# SUB-ADDITIVE PRESSURE FOR TRIANGULAR MAPS

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ABSTRACT. We investigate properties of the zero of the subadditive pressure used by Falconer, Barreira and Zhang to estimate the box and Hausdorff dimension of a non-conformal repeller. In the conformal case, and in Falconer's 1-bunched non-conformal case, the contraction rates satisfy bounded distortion and so this zero is insensitive to where on each cylinder the contraction is evaluated. We study some non-linear two-dimensional examples which do not satisfy bounded distortion but do exhibit the same insensitivity. Here the contraction rate fails to specify ellipses that can be used to cover cylinders.

## 1. INTRODUCTION

Pressure and subadditive pressure have proved valuable tools for obtaining (at least) upper bounds on the Hausdorff and box dimension of an invariant set of a non-conformal dynamical system. Consider a repeller $\Lambda$ of a $\mathcal{C}^2$ expanding map $E : M \to M$ or (by constructing a Markov partition [3, p. 79] and taking local inverses) the invariant set of a (possibly graph-directed) iterated function system (IFS).

When the maps are conformal the Hausdorff dimension is given by the zero of the pressure function $P(s)$ defined as $P_f(-s \log \|DE\|)$, [4, 10, 12]. The pressure, defined, for example, in [15, §9], is given by using one point from each $n$-cylinder and incorporating an approximation to the diameter of the cylinder (whose $s$-th power is used in calculating the upper bound for Hausdorff $s$-measure given by the covering by $n$-cylinders). In the conformal case one just multiplies the various contraction rates to get this approximation.

But, if the maps are not conformal, their composition can contract more strongly. An attempt to capture this is the subadditive pressure of Falconer, [6, 7], which uses a function $\phi^s$ in place of $-s \log \|DE\|$ that combines the singular values of the derivative of local inverses of $E$. The norm of the derivative of such inverses of $E^N$ can be very different from the product of norms, and subadditive pressure incorporates this.

Falconer considered a map for which the strongest expansion is less than the square of the weakest expansion, called the 1-*bunched* case [1, p. 903]. In the 1-bunched case the cylinders are convex and the singular values satisfy the bounded distortion property. Even when the 1-bunched condition is not

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satisfied Zhang [16] found that this zero is an upper bound for the Hausdorff dimension, but little else is known, although Barreira addresses this case in [2]. The approach of [2] is to cover by ellipsoids whose semiaxes come from certain singular values and then relate box dimension to a notion of elliptic box dimension. In §7 we point out an error in the proof of that relation and present a counterexample showing that a weaker statement does not hold.

The structure of our paper is as follows: we study examples in the plane of a triangular type, in fact a skew-product over an affine Cantor set. In Theorem 2 (§5) we show that, just beyond the 1-bunched condition (see Remark 1), it can happen that, on some cylinders, the singular values fail to satisfy bounded distortion. In Theorem 3 (§6) we show that the values of $\phi^s$ on a cylinder can fail to determine an ellipse or rectangle that contains the cylinder. In Theorem 1 (developed in §2 and proved in §4 using estimates obtained in §3) we show that, for a wider class of skew-products, the subadditive pressure does *not* depend on whether we use the maximum or minimum of $\phi^s$ on each cylinder.

## 2. Insensitivity of some two-dimensional examples

In this section we introduce our class of two-dimensional examples, define subadditive pressure for them and state our insensitivity theorem.

Let $M \subset \mathbb{R}^2$ be non-empty and open. Let $E : M \to M$ be a $\mathcal{C}^2$ map. We use notation very similar to that of Falconer in [7]. We say that a compact subset $\Lambda \subset M$ is a mixing repeller for $E$ if

**(a):** $E|\Lambda$ is expanding,

**(b):** there is an open set $V$, $\Lambda \subset V \subset M$ such that

$$\Lambda = \{(x_1, x_2) \in V : E^n(x_1, x_2) \in V \text{ for all } n \geq 0\}. \tag{1}$$

**(c):** $E|\Lambda$ is topologically mixing.

Let $\Lambda_1, \dots \Lambda_m$ be a Markov partition of $\Lambda$ into small subsets on each of which $E$ is injective. Let $\widetilde{\Lambda}_k$ be the closure of the $\delta$ neighbourhood of $\Lambda_k$, where $\delta$ is so small that $\widetilde{\Lambda}_k \subset V$ and

$$\widetilde{\Lambda}_i \cap \widetilde{\Lambda}_j = \emptyset \text{ if and only if } \Lambda_i \cap \Lambda_j = \emptyset.$$

A sequence $\mathbf{i} = (i_0, \dots, i_n) \in \{1, \dots, m\}^{n+1}$ is called *admissible* if $E(\Lambda_{i_j}) \supset \Lambda_{i_{j+1}}$ for $0 \leq j < n$. Let

$$S_n := \{\mathbf{i} : |\mathbf{i}| = n + 1, \mathbf{i} \text{ is admissible}\}$$

For $\mathbf{i} \in S_n$ we write

$$\Lambda_{\mathbf{i}} := \bigcap_{k=0}^{n} E^{-k}(\Lambda_{i_k}) \text{ and } \widetilde{\Lambda}_{\mathbf{i}} := \bigcap_{k=0}^{n} E^{-k}(\widetilde{\Lambda}_{i_k}).$$

As in [7, p. 321] we denote the local inverse of the map $E^n|\widetilde{\Lambda}_{i_0 i_1 \dots i_n}$ by

$$F_{i_0, i_1 \dots i_n} : \widetilde{\Lambda}_{i_n} \to \widetilde{\Lambda}_{i_0, i_1 \dots i_n}.$$

Then, as in [7], we can find $0 < \beta_1 < \beta_2 < 1$ such that for every $(x_1, x_2) \in \widetilde{\Lambda}_{i_n}$ and for all $\mathbf{u} \in \mathbb{R}^2$

$$\beta_1^n \|\mathbf{u}\| \leq \|(D_{(x_1,x_2)} F_{i_0,i_1\dots i_n})\mathbf{u}\| \leq \beta_2^n \|\mathbf{u}\|. \tag{2}$$

We now make the *assumption* that the function $E|V$ is given in the (skew-product) form

$$E(x_1, x_2) := (e_1(x_1), e(x_1, x_2)). \tag{3}$$

So the local inverses are given in the form

$$F_{i_0,i_1}(x_1, x_2) := (f_{i_0,i_1}(x_1), g_{i_0,i_1}(x_1, x_2)) \quad (x_1, x_2) \in \widetilde{\Lambda}_{i_1}. \tag{4}$$

Then, for $(i_0, i_1)$ admissible and $(x_1, x_2) \in \widetilde{\Lambda}_{i_1}$, $D_{(x_1,x_2)} F_{i_0,i_1}$ is non-singular. For $\mathbf{i} \in \Sigma^* := \bigcup_{\ell=1}^{\infty} S_\ell$ and $(x_1, x_2) \in \widetilde{\Lambda}_{i_n}$ we write $\alpha_k(\mathbf{i}, (x_1, x_2))$ for the $k$-th biggest singular value of the matrix $D_{(x_1,x_2)} F_{\mathbf{i}}$. For $k = 1, 2$ put

$$\overline{\alpha}_k(\mathbf{i}) := \max_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} \alpha_k(\mathbf{i}, (x_1, x_2)) \;\; \underline{\alpha}_k(\mathbf{i}) := \min_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} \alpha_k(\mathbf{i}, (x_1, x_2)).$$

For $0 < s \leq 2$ the *singular value function* is defined by

$$\phi^s(\mathbf{i}, (x_1, x_2)) := \begin{cases} \alpha_1(\mathbf{i}, (x_1, x_2))^s, & \text{if } s \leq 1; \\ \alpha_1(\mathbf{i}, (x_1, x_2)) \cdot \alpha_2(\mathbf{i}, (x_1, x_2))^{s-1}, & \text{if } 1 < s \leq 2. \end{cases} \tag{5}$$

As with $\alpha_k$ above, let

$$\overline{\phi}^s(\mathbf{i}) := \max_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} \phi^s(\mathbf{i}, (x_1, x_2)), \text{ and } \underline{\phi}^s(\mathbf{i}) := \min_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} \phi^s(\mathbf{i}, (x_1, x_2)).$$

Following [7, p. 322] we define the subadditive pressure

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{\phi}^s(\mathbf{i}) = \inf_n \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{\phi}^s(\mathbf{i}).$$

That this limit exists was proved in [7, p. 322]. Further put

$$\underline{P}(s) := \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \underline{\phi}^s(\mathbf{i}). \tag{6}$$

**Theorem 1.** *For every $0 < s \leq 2$ the limit exists in (6). For $E$ as in (3) we have*

$$\underline{P}(s) = P(s). \tag{7}$$

That is the sub-additive pressure is not sensitive to the choice of the points of the cylinders at which the singular value function is evaluated.

## 3. Estimating lower triangular matrices

In this section we study the diagonal terms in the derivative of an $n$-fold composition of our contractions and we estimate the off-diagonal terms using the diagonal ones.

To simplify the notation for the derivative of $F_{i_0,i_1}$ in (4) we write

$$u_{i_0,i_1}(x_1,x_2) := \frac{\partial g_{i_0,i_1}}{\partial x_1}(x_1,x_2) \text{ and } v_{i_0,i_1}(x_1,x_2) := \frac{\partial g_{i_0,i_1}}{\partial x_2}(x_1,x_2),$$

so that

$$DF_{i_0,i_1}(x_1,x_2) = \begin{bmatrix} f'_{i_0,i_1}(x_1) & 0 \\ u_{i_0,i_1}(x_1,x_2) & v_{i_0,i_1}(x_1,x_2) \end{bmatrix}. \tag{8}$$

Since each $F_{i_0,i_1}$ is contracting, for admissible $(i_0,i_1)$ and $(x_1,x_2) \in \widetilde{\Lambda}_{i_1}$ we have

$$|f'_{i_0,i_1}(x_1)| < 1,\ |v_{i_0,i_1}(x_1,x_2)| < 1 \text{ and } |u_{i_0,i_1}(x_1,x_2)| < 1. \tag{9}$$

Further, for every $n \geq 2$ and admissible $(i_0,\dots,i_n)$ we define

$$a_{i_0\dots i_n}(x_1) := \prod_{k=0}^{n-2} f'_{i_k,i_{k+1}}(f_{i_{k+1}\dots i_n}(x_1)) \cdot f'_{i_{n-1},i_n}(x_1)$$

and

$$c_{i_0\dots i_n}(x_1,x_2) := \prod_{k=0}^{n-2} v_{i_k,i_{k+1}}(F_{i_{k+1}\dots i_n}(x_1,x_2)) \cdot v_{i_{n-1},i_n}(x_1,x_2).$$

Then

$$DF_{i_0\dots i_n}(x_1,x_2) = \begin{bmatrix} a_{i_0\dots i_n}(x_1) & 0 \\ b_{i_0\dots i_n}(x_1,x_2) & c_{i_0\dots i_n}(x_1,x_2) \end{bmatrix}, \tag{10}$$

where

$$b_{i_0\dots i_n}(x_1,x_2) := \sum_{k=1}^{n} b^{(k)}_{i_0\dots i_n}(x_1,x_2), \tag{11}$$

and

(12)

$$\begin{aligned} b^{(k)}_{i_0\dots i_n}(x_1,x_2) :&= v_{i_0,i_1}(F_{i_1\dots i_n}(x_1,x_2)) \cdots v_{i_{k-2},i_{k-1}}(F_{i_{k-1}\dots i_n}(x_1,x_2)) \\ &\cdot u_{i_{k-1},i_k}(F_{i_k\dots i_n}(x_1,x_2)) \cdot f'_{i_k,i_{k+1}}(f_{i_{k+1}\dots i_n}(x_1)) \cdots f'_{i_{n-1},i_n}(x_1). \end{aligned}$$

For every $n$ and admissible $(i_0,\dots,i_n)$ we write

$$\overline{a}_{i_0\dots i_n} := \max_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} |a_{i_0\dots i_n}(x_1)| \text{ and } \overline{c}_{i_0\dots i_n} := \max_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} |c_{i_0\dots i_n}(x_1,x_2)|.$$

Further let

$$\underline{a}_{i_0\dots i_n} := \min_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} |a_{i_0\dots i_n}(x_1)| \text{ and } \underline{c}_{i_0\dots i_n} := \min_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} |c_{i_0\dots i_n}(x_1,x_2)|.$$

We write

$$\overline{b}^{(k)}_{i_0\dots i_n} := \overline{c}_{i_0\dots i_{k-1}}\overline{a}_{i_k\dots i_n}. \tag{13}$$

Then it follows from (9) and (12) that

$$\max_{(x_1,x_2)\in\widetilde{\Lambda}_{i_n}} |b^{(k)}_{i_0\dots i_n}(x_1,x_2)| \le \overline{b}^{(k)}_{i_0\dots i_n} \tag{14}$$

Finally we define

$$\overline{b}_{i_0\dots i_n} := \sum_{k=1}^{n} \overline{b}^{(k)}_{i_0\dots i_n}.$$

Then by definition for every $(x_1,x_2)$ we have

$$|b_{i_0\dots i_n}(x_1,x_2)| \le \overline{b}_{i_0\dots i_n}. \tag{15}$$

Using [11, Proposition 20.1] we obtain the bounded distortion properties that there exist $C_1>0$ such that for every $n$ and admissible $(i_0,\dots,i_n)$ we have

$$C_1^{-1} < \frac{\overline{a}_{i_0\dots i_n}}{\underline{a}_{i_0\dots i_n}} < C_1 \text{ and } C_1^{-1} < \frac{\overline{c}_{i_0\dots i_n}}{\underline{c}_{i_0\dots i_n}} < C_1. \tag{16}$$

It follows that

$$C_1^{-2}\overline{a}_{i_0\dots i_n}\cdot\overline{a}_{i_n\dots i_{n+k}} \le \overline{a}_{i_0\dots i_{n+k}} \le \overline{a}_{i_0\dots i_n}\cdot\overline{a}_{i_n\dots i_{n+k}}. \tag{17}$$

and

$$C_1^{-2}\overline{c}_{i_0\dots i_n}\cdot\overline{c}_{i_n\dots i_{n+k}} \le \overline{c}_{i_0\dots i_{n+k}} \le \overline{c}_{i_0\dots i_n}\cdot\overline{c}_{i_n\dots i_{n+k}}. \tag{18}$$

## 4. Singular values of our triangular maps

In this section we prove Theorem 1 by controlling the contribution of the off-diagonal terms to the singular value function and its pressure.

For a lower triangular $2\times 2$ matrix $A=\begin{bmatrix} a & 0\\ b & c\end{bmatrix}$ one can immediately see that the singular values $\lambda,\mu$, which are the positive square roots of the eigenvalues of the matrix $A^*A=\begin{bmatrix} a^2+b^2 & bc\\ bc & c^2\end{bmatrix}$, satisfy:

$$\lambda^2+\mu^2=a^2+b^2+c^2 \text{ and } \lambda\mu=ac.$$

Hence, for all $0<s\le 1$, we have

$$\frac{1}{3\cdot 2^s}\left\{|a|^s+|b|^s+|c|^s\right\} \le \max\left\{\lambda^s,\mu^s\right\} \le 4^s\cdot\left\{|a|^s+|b|^s+|c|^s\right\}. \tag{19}$$

It follows that for every $n$ and admissible $(i_0,\dots,i_n)$, $s>0$ we have

$$\begin{aligned}\frac{C_1^{-s}}{3\cdot 2^s}\left(\overline{a}^s_{i_0\dots i_n}+\overline{c}^s_{i_0\dots i_n}\right) &\le \frac{1}{3\cdot 2^s}\left(\underline{a}^s_{i_0\dots i_n}+\underline{c}^s_{i_0\dots i_n}\right)\\ &\le \underline{\alpha}^s_1(i_0\dots i_n) \le \overline{\alpha}^s_1(i_0\dots i_n)\\ &\le 4^s\left(\overline{a}^s_{i_0\dots i_n}+\overline{b}^s_{i_0\dots i_n}+\overline{c}^s_{i_0\dots i_n}\right).\end{aligned} \tag{20}$$

For every admissible $\mathbf{i} = (i_0, \dots, i_n)$ and every $(x_1, x_2)$ we have

$$a_{\mathbf{i}}(x_1,x_2) \cdot c_{\mathbf{i}}(x_1,x_2) = \det\left(DF_{\mathbf{i}}(x_1,x_2)\right) = \alpha_1(\mathbf{i},(x_1,x_2)) \cdot \alpha_2(\mathbf{i},(x_1,x_2)).$$

In this way for $1 < s \le 2$ we have

$$\begin{aligned} \phi^s(\mathbf{i},(x_1,x_2)) &= \alpha_1(\mathbf{i},(x_1,x_2)) \cdot \alpha_2(\mathbf{i},(x_1,x_2))^{s-1} \\ &= \alpha_1(\mathbf{i},(x_1,x_2))^{2-s} a_{\mathbf{i}}(x_1,x_2)^{s-1} c_{\mathbf{i}}(x_1,x_2)^{s-1} \end{aligned} \tag{21}$$

This implies that for $1 < s \le 2$ we have

$$\begin{aligned} \overline{\phi}^s(\mathbf{i}) &\le 4^{2-s} \cdot \left(\overline{a}_{\mathbf{i}}^{2-s} + \overline{b}_{\mathbf{i}}^{2-s} + \overline{c}_{\mathbf{i}}^{2-s}\right) \cdot \overline{a}_{\mathbf{i}}^{s-1} \overline{c}_{\mathbf{i}}^{s-1} \\ &= 4^{2-s}\left(\overline{a}_{\mathbf{i}} \overline{c}_{\mathbf{i}}^{s-1} + \overline{c}_{\mathbf{i}} \overline{a}_{\mathbf{i}}^{s-1} + \overline{b}_{\mathbf{i}}^{2-s} \overline{a}_{\mathbf{i}}^{s-1} \overline{c}_{\mathbf{i}}^{s-1}\right). \end{aligned} \tag{22}$$

We obtain from (20), (21) and (16) that there exists $C_2 > 0$ such that for $1 < s \le 2$

$$\underline{\phi}^s(\mathbf{i}) \ge C_2 \left[\overline{a}_{\mathbf{i}} \overline{c}_{\mathbf{i}}^{s-1} + \overline{c}_{\mathbf{i}} \overline{a}_{\mathbf{i}}^{s-1}\right]. \tag{23}$$

Similarly, for $0 < s \le 1$, we have

$$\frac{C_1^{-s}}{3 \cdot 2^s}\left(\overline{a}_{\mathbf{i}}^s + \overline{c}_{\mathbf{i}}^s\right) \le \underline{\phi}^s(\mathbf{i}) \le \overline{\phi}^s(\mathbf{i}) \le 4^s \cdot \left(\overline{a}_{\mathbf{i}}^s + \overline{b}_{\mathbf{i}}^s + \overline{c}_{\mathbf{i}}^s\right). \tag{24}$$

Motivated by formulae (22) and (24) we define

$$\psi_a^s(\mathbf{i}) := \begin{cases} \overline{a}_{\mathbf{i}}^s, & \text{if } 0 < s \le 1 \\ \overline{a}_{\mathbf{i}} \overline{c}_{\mathbf{i}}^{s-1}, & 1 < s \le 2 \end{cases} \quad \text{and } \psi_c^s(\mathbf{i}) := \begin{cases} \overline{c}_{\mathbf{i}}^s, & \text{if } 0 < s \le 1 \\ \overline{c}_{\mathbf{i}} \overline{a}_{\mathbf{i}}^{s-1}, & 1 < s \le 2. \end{cases}$$

It follows from (17) and (18) that, for $0 < s \le 2$, $\sum_{\mathbf{i} \in S_n} \psi_a^s(\mathbf{i})$ and $\sum_{\mathbf{i} \in S_n} \psi_c^s(\mathbf{i})$ are submultiplicative so the following limits exist:

$$P_a(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in S_n} \psi_a^s(\mathbf{i}) \text{ and } P_c(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in S_n} \psi_c^s(\mathbf{i}). \tag{25}$$

Let

$$P^{\Delta}(s) := \max\left\{P_a(s), P_c(s)\right\}.$$

*Proof of Theorem 1.* It follows from (23) and (24) that for every $0 < s \le 2$ we have

$$P^{\Delta}(s) \le \underline{P}(s). \tag{26}$$

To verify our theorem we only need to prove that for every $0 < s \le 2$ we have

$$P(s) \le P^{\Delta}(s). \tag{27}$$

For the rest of the proof we fix $\varepsilon > 0$ and we choose $K$ such that for every $n \ge K$ we have

$$\sum_{\mathbf{i} \in S_n} \psi_a^s(\mathbf{i}) < e^{n(P_a(s)+\varepsilon)} \text{ and } \sum_{\mathbf{i} \in S_n} \psi_c^s(\mathbf{i}) < e^{n(P_c(s)+\varepsilon)}. \tag{28}$$

*First we assume that* $0 < s \leq 1$. It follows from (24) that

$$P(s) \leq \max\left\{P^{\Delta}(s), \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^s\right\}. \tag{29}$$

So, it is enough to prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^s \leq P^{\Delta}(s) + \varepsilon. \tag{30}$$

Now

$$\begin{aligned}\sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^s &= \sum_{\mathbf{i}\in S_n} \left(\sum_{k=1}^{n} \overline{b}_{\mathbf{i}}^{(k)}\right)^s \leq \sum_{k=1}^{n} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(k)}\right)^s \\ &= \underbrace{\sum_{j=1}^{K-1} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(j)}\right)^s}_{S_1} + \underbrace{\sum_{j=K}^{n-K} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(j)}\right)^s}_{S_2} + \underbrace{\sum_{j=n-K+1}^{n} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(j)}\right)^s}_{S_3}.\end{aligned} \tag{31}$$

Using (13) and (9) we get that

$$S_1 \leq K \cdot m^K \cdot \sum_{(i_K,\dots,i_n)\in S_{n-K}} \overline{a}_{i_K\dots i_n}^s \leq K \cdot m^K \cdot e^{(n-K)\cdot(P_a(s)+\varepsilon)}. \tag{32}$$

$$S_3 \leq K \cdot m^K \cdot \sum_{\mathbf{i}\in S_{n-K}} \overline{c}_{i_0\dots i_{n-K}}^s \leq K \cdot m^K \cdot e^{(n-K)\cdot(P_c(s)+\varepsilon)}. \tag{33}$$

$$S_2 \leq \sum_{j=K}^{n-K} e^{j(P_c(s)+\varepsilon)+(n-j)(P_a(s)+\varepsilon)} < n \cdot e^{n(P^{\Delta}(s)+\varepsilon)}. \tag{34}$$

Putting the last three inequalities together we immediately get that (30) holds.

*Now we assume that* $1 < s \leq 2$. Then for every admissible word $\mathbf{i}$ and for every $(x_1, x_2)$

$$\alpha_1(\mathbf{i},(x_1,x_2)) \cdot \alpha_2(\mathbf{i},(x_1,x_2)) = \det\left(DF_{\mathbf{i}}(x_1,x_2)\right) = a_{\mathbf{i}}(x_1,x_2) \cdot c_{\mathbf{i}}(x_1,x_2). \tag{35}$$

This implies that

$$\phi^s(\mathbf{i},(x_1,x_2)) = \alpha_1(\mathbf{i},(x_1,x_2))^{2-s} \cdot a_{\mathbf{i}}(x_1,x_2)^{s-1} \cdot c_{\mathbf{i}}(x_1,x_2)^{s-1}. \tag{36}$$

This and (20) imply that

$$\overline{\phi}^s(\mathbf{i}) \leq 4^{2-s}(\psi_a^s(\mathbf{i}) + \overline{b}_{\mathbf{i}}^{2-s} \cdot \overline{a}_{\mathbf{i}}^{s-1} \cdot \overline{c}_{\mathbf{i}}^{s-1} + \psi_c^s(\mathbf{i})). \tag{37}$$

It follows that

$$P(s) \leq \max\left\{P^{\Delta}(s), \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^{2-s} \cdot \overline{a}_{\mathbf{i}}^{s-1} \cdot \overline{c}_{\mathbf{i}}^{s-1}\right\}. \tag{38}$$

In this way it is enough to verify that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^{2-s} \cdot \overline{a}_{\mathbf{i}}^{s-1} \cdot \overline{c}_{\mathbf{i}}^{s-1} \leq P^{\Delta}(s) + \varepsilon. \tag{39}$$

Using that $0 \leq 2-s < 1$ we obtain that

$$\begin{aligned} \sum_{\mathbf{i}\in S_n} \overline{b}_{\mathbf{i}}^{2-s} \cdot \overline{a}_{\mathbf{i}}^{s-1} \cdot \overline{c}_{\mathbf{i}}^{s-1} &\leq \sum_{k=1}^{n} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(k)}\right)^{2-s} \cdot \overline{a}_{\mathbf{i}}^{s-1} \cdot \overline{c}_{\mathbf{i}}^{s-1} \\ &= U_1 + U_2 + U_3 \end{aligned}$$

where

$$U_1 := \sum_{k=1}^{K-1} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(k)}\right)^{2-s} \overline{a}_{\mathbf{i}}^{s-1}\overline{c}_{\mathbf{i}}^{s-1}, \ U_2 := \sum_{k=K}^{n-K} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(k)}\right)^{2-s} \overline{a}_{\mathbf{i}}^{s-1}\overline{c}_{\mathbf{i}}^{s-1}$$

and

$$U_3 := \sum_{k=n-K+1}^{n} \sum_{\mathbf{i}\in S_n} \left(\overline{b}_{\mathbf{i}}^{(k)}\right)^{2-s} \overline{a}_{\mathbf{i}}^{s-1}\overline{c}_{\mathbf{i}}^{s-1}.$$

We can estimate $U_1, U_2$ and $U_3$ in way similar to that used for $S_1, S_2$ and $S_3$ above. Namely using (9) we obtain the existence of constants $C_4 = C_4(K), C_5 = C_5(K)$ and $C_6 = C_6(K)$ such that

$$U_1 \leq C_4 \sum_{k=1}^{K-1} \sum_{\mathbf{i}\in S_n} \psi_a^s(\mathbf{i}) \text{ and } U_3 \leq C_5 \sum_{k=n-K+1}^{n} \sum_{\mathbf{i}\in S_n} \psi_c^s(\mathbf{i}) \tag{40}$$

and, using (17) and (18),

$$\begin{aligned} U_2 &\leq C_6 \sum_{j=K}^{n-K} \left( \sum_{(i_0,\dots,i_j)\in S_j} \overline{c}_{i_0\dots i_j} \left(\overline{a}_{i_0\dots i_j}\right)^{s-1} \cdot \sum_{(i_j,\dots,i_n)\in S_{n-j}} \overline{a}_{i_j\dots i_n} \left(\overline{c}_{i_j\dots i_n}\right)^{s-1} \right) \\ &\leq C_6 \sum_{j=K}^{n-K} e^{j(P^{\Delta}(s)+\varepsilon)+(n-j)(P^{\Delta}(s)+\varepsilon)} \leq C_6 \cdot n \cdot e^{n(P^{\Delta}(s)+\varepsilon)}. \end{aligned} \tag{41}$$

Putting together (40) and (41) we see that (39) holds. This completes the proof of Theorem 1. □

## 5. Failure of the bounded distortion property

In this section we will present an IFS of the form (4) for which the largest singular value does not satisfy the bounded distortion property. Our IFS will be $\{F_1, F_2\}$ modeled by the full 2-shift and using maps $F_i$ instead of $F_{i_0,i_1}$ so the admissible $n$-tuples $(i_1, \dots, i_n)$ do not need the symbol $i_0$.

**Theorem 2.** *There exists an IFS of the form (4) such that*

$$\sup_{\mathbf{i}\in\Sigma^*} \frac{\overline{\alpha}_1(\mathbf{i})}{\underline{\alpha}_1(\mathbf{i})} = \sup_{\mathbf{i}\in\Sigma^*} \frac{\sup_{(x_1,x_2)\in\widetilde{\Lambda}_{\mathbf{i}}} \|DF_{\mathbf{i}}(x_1,x_2)\|}{\inf_{(x_1,x_2)\in\widetilde{\Lambda}_{\mathbf{i}}} \|DF_{\mathbf{i}}(x_1,x_2)\|} = \infty. \tag{42}$$

*Proof.* Using the notation of (4) let $m = 2$. We fix for the rest of the proof constants $\lambda_1, \widetilde{\lambda}_1, \lambda_2, \widetilde{\lambda}_2$ less than 1 that satisfy:

$$\lambda_1 > \frac{\lambda_2}{\widetilde{\lambda}_2}, \quad \lambda_1 > \sqrt{\widetilde{\lambda}_1} \text{ and } \widetilde{\lambda}_2 < \lambda_1. \tag{43}$$

Note that for any $\tau > 0$ the following choice will do:

$$\lambda_1 = \frac{1}{2}, \ \widetilde{\lambda}_1 = \frac{1}{2^{2+2\tau}}, \ \lambda_2 = \frac{1}{2^{2+4\tau}}, \ \widetilde{\lambda}_2 = \frac{1}{2^{1+\tau}}. \tag{44}$$

First we define affine contractions $f_1, f_2 : [0,1] \to [0,1]$ by

$$f_1(x) := \lambda_1 x \quad \text{and} \quad f_2(x) := \lambda_2 x + 1 - \lambda_2.$$

Let $C$ be the attractor of the IFS $\{f_1, f_2\}$. As a preparation for the construction of $g_i(x_1, x_2)$ for $i = 1, 2$ and $0 \le x_1, x_2 \le 1$ we construct a strictly increasing function $h \in \mathcal{C}^2([0,1])$ which has zero derivative on $C$ so that, on every interval complementary to $C$, $h'$ has maximum a little less than the length of this interval. We use

**Lemma 1.** *Let $J_{\mathbf{i}} := f_{\mathbf{i}}([0,1])$ and let $I_{\mathbf{i}} := J_{\mathbf{i}} \setminus (\mathrm{int}(J_{\mathbf{i},1} \cup J_{\mathbf{i},2}))$. There exists a $\mathcal{C}^1$ function $\eta : [0,1] \to \mathbb{R}^+$ and $z \in (0,1)$ such that*

**(a):**

$$\eta(x) = 0 \text{ if and only if } x \in C. \tag{45}$$

**(b):** *for every $\mathbf{i} \in \Sigma^*$ and $z_{\mathbf{i}} := f_{\mathbf{i}}(z)$ we have*

$$\eta(z_{\mathbf{i}}) = \frac{|I_{\mathbf{i}}|}{\log(n+2)}, \tag{46}$$

*where $n = |\mathbf{i}|$.*

*Proof.* We put $z := \lambda_1 + \frac{1-(\lambda_1+\lambda_2)}{2}$. That is $z$ is the midpoint of $I := [0,1] \setminus \mathrm{int}(J_1 \cup J_2)$. Let $u$ and $v$ be the left and right end points of the interval $I$ and for every $\mathbf{i} \in \Sigma^*$ let $u_{\mathbf{i}}$ and $v_{\mathbf{i}}$ be the left and right end points of the interval $I_{\mathbf{i}}$.

For each $\mathbf{i} \in \Sigma^*$ we define the function $\eta$ on $I_{\mathbf{i}}$ as follows: if $x$ is chosen from the first quarter of the interval $I_{\mathbf{i}}$ then let

$$\eta(x) := \frac{8}{\log(n+2)} \frac{(x-u_{\mathbf{i}})^2}{|I_{\mathbf{i}}|}. \tag{47}$$

Then for such an $x$ we have

$$\eta'(x) = \frac{16}{\log(n+2)} \frac{(x-u_{\mathbf{i}})}{|I_{\mathbf{i}}|} \le \frac{4}{\log(n+2)}. \tag{48}$$

Then we define the function $\eta$ on the second quarter of the interval $I_{\mathbf{i}}$ by

$$\eta(x) := |I_{\mathbf{i}}| / \log(n+2) - \eta\left(u_{\mathbf{i}} + |I_{\mathbf{i}}|/2 - x\right)$$

making the graph of $\eta|_{[u_{\mathbf{i}}, z_{\mathbf{i}}]}$ symmetrical about its midpoint. Then by the symmetry, $\eta'(z_{\mathbf{i}}) = 0$. Therefore, we can define the function $\eta$ on the second half of the interval $I_{\mathbf{i}}$ by reflecting the graph of $\eta|_{[u_{\mathbf{i}},z_{\mathbf{i}}]}$ in the vertical line $x \equiv z_{\mathbf{i}}$. Then

$$\max \eta|_{I_{\mathbf{i}}} = \eta(z_{\mathbf{i}}) = \frac{|I_{\mathbf{i}}|}{\log(n+2)}. \tag{49}$$

In this way we have defined $\eta$ on $\cup_{\mathbf{i}} I_{\mathbf{i}}$ such that at all endpoints $\eta$ is 0. Let $\eta|_C :\equiv 0$. Then $\eta$ is differentiable on $C$ and $\eta'|_C \equiv 0$.

□

Choose $c_1$ such that for

$$h(x) := c_1 \int_0^x \eta(t)dt, \tag{50}$$

we have

$$h(1) := \min\left\{1-\widetilde{\lambda}_1, 1-\widetilde{\lambda}_2\right\} = 1-\widetilde{\lambda}_2.$$

Then $h \in \mathcal{C}^2[0,1]$ is strictly monotonic and

$$\forall x \in C \text{ we have } h'(x) = 0. \tag{51}$$

Further, for every $\mathbf{j} \in \Sigma^*$ with $|\mathbf{j}| = \ell$ we have

$$h'(z_{\mathbf{j}}) = \frac{4c_1|I_{\mathbf{j}}|}{\log(\ell+2)}. \tag{52}$$

We choose $c_2 := 4c_1 \cdot |I|$. Then for $\mathbf{j} = (j_1, \dots, j_\ell)$

$$h'(z_{\mathbf{j}}) = \frac{c_2 \cdot \lambda_{j_1} \cdots \lambda_{j_\ell}}{\log(\ell+2)}. \tag{53}$$

Now we can define our IFS as:

$$\begin{aligned} F_1(x_1,x_2) &:= (f_1(x_1), h(x_1) + \widetilde{\lambda}_1 x_2), \\ F_2(x_1,x_2) &:= (f_2(x_1), h(x_1) + \widetilde{\lambda}_2 x_2 + 1 - h(1) - \widetilde{\lambda}_2). \end{aligned} \tag{54}$$

Then $F_i : [0,1]^2 \to [0,1]^2$ for $i = 1,2$ are $\mathcal{C}^2$ functions and $F_1(0,0) = (0,0)$, $F_2(1,1) = (1,1)$. We write $\Lambda$ for the attractor of $\{F_1, F_2\}$; $\Lambda \subset [0,1]^2$. Since $\lambda_1 + \lambda_2 < 1$ we have $F_1\left([0,1]^2\right) \cap F_2\left([0,1]^2\right) = \emptyset$

and the strong separation property, see [9, p. 35], holds. Further,

$$DF_1(x_1,x_2) = \begin{bmatrix} \lambda_1 & 0 \\ h'(x_1) & \widetilde{\lambda}_1 \end{bmatrix}, \; DF_2(x_1,x_2) = \begin{bmatrix} \lambda_2 & 0 \\ h'(x_1) & \widetilde{\lambda}_2 \end{bmatrix}.$$

Because

$$DF_{i_k}(F_{i_{k+1}\dots i_n}(x_1,x_2)) = \begin{bmatrix} \lambda_{i_k} & 0 \\ h'(f_{i_{k+1}\dots i_n}(x_1)) & \widetilde{\lambda}_{i_k} \end{bmatrix}$$

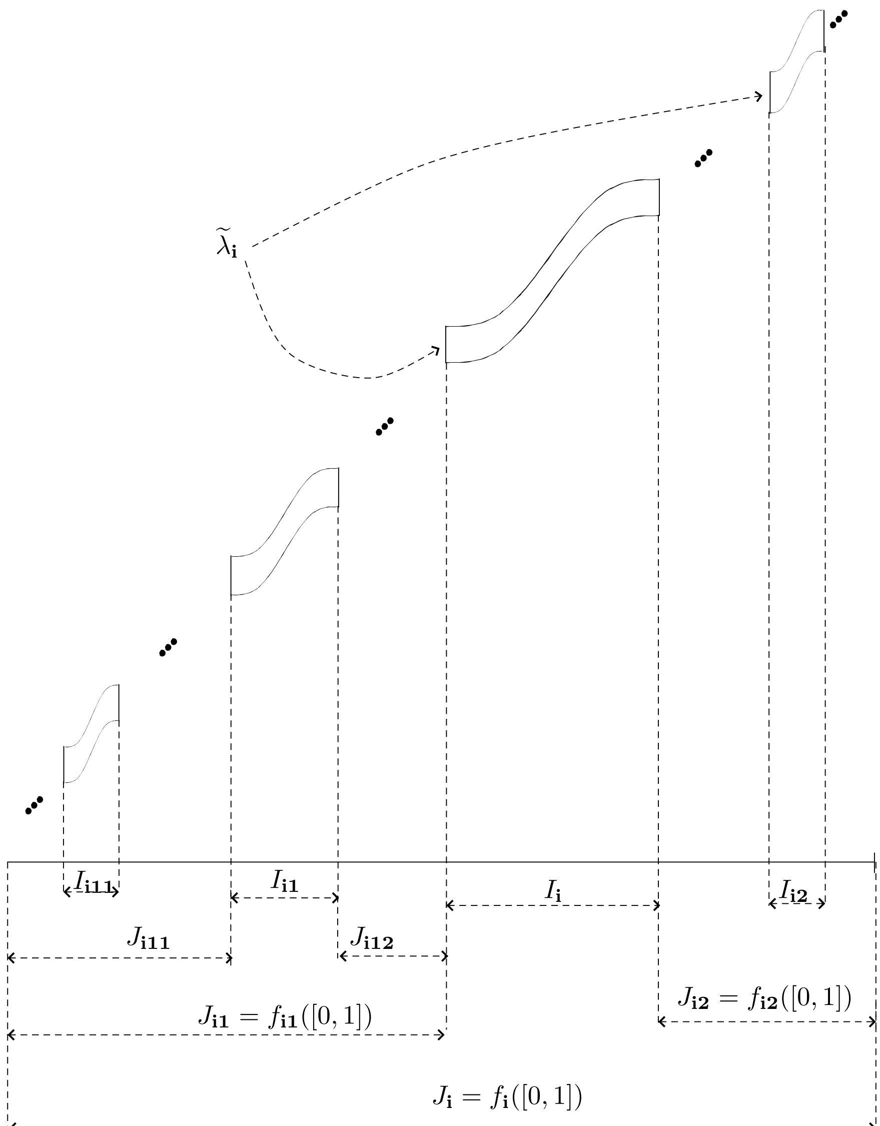


FIGURE 1. The cylinder $F_{\mathbf{i}}\left([0,1]^2\right)$ contains infinitely many smooth strips of height $\widetilde{\lambda}_{\mathbf{i}}$

we get

$$
\begin{aligned}
D(F_{i_1\dots i_n}(x_1,x_2)) &= DF_{i_1}(F_{i_2\dots i_n}(x_1,x_2))\cdots DF_{i_n}(x_1,x_2)\\
&= \begin{bmatrix} \lambda_{i_1} & 0 \\ h'(f_{i_2\dots i_n}(x_1)) & \widetilde{\lambda}_{i_1} \end{bmatrix}\cdots\begin{bmatrix} \lambda_{i_n} & 0 \\ h'(x_1) & \widetilde{\lambda}_{i_n} \end{bmatrix}\\
&= \begin{bmatrix} \lambda_{i_1\dots i_n} & 0 \\ b_{i_1\dots i_n}(x_1) & \widetilde{\lambda}_{i_1\dots i_n} \end{bmatrix}, \qquad (55)
\end{aligned}
$$

where $\lambda_{\mathbf{i}} = \lambda_{i_1\dots i_n} := \lambda_{i_1}\cdots\lambda_{i_n}$, $\widetilde{\lambda}_{\mathbf{i}} = \widetilde{\lambda}_{i_1\dots i_n} := \widetilde{\lambda}_{i_1}\cdots\widetilde{\lambda}_{i_n}$. As in (11),

$$b_{i_1\dots i_n}(x_1) := \sum_{k=1}^{n} b^{(k)}_{i_1\dots i_n}(x_1), \tag{56}$$

where, as in (12), for $1 \leq k < n$

$$b^{(k)}_{i_1\dots i_n}(x_1) = \widetilde{\lambda}_{i_1\dots i_{k-1}} \cdot h'(f_{i_{k+1}\dots i_n}(x_1)) \cdot \lambda_{i_{k+1}\dots i_n} \tag{57}$$

and

$$b^{(n)}_{i_1\dots i_n}(x_1) = \widetilde{\lambda}_{i_1\dots i_{k-1}} \cdot h'(x_1). \tag{58}$$

Using this and the fact that for every $\ell$ and for every $(j_1,\dots,j_\ell)$ we have $f_{j_1\dots j_\ell}(u) \in C$, (51) implies

$$\forall \mathbf{i} \in \Sigma^* \quad b_{\mathbf{i}}(u) = 0. \tag{59}$$

It follows from (53) that, for $k < n$, we have

$$h'(f_{i_{k+1}\dots i_n}(z)) = \frac{c_2}{\log(n-k+2)} \lambda_{i_{k+1}\dots i_n}, \tag{60}$$

while, for $k = n$, we have $h'(z) = c_2/\log 2$. In this way, for $1 \leq k < n$, we have

$$b^{(k)}_{i_1\dots i_n}(z) = \widetilde{\lambda}_{i_1\dots i_{k-1}} \cdot \frac{c_2}{\log(n-k+2)} \cdot \left(\lambda_{i_{k+1}\dots i_n}\right)^2, \tag{61}$$

and for $k = n$ we have

$$b^{(n)}_{i_1\dots i_n}(z) = \widetilde{\lambda}_{i_1\dots i_{n-1}} \cdot \frac{c_2}{\log 2}. \tag{62}$$

Let $N := 2n+1$. Then

$$b^{(n+1)}_{i_1\dots i_N}(z) = \widetilde{\lambda}_{i_1\dots i_n} \cdot \frac{c_2}{\log(n+2)} \cdot \left(\lambda_{i_{n+2}\dots i_N}\right)^2. \tag{63}$$

Thus

$$\frac{b^{(n+1)}_{i_1\dots i_N}(z)}{\lambda_{i_1\dots i_N}} = \frac{c_2}{\log(n+2)} \frac{\widetilde{\lambda}_{i_1\dots i_n}\lambda_{i_{n+2}\dots i_N}}{\lambda_{i_1\dots i_{n+1}}}, \tag{64}$$

while

$$\frac{b^{(n+1)}_{i_1\dots i_N}(z)}{\widetilde{\lambda}_{i_1\dots i_N}} = \frac{c_2}{\log(n+2)} \frac{\left(\lambda_{i_{n+2}\dots i_N}\right)^2}{\widetilde{\lambda}_{i_{n+1}\dots i_N}}. \tag{65}$$

Let us choose

$$\mathbf{i} := (\underbrace{2,\dots,2}_{n}, \underbrace{1,\dots,1}_{n+1}), \quad |\mathbf{i}| = 2n+1 = N. \tag{66}$$

Then, using (43), for this particular choice of $\mathbf{i}$ we have

$$\frac{b^{(n+1)}_{\mathbf{i}}(z)}{\lambda_{\mathbf{i}}} = \frac{c_2}{\lambda_2 \log(n+2)} \cdot \left(\frac{\widetilde{\lambda}_2\lambda_1}{\lambda_2}\right)^n \to \infty \tag{67}$$

as $N \to \infty$. Similarly we have

$$\frac{b_{\mathbf{i}}^{(n+1)}(z)}{\widetilde{\lambda}_{\mathbf{i}}} = \frac{c_2}{\widetilde{\lambda}_1 \log(n+2)} \cdot \left(\frac{\lambda_1^2}{\widetilde{\lambda}_1}\right)^n \to \infty, \tag{68}$$

as $N \to \infty$.

Now for the $\mathbf{i}$ defined in (66) we use (19) with $s = 1$ to obtain

$$\alpha_1\left(\mathbf{i}, \left(z, \frac{1}{2}\right)\right) \geq \frac{1}{6} b_{\mathbf{i}}^{(n+1)}(z) = \widetilde{\lambda}_{i_1 \dots i_n} \cdot \frac{c_2}{6\log(n+2)} \cdot \left(\lambda_{i_{n+2}\dots i_N}\right)^2, \tag{69}$$

where we could have chosen any other $x_2 \in [0,1]$ instead of $1/2$. Using (59) it follows from (19) again that

$$\alpha_1(\mathbf{i}, (u, \frac{1}{2})) \leq 8 \max\left\{\lambda_{\mathbf{i}}, \widetilde{\lambda}_{\mathbf{i}}\right\}. \tag{70}$$

Now putting together (67), (68), (69) and (70) we obtain

$$\frac{\alpha_1(\mathbf{i}, (z, \frac{1}{2}))}{\alpha_1(\mathbf{i}, (u, \frac{1}{2}))} > \frac{\text{const}}{\log(n+2)} \left(\min\left\{\lambda_1 \widetilde{\lambda}_2 / \lambda_2, \lambda_1^2 / \widetilde{\lambda}_1\right\}\right)^n \to \infty, \tag{71}$$

as $N = 2n + 1$ tends to infinity. This completes the proof of Theorem 2. $\square$

**Remark 1.** *For any $\tau > 0$ we can use $\lambda_1$ etc as in (44) and see that this IFS is $1 + 3\tau$-bunched on a small neighborhood of the attractor. Falconer* [7] *proved that a 1-bunched IFS satisfies the bounded distortion property and has convex cylinders. Our example shows that these properties may not hold without the 1-bunched assumption.*

**Corollary 1.** *Here we use the $\mathbf{i} \in \Sigma^*$ defined in (66). Let $\mathbf{x_i} \in [0,1]^2$ denote a point at which the singular value function $\phi^s(\mathbf{i}, (x_1, x_2))$ (defined in (5)) attains its maximum on the cylinder $F_{\mathbf{i}}([0,1]^2)$. Then we have*

$$\frac{\alpha_2(\mathbf{i}, (u, \frac{1}{2}))}{\alpha_2(\mathbf{i}, \mathbf{x_i})} \to 0, \tag{72}$$

*as $N$ tends to infinity.*

*Proof.* $\alpha_1 \alpha_2$ is the determinant $\lambda_{\mathbf{i}} \widetilde{\lambda}_{\mathbf{i}}$, which is constant on $F_{\mathbf{i}}([0,1]^2)$. $\square$

## 6. Cylinders that cannot be covered by ellipses suggested by the singular values

In [2, Th. 3] Barreira claimed that the zero of the subadditive pressure is an upper bound for the upper box dimension of a non-conformal repeller $\Lambda$. His method of proof in §5.2 was to cover each $N$-cylinder by an ellipsoid with semiaxes given by the singular values at a point where $\phi^s$ takes its maximum. In this section we exhibit a cylinder for which this is *not* even possible using the singular values at any of its points.

Here we use the same $\mathbf{i}$ as above, defined in (66). Similarly we use the same maps $F_1, F_2$ as in the proof of Theorem 2. The idea is that $F_1$ contracts more strongly vertically than horizontally while, for $F_2$ it is the other way round.

The cylinder $\mathbf{i}$ is strongly distorted because of the non-linear term coming after weak horizontal contraction and before weak vertical contraction.

Since throughout the proof $\mathbf{i}$ depends only on $n$ and $\alpha_1\left(\mathbf{i},(x_1,x_2)\right)$, $\alpha_2\left(\mathbf{i},(x_1,x_2)\right)$ do not depend on $x_2\in[0,1]$, we will write for $k=1,2$

$$\alpha_k^{(n)}(x_1):=\alpha_k\left(\mathbf{i},(x_1,x_2)\right).$$

**Theorem 3.** *For every constant $c_{10}$ we can find $L$ such that for every $\mathbf{x}=(x_1,x_2)\in[0,1]^2$ and for every $n>L$ the cylinder $F_{\mathbf{i}}\left([0,1]^2\right)$ cannot be covered by any rectangle with sides $c_{10}\cdot\alpha_1^{(n)}(x_1)$ and $c_{10}\cdot\alpha_2^{(n)}(x_1)$.*

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive numbers. We say that

$$a_n\lll b_n \text{ if } \limsup_{n\to\infty}\frac{1}{n}\log a_n<\liminf_{n\to\infty}\frac{1}{n}\log b_n.$$

Similarly we write

$$a_n\approxeq b_n \text{ if } \lim_{n\to\infty}\frac{1}{n}\log a_n=\lim_{n\to\infty}\frac{1}{n}\log b_n.$$

A simple calculation shows that for every $1\le k\le 2n+1$ we have

$$\log(n+2)b_{\mathbf{i}}^{(n+1)}(z)\ge\widetilde{\lambda}_2\log(2n+1-k+2)b_{\mathbf{i}}^{(k)}(z). \tag{73}$$

Thus in the sum (56) no summand is larger asymptotically than $b_{\mathbf{i}}^{(n+1)}(z)$. Thus using (63) and (19) we obtain

$$\alpha_1^{(n)}(z)\approxeq b_{\mathbf{i}}^{(n+1)}(z)\approxeq\left(\lambda_1^2\widetilde{\lambda}_2\right)^n. \tag{74}$$

It follows from (35) and (55) that

$$\alpha_2^{(n)}(z)=\frac{\lambda_{\mathbf{i}}\widetilde{\lambda}_{\mathbf{i}}}{\alpha_1^{(n)}(z)}\approxeq\frac{\lambda_1^{n+1}\lambda_2^n\widetilde{\lambda}_1^{n+1}\widetilde{\lambda}_2^n}{\left(\lambda_1^2\widetilde{\lambda}_2\right)^n}\approxeq\left(\frac{\widetilde{\lambda}_1\lambda_2}{\lambda_1}\right)^n. \tag{75}$$

**Lemma 2.** *There exists a constant $C_3$ such that for every $n$ and every $0\le k\le 2n$ and for every $x\in(u,v)$ we have*

- **(a):** *If $\left|x-\frac{u+v}{2}\right|<\frac{|I|}{4}$ then*
$$\frac{h'(f_{\sigma^k\mathbf{i}}x)}{h'(f_{\sigma^k\mathbf{i}}z)}\in\left[\frac{1}{2},1\right].$$
- **(b):** *If $u<x<u+\frac{|I|}{4}$ or $v-\frac{|I|}{4}<x<v$ then*
$$\frac{h'(f_{\sigma^k\mathbf{i}}x)}{h'(f_{\sigma^k\mathbf{i}}z)}=C_3\cdot(x-u)^2 \text{ or } \frac{h'(f_{\sigma^k\mathbf{i}}x)}{h'(f_{\sigma^k\mathbf{i}}z)}=C_3\cdot(v-x)^2,$$
*respectively.*

*Proof.* The proof follows immediately from (47) and (49). $\square$

Using this, (57), (74) and (75) we see that

$$\forall x \in (u,v),\ \alpha_1^{(n)}(x) \approxeq \left(\lambda_1^2\widetilde{\lambda}_2\right)^n \text{ and } \alpha_2^{(n)}(x) \approxeq \left(\frac{\widetilde{\lambda}_1\lambda_2}{\lambda_1}\right)^n. \tag{76}$$

Further, putting together Lemma 2, part (b) and (53) we obtain that for $0 \le k \le 2n+1$ we have

$$u \le x \le u + \frac{|I|}{4} \Rightarrow h'\left(f_{\sigma^k\mathbf{i}}(x)\right) = c_2C_3 \cdot (x-u)^2 \cdot \frac{\lambda_{\sigma^k\mathbf{i}}}{\log(2n+3-k)} \tag{77}$$

and

$$v - \frac{|I|}{4} \le x \le v \Rightarrow h'\left(f_{\sigma^k\mathbf{i}}(x)\right) = c_2C_3 \cdot (v-x)^2 \cdot \frac{\lambda_{\sigma^k\mathbf{i}}}{\log(2n+3-k)}. \tag{78}$$

Put $c_4 := c_2C_3$. Then this, (12), (57) and (73) imply that if $u \le x \le u + \frac{|I|}{4}$ then

$$\frac{c_4}{\log(n+2)}(x-u)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} \le b_{\mathbf{i}}(x) \le (2n+1)c_4(x-u)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{79}$$

Further, if $v - \frac{|I|}{4} \le x \le v$ then

$$\frac{c_4}{\log(n+2)}(v-x)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} \le b_{\mathbf{i}}(x) \le (2n+1)c_4(v-x)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{80}$$

Finally for $|x-z| < \frac{|I|}{4}$ we have

$$\frac{c_2}{2\log(n+2)}\widetilde{\lambda}_2^n \cdot \lambda_1^{2n} \le b_{\mathbf{i}}(x) \le (2n+1)c_2\widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{81}$$

**Lemma 3.** *For* $(x_1,x_2) \in [0,1]^2$ *we have*

$$F_{\mathbf{i}}(x_1,x_2) = \left[f_2^n \circ f_1^{n+1}(x_1), p_n(x_1) + \widetilde{\lambda}_1^{n+1}\widetilde{\lambda}_2^n x_2 + a_n\right], \tag{82}$$

*where* $a_n := (1-h(1)-\widetilde{\lambda}_2)(\widetilde{\lambda}_2^{n-1} + \dots + \widetilde{\lambda}_2 + 1)$ *and for* $x \in [0,1]$

$$p_n(x) := \sum_{\ell=0}^{2n} \widetilde{\lambda}_{\mathbf{i}|\ell} \cdot h(f_{\sigma^{\ell+1}\mathbf{i}}(x)), \tag{83}$$

*where* $\mathbf{i}|\ell := (i_1,\dots,i_\ell)$.

*Proof.* The statement follows by induction from the definitions of $\mathbf{i}, F_1, F_2$. In fact we can write

$$p_n(x) := \sum_{k=0}^{n-1} q_n^{(k)}(x) + \sum_{\ell=0}^{n} r_n^{(\ell)}(x),$$

where, for $0 \le k \le n-1$,

$$q_n^{(k)}(x) := \widetilde{\lambda}_2^k \cdot h\left(f_2^{n-1-k}\left(f_1^{n+1}(x)\right)\right) = \widetilde{\lambda}_2^k \cdot h\left(f_{\sigma^{k+1}\mathbf{i}}(x)\right)$$

and, for $0 \le \ell \le n$,

$$r_n^{(\ell)}(x) := \widetilde{\lambda}_2^n \cdot \widetilde{\lambda}_1^\ell \cdot h\left(f_1^{n-\ell}(x)\right) = \widetilde{\lambda}_2^n \cdot \widetilde{\lambda}_1^\ell \cdot h\left(f_{\sigma^{n+\ell+1}\mathbf{i}}(x)\right).$$

□

Observe that it follows from (77) that for $u \leq x \leq x + \frac{|I|}{4}$

$$p_n'(x) = \sum_{\ell=0}^{2n} \frac{c_2 C_3}{\log(2n+2-\ell)} \cdot (x-u)^2 \cdot \widetilde{\lambda}_{\mathbf{i}|\ell} \cdot (\lambda_{\sigma^{\ell+1}\mathbf{i}})^2 . \tag{84}$$

An easy calculation yields $\widetilde{\lambda}_{\mathbf{i}|\ell} \cdot (\lambda_{\sigma^{\ell+1}\mathbf{i}})^2 \leq \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} \cdot \widetilde{\lambda}_2^{-2}$. Thus with $c_4 := c_2 C_3$ and $c_5 := 2 \frac{c_2 C_3}{\widetilde{\lambda}_2^2 \log 3}$ we obtain that, for $u \leq x \leq u + \frac{|I|}{4}$,

$$\frac{c_4}{\log(n+2)} \cdot (x-u)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} < p_n'(x) < c_5 \cdot n \cdot (x-u)^2 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{85}$$

Thus we obtain that, for $u \leq x \leq u + \frac{|I|}{4}$,

$$p_n(x) \leq p_n(u) + \frac{c_5}{3} \cdot n \cdot (x-u)^3 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{86}$$

Similarly if $\left| x - \frac{u+v}{2} \right| < \frac{|I|}{4}$ then by Lemma 2 part (b) we have

$$\frac{1}{2} \cdot \frac{c_4}{\log(n+2)} \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} < p_n'(x) < c_5 \cdot n \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}. \tag{87}$$

Now we can prove that

**Lemma 4.** *There exist $c_6$ and $c_7$ such that*

**(a):** $|p_n(v - \frac{|I|}{4}) - p_n(u + \frac{|I|}{4})| \geq \frac{c_6}{\log(n+2)} \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n}$.

**(b):** *For $u < x < u + \frac{c_7}{\sqrt[3]{n}} \cdot \left( \frac{\widetilde{\lambda}_1}{\lambda_1^2} \right)^{n/3}$ we have*

$$p_n(u) < p_n(x) < p_n(u) + \frac{1}{2} \widetilde{\lambda}_1^{n+1} \widetilde{\lambda}_2^n. \tag{88}$$

*Proof. Part (a)* immediately follows from the left hand side of the inequality (87).

Now we prove *Part (b).* We know that the function $p_n(x)$ is monotone increasing. It follows from a straightforward calculation that for $c_7 := \sqrt[3]{\frac{3\widetilde{\lambda}_1}{2c_5}}$ and $x = u + \frac{c_7}{\sqrt[3]{n}} \cdot \left( \frac{\widetilde{\lambda}_1}{\lambda_1^2} \right)^{n/3}$ we have

$$\frac{c_5}{3} \cdot n \cdot (x-u)^3 \cdot \widetilde{\lambda}_2^n \cdot \lambda_1^{2n} = \frac{1}{2} \cdot \widetilde{\lambda}_1^{n+1} \cdot \widetilde{\lambda}_2^n.$$

It follows from (43) that the assumption of (86) holds which immediately implies the assertion of Part (b). □

Observe that it follows from (43) that $\lambda_1^{1/3} \lambda_2 \widetilde{\lambda}_1^{1/3} > \frac{\widetilde{\lambda}_1 \lambda_2}{\lambda_1}$ and $\widetilde{\lambda}_1 \widetilde{\lambda}_2 > \frac{\widetilde{\lambda}_1 \lambda_2}{\lambda_1}$. Choose $\delta$ such that

$$\frac{\widetilde{\lambda}_1 \lambda_2}{\lambda_1} < \delta < \min \left\{ \lambda_1^{1/3} \lambda_2 \widetilde{\lambda}_1^{1/3}, \widetilde{\lambda}_1 \widetilde{\lambda}_2 \right\}. \tag{89}$$

Then we obtain from (76) that for all $u < x < v$

$$\alpha_2^{(n)}(x) \lll \delta^n \lll \left(\lambda_1^{1/3}\lambda_2\widetilde{\lambda}_1^{1/3}\right)^n. \tag{90}$$

**Lemma 5.** *Put* $H := \left[u, u + \frac{c_7}{\sqrt[3]{n}} \cdot \left(\frac{\widetilde{\lambda}_1}{\lambda_1^2}\right)^{n/3}\right]$. *Then for all $n$ big enough there exists a ball of diameter $\delta^n$ contained in* $F_{\mathbf{i}}\left(H \times \left[\frac{1}{2}, 1\right]\right)$.

*Proof.* It follows from (82) and Lemma 4 Part (b) that the rectangle

$$\left[u_{\mathbf{i}}, u_{\mathbf{i}} + \frac{c_7}{\sqrt[3]{n}} \cdot \left(\frac{\widetilde{\lambda}_1}{\lambda_1^2}\right)^{n/3} \cdot \lambda_1^{n+1} \cdot \lambda_2^n\right] \times \left[p_n(u) + a_n + \frac{1}{2}\widetilde{\lambda}_1^{n+1}\widetilde{\lambda}_2^n, p_n(u) + a_n + \widetilde{\lambda}_1^{n+1}\widetilde{\lambda}_2^n\right]$$

is contained in $F_{\mathbf{i}}\left[H \times [0,1]\right]$. Since we assumed that $\left(\frac{\widetilde{\lambda}_1}{\lambda_1^2}\right)^{1/3}\lambda_1\lambda_2 = \lambda_1^{1/3}\lambda_2\widetilde{\lambda}_1^{1/3} > \delta$ and $\widetilde{\lambda}_1\widetilde{\lambda}_2 > \delta$ this completes the proof of the Lemma. □

*Proof of Theorem 3.* Let $L$ be so big that

$$n > L \Rightarrow \lambda_1\widetilde{\lambda}_1 \frac{c_{10}\log(n+2)}{c_6}\left(\frac{\widetilde{\lambda}_1\lambda_2}{\lambda_1}\right)^n < \delta^n. \tag{91}$$

To get a contradiction we assume that there exists $n > L$, $\mathbf{x} = (x_1, x_2) \in [0,1]^2$ and a rectangle $R$ with sides $c_{10}\alpha_k^{(n)}(x_1)$ $k = 1, 2$ which covers $F_{\mathbf{i}}([0,1]^2)$. The matrix $D_{\mathbf{x}}F_{\mathbf{i}}$ is a triangular matrix with diagonal entries $\lambda_{\mathbf{i}}$ and $\widetilde{\lambda}_{\mathbf{i}}$ and so

$$\alpha_1^{(n)}(x_1) \cdot \alpha_2^{(n)}(x_1) = \lambda_{\mathbf{i}}\widetilde{\lambda}_{\mathbf{i}}. \tag{92}$$

It follows from Lemma 4 (a) that the vertical size of $F_{\mathbf{i}}([0,1]^2)$ is greater than $\frac{c_6}{\log(n+2)}\widetilde{\lambda}_2^n\lambda_1^{2n}$. Thus $\alpha_1^{(n)}(x_1) > \frac{c_6}{\log(n+2)}\widetilde{\lambda}_2^n\lambda_1^{2n}$. This and (92) implies that

$$c_{10}\alpha_2^{(n)}(x_1) < \lambda_1\widetilde{\lambda}_1 \frac{c_{10}\log(n+2)}{c_6}\left(\frac{\widetilde{\lambda}_1\lambda_2}{\lambda_1}\right)^n < \delta^n. \tag{93}$$

However, according to Lemma 5 there is a ball with radius $\delta^n$ contained in $F_{\mathbf{i}}([0,1]^2)$. This contradiction completes the proof of Theorem 3. □

## 7. Elliptic Box dimension

For $E \subset \mathbb{R}^d$ and $r > 0$ we denote the minimal number of balls of radius $r$ required to cover the set $E$ by $N_r(E)$. In [5] Douady and Oesterlé write

$$\xi_t(E) := a_1 \cdots a_k a_{k+1}^{t-k}$$

for an ellipsoid $E \subset \mathbb{R}^d$ with semi-axes $a_1 \geq \cdots \geq a_d > 0$ where $k < t \leq k+1$ and show that the associated elliptic Hausdorff dimension is the same as the usual Hausdorff dimension.

Following [2] define $N_{r,t}(A)$ as the minimal number of ellipsoids $E$ satisfying $\xi_t(E)^{1/t} = r$ required to cover the set $A$. If $E$ is a ball of radius

$r$ then $\xi_t(E) = r$ so $N_{r,t}(A) \le N_r(A)$. In [2, Theorem 11] Barreira claims that there exists a constant $c = c(t)$ depending only on $t$ such that for every $A \subset \mathbb{R}^d$ and $t \in [0, d]$ we have

$$\frac{N_r(A)}{N_{r,t}(A)} < c. \tag{94}$$

Here by constructing a counter example we prove that a much weaker statement is false.

**Example 1** (Elliptic Box dimension)**.** *Let $H \subset [0,1]$ be a compact set such that for $s = \underline{\dim}_{\mathrm{B}} H$ we have $0 < s < 1$. Put $Z := [0,1] \times H \subset \mathbb{R}^2$. Then it is easy to see that*

$$\underline{\dim}_{\mathrm{B}} Z = 1 + s. \tag{95}$$

*If $1 + s < t \le 2$ then*

$$\limsup_{r \to 0} \frac{N_r(Z)}{N_{r,t}(Z)} = \infty. \tag{96}$$

*Proof.* Using

$$\frac{t}{t-1} < \frac{1+s}{s}, \tag{97}$$

we can choose $1 + s < t' < t$ and we can also choose $\varepsilon > 0$ such that

$$\frac{t'-1}{t-1} \cdot t < 1 + s - \varepsilon. \tag{98}$$

Using (95), for every $r > 0$ small enough we have

$$N_r(Z) \ge r^{-1-s+\varepsilon} = \left(\frac{1}{r}\right)^{1+s-\varepsilon}. \tag{99}$$

Since $t' - 1 > s = \underline{\dim}_{\mathrm{B}} H$ there exists a sequence $\rho_n \downarrow 0$ such that for every $n$ the set $H$ can be covered by

$$M_n = \rho_n^{-(t'-1)}$$

intervals of length $\rho_n$, $I_1^{(n)}, \ldots, I_{M_n}^{(n)}$. Let $r_n := \rho_n^{(t-1)/t}$. Each rectangle

$$J_k^{(n)} := [0,1] \times I_k^{(n)}, \quad k = 1, \ldots, M_n$$

is contained in an ellipse $E_k^{(n)}$ having the same center and semi-axes 1 and $r_n^{t/(t-1)}$ and so

$$\xi_t(E_k^{(n)}) = r_n^t.$$

Therefore

$$N_{r_n,t}(Z) \le M_n. \tag{100}$$

Now (98), (99) and (100) imply that

$$\frac{N_{r_n}(Z)}{N_{r_n,t}(Z)} \ge r_n^{-1-s+\varepsilon+t(t'-1)/(t-1)} \to \infty, \quad \text{as } n \to \infty. \tag{101}$$

□

Barreira also asserts in [2, Theorem 11] that

$$\underline{\dim}_{\mathrm{B}} Z = \liminf_{r\to 0} \frac{\log N_{r,t}(Z)}{-\log r} \tag{102}$$

for any $Z \subset \mathbb{R}^d$ and any $t \in [0,d]$. (There is a mistake in the proof in that the sentence before (A3) may hold only for $\delta$ near 1.) We finish our paper with the following counterexample.

**Example 2.** *Assume for $Z \subset \mathbb{R}^2$ in Example 1 that*

$$\underline{\dim}_{\mathrm{B}} Z = \overline{\dim}_{\mathrm{B}} Z. \tag{103}$$

*Then for every $\dim_{\mathrm{B}} Z < t \le 2$ we have*

$$\liminf_{r\to 0} \frac{\log N_{r,t}(Z)}{-\log r} < \dim_{\mathrm{B}} Z. \tag{104}$$

*Proof.* Using the notation of Example 1 above we write

$$q := -1 - s + \varepsilon + t(t'-1)/(t-1) < 0.$$

It follows from (101) that

$$N_{r_n}(Z) \ge r_n^q \cdot N_{r_n,t}(Z). \tag{105}$$

Thus

$$\begin{aligned} \dim_{\mathrm{B}} Z &= \lim_{n\to\infty} \frac{\log N_{r_n}(Z)}{-\log r_n} \ge -q + \liminf_{n\to\infty} \frac{\log N_{r_n,t}(Z)}{-\log r_n} \\ &> \liminf_{r\to 0} \frac{\log N_{r,t}(Z)}{-\log r}. \end{aligned}$$

□

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