 1, \ldots, N$. By Claim 1, the maps S_{wu_j} are pairwise C_2C_4-relatively close. By (1.2), this implies that for $x = \Pi(\omega) \in \mathcal{K}_{wu_1}$ we have

$$B(x,r) \supset \bigcup_{j=1}^{N}\mathcal{K}_{wu_j}, \quad \text{where} \quad r = (2 + C_2C_4)\max_{j\leq N} d_{wu_j}.$$

Thus, by (2.7) and (2.4),

$$\nu B(x,r) \geq \sum_{j=1}^{N} \mu[wu_j] \geq C_3^{-1} C_5^{-1} N \min_{j \leq N} d_{wu_j}^s .$$

Combining this with Claim 3, we obtain

$$\frac{\nu B(x,r)}{r^s} \geq \frac{C_3^{-1} C_5^{-1} N}{(2 + C_2 C_4)^s (1 + 2C_2 C_4)^s} .$$

Since r in the last formula can be arbitrarily small, (2.8) follows.

We have verified (2.8) for $x \in \Pi(\Omega)$ which is a set of full ν-measure. Now $\mathcal{H}^s(\Pi(\Omega)) \leq 2^s (cN)^{-1} \nu(\Pi(\Omega))$ by the Rogers-Taylor density theorem (see [9] or [4, Proposition 2.2]). On the other hand, $\nu(\mathcal{K} \setminus \Pi(\Omega)) = 0$, so $\mathcal{H}^s(\mathcal{K} \setminus \Pi(\Omega)) = 0$ by Lemma 2.1(ii). Thus, $\mathcal{H}^s(\mathcal{K}) \leq 2^s (cN)^{-1} \nu(\mathcal{K})$, and since N was arbitrary we conclude that $\mathcal{H}^s(\mathcal{K}) = 0$. □

3. Generalizing Schief's theorem

In this section we prove the implication (c) ⇒ (d) in Theorem 1.1 and Corollary 1.2, generalizing results of Schief [11]. For $T \geq 1, a \geq 0$ and $u \in \mathcal{A}^*$ let

$$W_{a,T}(u) = \left\{ v \in \mathcal{A}^* : \ \frac{1}{T} \leq \frac{d_v}{d_u} \leq T, \ \mathrm{dist}(\mathcal{K}_v, \mathcal{K}_u) \leq a d_u \right\} . \tag{3.1}$$

Lemma 3.1. *Suppose that the Bandt-Graf condition holds, that is, there exists $\varepsilon > 0$ such that for any distinct $v, w \in \mathcal{A}^*$,*

$$\exists\, x \in \mathcal{K} : \ |S_v(x) - S_w(x)| \geq \varepsilon \min\{d_v, d_w\}. \tag{3.2}$$

Then for any $a > 0$ and $T \geq 1$ there exists $C(a,T) < \infty$ such that

$$\# W_{a,T}(u) \leq C(a,T) \quad \textit{for all} \ \ u \in \mathcal{A}^*.$$

Remark. This lemma is the only place in this section where the Bandt-Graf condition is used. It is easy to see that the statement of the lemma holds if the Bandt-Graf condition is replaced by the OSC, thus providing a direct derivation of the implication OSC ⇒ SOSC (the strong OSC).

Proof of Lemma 3.1. Let $\delta = \frac{\varepsilon}{4 C_2 C_3 T^2}$. It follows from (3.2) that if $\tilde{x} \in \mathcal{K}$ and $|x - \tilde{x}| \leq \delta$, then for $v, w \in W_{a,T}(u)$, in view of (2.2) and (2.4),

$$\begin{aligned} |S_v(\tilde{x}) - S_w(\tilde{x})| &\geq |S_v(x) - S_w(x)| - |S_v(x) - S_v(\tilde{x})| - |S_w(x) - S_w(\tilde{x})| \\ &\geq \varepsilon \min\{d_v, d_w\} - C_2 \delta (\|S_v\| + \|S_w\|) \\ &\geq \varepsilon \min\{d_v, d_w\} - C_2 C_3 \delta (d_v + d_w) \\ &\geq d_u (\varepsilon T^{-1} - C_2 C_3 \cdot 2 \delta T) \\ &= (1/2) d_u \varepsilon T^{-1}. \end{aligned} \tag{3.3}$$

Fix a finite set $\{x_1, \ldots, x_N\} \subset \mathcal{K}$ so that $\bigcup_{i=1}^N B(x_i, \delta) \supset \mathcal{K}$. For each $v \in W_{a,T}(u)$ let $\xi_v = [S_v(x_i)]_{i \leq N} \in \mathbb{R}^{dN}$. By (3.3),

$$|\xi_v - \xi_w| \geq (1/2) d_u \varepsilon T^{-1} \quad \text{for all} \ \ v, w \in W_{a,T}(u).$$

On the other hand, if $v \in W_{a,T}(u)$, then $\mathrm{dist}(\mathcal{K}_v, \mathcal{K}_u) \leq a d_u$; hence

$$|S_u(x) - S_v(x)| \leq a d_u + d_u + d_v \leq (a + 1 + T) d_u \quad \text{for all} \ \ x \in \mathcal{K}.$$

It follows that $|\xi_u - \xi_v| \le \sqrt{N}(a+1+T)d_u$. Thus, open balls in $\mathbb{R}^{dN}$ of radius $\frac{1}{4}d_u\varepsilon T^{-1}$ around ξ_v for $v \in W_{a,T}(u)$ are all disjoint and lie in the ball of radius $(\sqrt{N}(a+1+T)+\frac{1}{4}\varepsilon T^{-1})d_u$ around ξ_u. It follows that

$$\#W_{a,T}(u) \le \left(\frac{\sqrt{N}(a+1+T)+\frac{1}{4}\varepsilon T^{-1}}{\frac{1}{4}\varepsilon T^{-1}}\right)^{dN},$$

which is a constant independent of u. □

We need a lemma on "local" bounded distortion. Recall that $V'' = \mathcal{O}(X, 2\delta_0) \subset V$.

Lemma 3.2. **(i)** *There exists $L_1 > 0$ such that for all $x, y \in V''$,*

$$\frac{|S_u'(x)|}{|S_u'(y)|} \le \exp[L_1|x-y|^\alpha] \qquad \textit{for all} \;\; u \in \mathcal{A}^*. \tag{3.4}$$

(ii) *There exists $L_2 > 0$ such that for all $u \in \mathcal{A}$ such that $d_u \le \delta_0$ and all $w \in \mathcal{A}^*$,*

$$\mathrm{dist}(z, \mathcal{K}_u) \le d_u \;\Rightarrow\; \exp[-L_2 d_u^\alpha] \le \frac{d_{wu}}{d_u|S_w'(z)|} \le \exp[L_2 d_u^\alpha]. \tag{3.5}$$

Proof. (i) is folklore; it is obtained in the course of the standard proof of "global" bounded distortion (see, *e.g.*, [2] or [8, Lemma 2.1]).

(ii) Note that $z \in \mathcal{O}(X, \delta_0) \subset V$; hence $|S_w'(z)|$ is well-defined. We can assume that d_u is sufficiently small, since otherwise (3.5) follows from (2.4) and (2.5). Suppose that $C_2C_4d_u \le \delta_0$. Then for any $x, y \in \mathcal{K}$ we have $[S_u(x), S_u(y)] \subset V'$; hence

$$|S_{wu}(x) - S_{wu}(y)| \le |S_w'(\zeta)| \cdot |S_u(x) - S_u(y)|$$

for some ζ satisfying $\mathrm{dist}(\zeta, \mathcal{K}_u) \le d_u$. If $\mathrm{dist}(z, \mathcal{K}_u) \le d_u$, then $|\zeta - z| \le 3d_u$ and $\zeta, z \in V'$. Thus,

$$d_{wu} \le d_u|S_w'(z)|\exp[L_1(3d_u)^\alpha]$$

by (3.4). To obtain the other inequality, observe that by (2.3) and (2.5),

$$S_wB(S_u(x), C_2C_4d_u) \supset B(S_{wu}(x), C_4\|S_w'\|d_u) \supset B(S_{wu}(x), d_{wu}).$$

Therefore, $[S_{wu}(x), S_{wu}(y)] \subset V'$ and we have

$$|S_u(x) - S_u(y)| \le |(S_w^{-1})'(\xi)| \cdot |S_{wu}(x) - S_{wu}(y)|,$$

for some $\xi \in B(S_{wu}(x), d_{wu}) \subset S_wB(S_u(x), C_2C_4d_u)$. We have $|z - S_w^{-1}(\xi)| \le 2d_u + C_2C_4d_u$; hence by (3.4),

$$d_u \le d_{wu}|S_w'(S_w^{-1}\xi)|^{-1} \le d_{wu}|S_w'(z)|^{-1}\exp[L_2d_u^\alpha],$$

with $L_2 = L_1(2 + C_2C_4)^\alpha$, as desired. □

Lemma 3.3. *Let $T_0 \ge 1$ and $\varepsilon > 0$. There exists $\delta = \delta(T_0, \varepsilon) > 0$ such that for all $u \in \mathcal{A}^*$ with $d_u \le \delta$, for all $a \in [0,1]$ and all $T \in [T_0, 2T_0]$,*

$$v \in W_{a,T}(u) \;\Rightarrow\; wv \in W_{a(1+\varepsilon),T(1+\varepsilon)}(wu) \qquad \textit{for all} \;\; w \in \mathcal{A}^*.$$

Proof. Suppose that $d_u \le \delta < \delta_0/(2T_0)$ and $v \in W_{a,T}(u)$. Fix $w \in \mathcal{A}^*$. We need to check that (i) $T^{-1}(1+\varepsilon)^{-1} \le \frac{d_{wu}}{d_{wv}} \le T(1+\varepsilon)$ and (ii) $\mathrm{dist}(\mathcal{K}_{wv}, \mathcal{K}_{wu}) \le a(1+\varepsilon)d_{wu}$.

(i) Let $z \in \mathcal{K}_v$ be such that $\mathrm{dist}(z, \mathcal{K}_u) \le ad_u \le d_u$. Then by (3.5), using that $d_u T^{-1} \le d_v \le d_u T \le 2\delta T_0 < \delta_0$, we obtain

$$\frac{d_{wu}}{d_{wv}} \le \frac{d_u |S'_w(z)| \exp[L_2 d_u^\alpha]}{d_v |S'_w(z)| \exp[-L_2 d_v^\alpha]} \le T e^{L_2 \delta^\alpha (1+(2T_0)^\alpha)} \le T(1+\varepsilon),$$

for $\delta > 0$ sufficiently small. The other inequality is obtained similarly.

(ii) Since $v \in W_{a,T}(u)$, there exist $x, y \in \mathcal{K}$ such that $|S_u(x) - S_v(y)| \le ad_u$. Then $[S_u(x), S_v(y)] \subset V'$; hence

$$|S_{wu}(x) - S_{wv}(y)| \le |S'_w(z)| \cdot |S_u(x) - S_v(y)|$$

for some z with $\mathrm{dist}(z, \mathcal{K}_u) \le ad_u \le d_u$. Therefore, by (3.5),

$$\mathrm{dist}(\mathcal{K}_{wv}, \mathcal{K}_{wu}) \le a|S'_w(z)| \cdot d_u \le ad_{wu} \exp[L_2 \delta^\alpha] \le a(1+\varepsilon)d_{wu},$$

for $\delta > 0$ sufficiently small, and we are done. □

Proof of (c) ⇒ (d) in Theorem 1.1. The scheme of the proof generally follows that of Schief's [11], but we have to be careful with distortion.

Fix $T_0 \ge 1$ so large that for all $j \in \mathcal{A}$ and all $v \in \mathcal{A}^*$,

$$d_v \le T_0^2 d_{vj} \quad \text{and} \quad T_0 d_j \ge 1 \tag{3.6}$$

(in fact, one can take $T_0 = \max\{d_j^{-1}, C_4^{1/2} \|S'_j\|^{-1/2},\ j \in \mathcal{A}\}$ by (2.5)). It follows from (3.6) that for any $r \le 1$ and any $w = w_1 \dots w_n \in \mathcal{A}^*$, with $d_w \le r$, there is $1 \le k \le n$ such that

$$T_0^{-1} \le d_{w'}/r \le T_0 \qquad \text{where} \quad w' = w_1 \dots w_k \tag{3.7}$$

(just take maximal $1 \le k \le n$ such that $d_{w'} \ge rT_0^{-1}$). To simplify notation, let

$$W_a(u) := W_{a,(1+a)T_0}(u) \qquad \text{and} \qquad M_a(u) = \#W_a(u).$$

By Lemma 3.1, there exists $C = C(1, 2T_0) > 0$ such that

$$M_a(u) \le C \qquad \text{for all} \ \ u \in \mathcal{A}^* \text{ and all } a \in [0,1].$$

By the definition (3.1), the function $a \mapsto M_a(u)$ is non-decreasing. For $r > 0$ consider

$$\widetilde{M}_a(r) := \sup\{M_a(u) : \ u \in \mathcal{A}^*,\ d_u \le r\}. \tag{3.8}$$

Let $\varepsilon = \frac{1}{2C}$ and fix $r = \min\{1, \delta(T_0, \varepsilon)\}$ where $\delta(T_0, \varepsilon)$ is from Lemma 3.3. The function $a \mapsto \widetilde{M}_a(r)$ on $[0,1]$ is non-decreasing, integer-valued, and is bounded above by C. Thus, we can find an interval $[a_1, a_2] \subset [0,1]$ with $a_2 - a_1 \ge \frac{1}{C}$ such that $\widetilde{M}_{a_1}(r) = \widetilde{M}_{a_2}(r)$. Clearly, the supremum in (3.8) is attained, so we can find $u \in \mathcal{A}^*$ with $d_u \le r$ such that

$$M_{a_1}(u) = \widetilde{M}_{a_1}(r).$$

Fix this u for the rest of the proof. Since, in addition, $\widetilde{M}_{a_2}(r) = \widetilde{M}_{a_1}(r)$ and $M_{a_2}(u) \ge M_{a_1}(u)$, we deduce that $M_{a_2}(u) = \widetilde{M}_{a_2}(r) = M_{a_1}(u)$. Observe that

$$a_2 \ge (1 + (2C)^{-1})a_1 = a_1(1+\varepsilon)$$

and

$$1 + a_2 \geq (1 + a_1)(1 + (2C)^{-1}) = (1 + a_1)(1 + \varepsilon);$$

hence

$$v \in W_{a_1}(u) \;\Rightarrow\; qv \in W_{a_2}(qu) \quad \text{for all} \;\; q \in \mathcal{A}^*$$

by Lemma 3.3. It follows that $M_{a_2}(qu) \geq M_{a_1}(u)$. But

$$M_{a_2}(qu) \leq \widetilde{M}_{a_2}(r) = \widetilde{M}_{a_1}(r) = M_{a_1}(u);$$

therefore, $M_{a_2}(u) = M_{a_1}(u) = M_{a_2}(qu)$ for all $q \in \mathcal{A}^*$. Thus,

$$W_{a_2}(qu) = \{qv : \; v \in W_{a_2}(u)\} \quad \text{for all} \;\; q \in \mathcal{A}^*. \tag{3.9}$$

Consider

$$U = \bigcup_{v \in \mathcal{A}^*} S_v \mathcal{O}(\mathcal{K}_u, \varepsilon'), \tag{3.10}$$

where $\varepsilon' > 0$ will be chosen later. This will be our open set in the strong OSC. Clearly, $U \cap \mathcal{K} \neq \emptyset$ and $S_i U \subset U$ for all $i \leq m$. It remains to check that $S_i U \cap S_j U = \emptyset$ for all $i \neq j$. This will follow if we prove that for all v, w in $\mathcal{A}^*$ and all $i \neq j$,

$$S_{iv}\mathcal{O}(\mathcal{K}_u, \varepsilon') \cap S_{jw}\mathcal{O}(\mathcal{K}_u, \varepsilon') = \emptyset. \tag{3.11}$$

If $\varepsilon' \leq \delta_0$, then

$$S_{iv}\mathcal{O}(\mathcal{K}_u, \varepsilon') \subset \mathcal{O}(\mathcal{K}_{ivu}, \|S'_{iv}\|\varepsilon') \subset \mathcal{O}(\mathcal{K}_{ivu}, \varepsilon'' d_{ivu}), \quad \text{with} \;\; \varepsilon'' = C_4 d_u^{-1} \varepsilon',$$

in view of (2.5). Similarly,

$$S_{jw}\mathcal{O}(\mathcal{K}_u, \varepsilon') \subset \mathcal{O}(\mathcal{K}_{jwu}, \varepsilon'' d_{jwu}).$$

Assume that $d_{ivu} \geq d_{jwu}$ without loss of generality. By (3.7), there is a prefix (initial block) jw' of the word jwu such that $T_0^{-1} \leq \frac{d_{jw'}}{d_{ivu}} \leq T_0$ (here w' may range from empty to wu). Now jw' satisfies the diameter condition for membership in $W_{a_2}(ivu)$ but $jw' \notin W_{a_2}(ivu)$ by (3.9). Therefore,

$$\operatorname{dist}(\mathcal{K}_{ivu}, \mathcal{K}_{jwu}) \geq \operatorname{dist}(\mathcal{K}_{ivu}, \mathcal{K}_{jw'}) > a_2 d_{ivu}.$$

Thus, if $\varepsilon'' \leq a_2/2$, then (3.11) holds. It suffices to take $\varepsilon' = \min\{\delta_0, \frac{1}{2} a_2 C_4^{-1} d_u\}$, and the proof is complete. □

Proof of Corollary 1.2. We want to show that if $s = d$, the dimension of the space, and $\mathcal{H}^s(\mathcal{K}) > 0$, then $\mathcal{K} = \operatorname{clos}(\operatorname{int} \mathcal{K})$. The proof is quite similar to the proof of [11, Cor. 2.3]. By Theorem 1.1, the OSC holds, and moreover, the open set U can be chosen so that $U \subset V' = \mathcal{O}(X, \delta_0)$ (see (3.10)). The OSC means that $S_i U$ are pairwise disjoint subsets of U, for $i \leq m$. Let

$$W = U \setminus \bigcup_{i=1}^{m} S_i U.$$

We claim that $\mathcal{L}_d(W) = \mathcal{H}^d(W) = 0$ where $\mathcal{L}_d$ is the Lebesgue measure in $\mathbb{R}^d$. Indeed, it is easy to see that the sets $S_v W$ are pairwise disjoint for all $v \in \mathcal{A}^*$, and they all lie in U. Thus,

$$\sum_{n \geq 1} \sum_{|v|=n} \mathcal{L}_d(S_v W) \leq \mathcal{L}_d(U) < \infty. \tag{3.12}$$

We have

$$\mathcal{L}_d(S_v W) = \int_W |S_v'(x)|^d \, dx \geq C_1^{-d} \|S_v'\|^d \mathcal{L}_d(W),$$

in view of (2.1). Therefore, by (2.7),

$$\sum_{|v|=n} \mathcal{L}_d(S_v W) \geq C_1^{-d} C_5^{-1} \mathcal{L}_d(W) \sum_{|v|=n} \mu[v] = C_1^{-d} C_5^{-1} \mathcal{L}_d(W);$$

hence $\mathcal{L}_d(W) = 0$ by (3.12). This implies that the open set $U \setminus \operatorname{clos}(\bigcup_{i=1}^m S_i U)$ is empty. Therefore,

$$\bigcup_{i=1}^m S_i(\operatorname{clos} U) = \bigcup_{i=1}^m \operatorname{clos}(S_i U) = \operatorname{clos}\left(\bigcup_{i=1}^m S_i U\right) = \operatorname{clos} U,$$

so $\operatorname{clos} U$ is an invariant compact set for the i.f.s. $\{S_i\}_{i \leq m}$. By uniqueness, $\operatorname{clos} U = \mathcal{K}$, and the proof is complete. □

In conclusion, we should comment on the paper of Fan and Lau [5] where the implication OSC ⇒ SOSC is stated in Lemma 2.6. However, as pointed out by N. Patzschke (personal communication), the proof of [5, Lemma 2.6] contains a gap. The formula on the second line of [5, p. 335] is unjustified; proving it involves checking two facts, one of which, that $IJ \in \Lambda_{|U_{IJ_0}|}$, may fail, due to distortion. Perhaps one could fix the proof, but this would require further arguments.

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