THE LEBESGUE MEASURE OF THE ALGEBRAIC DIFFERENCE OF TWO RANDOM CANTOR SETS

PÉTER MÓRA, KÁROLY SIMON, AND BORIS SOLOMYAK

Abstract. In this paper we consider a family of random Cantor sets on the line. We give some sufficient conditions when the Lebesgue measure of the arithmetic difference is positive. Combining this with the main result of a recent joint paper of the second author with M. Dekking we construct random Cantor sets F_1, F_2 such that the arithmetic difference set $F_2 - F_1$ does not contain any intervals but $\mathcal{L}\mathrm{eb}(F_2 - F_1) > 0$ almost surely, conditioned on non-extinction.

1. Introduction

This note is a continuation of a joint work of the second author with M. Dekking [4]. Both papers deal with a random version of the following problem asked by J. Palis related to the arithmetic difference

$$F_2 - F_1 = \{y - x : x \in F_1, y \in F_2\}$$

of the dynamically defined Cantor sets $F_1, F_2 \subset \mathbb{R}$.

Conjecture 1 (Palis). *"Typically" either the set $F_2 - F_1$ is "small" in the sense that $\mathcal{L}\mathrm{eb}(F_2 - F_1) = 0$ or $F_2 - F_1$ is a "big" set in the sense that $F_2 - F_1$ contains some intervals.*

In this paper we show (Corollary 1) that within a natural family of self-similar random Cantor sets it can happen that $F_2 - F_1$ has positive Lebesgue measure but contains no intervals almost surely.

In [2] T.A. Moreira and J.C. Yoccoz answered Palis' problem positively for "typical" non-linear deterministic $\mathcal{C}^2$ Cantors sets on the line. However the problem is still open for linear Cantor sets.
The authors of [4] considered a natural family of random Cantor sets and they gave a condition (see Theorem 1(a)) under which $F_2 - F_1$ contains some intervals (conditional on $F_1, F_2 \neq \emptyset$). On the other hand, the authors of [4] also gave a condition (see Theorem 1(b)) which implies that $int(F_2 - F_1) = \emptyset$. Continuing this line of research in this paper we consider the same

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family of random Cantor sets and we give a condition which implies that the arithmetic difference set $F_2 - F_1$ has positive Lebesgue measure. Using a combination of these two results, we construct some families of random Cantor sets for which the Palis conjecture above fails.

2. Results

Our main result is about the Lebesgue measure of the set $F_2 - F_1$, where F_1, F_2 are independent copies of the random Cantor sets constructed below. We have analogous results for the $F - F$ type random Cantor sets and in the deterministic cases.

2.1. **Preliminaries.** We use the same definition of the random Cantor set as in [4, p. 206]. For the convenience of the reader here we sketch the idea of the construction. We are given a natural number $M \geq 2$ and a vector $\mathbf{p} = (p_0, \dots, p_{M-1}) \in [0,1]^M$ which is not a probability vector in general. In the first step of the construction we partition the unit interval $I = [0,1]$ into M equal sub intervals $I_0, \dots, I_{M-1}$. We choose interval $I_k = \left[\frac{k}{M}, \frac{k+1}{M}\right]$ with probability p_k independently for each $k = 0, \dots, M-1$. The first approximation F^1 of our random Cantor set is the union of the intervals chosen in the first step. In the second step for all of the intervals I_k which were chosen in the first step we repeat the same process for I_k instead of I independently. So, the level 2 interval

$$I_{k_1 k_2} := \left[\frac{k_1}{M} + \frac{k_2}{M^2}, \frac{k_1}{M} + \frac{k_2}{M^2} + \frac{1}{M^2}\right]$$

can be chosen in the second step of the construction only if we selected I_{k_1} in the first step. In this case the conditional probability that we select $I_{k_1 k_2}$ conditioned on the event that I_{k_1} was selected is equal to p_{k_2}. All selections made are independent of everything. The union of all of these randomly selected intervals $I_{k_1 k_2}$ is denoted by F^2 and is called the level 2 approximation of our random Cantor set. We continue this process in the same way to define the level n approximation F^n as a union of randomly selected level n intervals of the form

$$I_{\underline{k}_n} := \left[k_1 \cdot M^{-1} + \dots + k_n \cdot M^{-n}, k_1 \cdot M^{-1} + \dots + k_n \cdot M^{-n} + M^{-n}\right],$$

where $\underline{k}_n = (k_1, \dots, k_n)$. Then the random Cantor set F is defined by

$$F := \bigcap_{n=1}^{\infty} F^n.$$

In this paper, if we do not say otherwise, we always consider the arithmetic difference of two independent copies F_1, F_2 of this random Cantor set. As above, the level n approximation of F_1, F_2 is denoted by F_1^n, F_2^n respectively.

For the precise definition of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of pairs of independent random Cantor sets see [4, p. 206]. It is well known (see e.g. [4, Fact 2]) that

$$F_2 - F_1 = Proj_{45^\circ}(F_1 \times F_2).$$

It follows (cf. [4, p. 207]) that whenever $\sum_{k=0}^{M-1} p_k < \sqrt{M}$ the difference of the Cantor sets $F_2 - F_1$ has Hausdorff dimension smaller than 1.
Now we can define the *cyclic autocorrelations* γ_k by

$$\gamma_k := \sum_{j=0}^{M-1} p_j p_{j+k \ (\bmod M)} \quad \text{for} \quad k = 0, \dots, M.$$

Theorem 1 (Dekking, Simon [4]). *Conditional on $F_1, F_2 \neq \emptyset$, we have*

(a): *If $\gamma_k > 1$ for all k, then $F_2 - F_1$ contains an interval almost surely.*

(b): *If there exists $k \in \{0, \dots, M-1\}$ such that γ_k and γ_{k+1} are both less than 1, then $F_2 - F_1$ almost surely does not contain any intervals.*

2.2. **The main result.** To state our main result we introduce

$$u_k := \begin{cases} p_0 p_k + \dots + p_{M-k-1} p_{M-1}, & \text{if } 0 \leq k < M; \\ 0, & \text{if } k = M. \end{cases} \tag{2.1}$$

note that $\gamma_k = u_k + u_{M-k}$.

Theorem 2. *We assume that*

(**A1**) $$\Gamma := \gamma_0 \cdots \gamma_{M-1} > 1,$$

(**A2**) *and for every $0 \leq k \leq M-1$ we have*

$$\min(u_k, u_{k+1}) > 0 \quad or \quad \min(u_{M-k}, u_{M-k-1}) > 0.$$

Then conditional on $F_1, F_2 \neq \emptyset$,

$$\mathcal{L}\mathrm{eb}(F_2 - F_1) > 0$$

holds almost surely.

Remark 1. *The second assumption of our theorem is rather technical. However, it always holds whenever all the probabilities $p_0, \dots p_{M-1}$ are positive.*

Remark 2. *Our result is close to be sharp. Namely, our theorem asserts that if the geometric mean of the γ_i's is greater than 1 and* (**A2**) *holds, then the difference set $F_2 - F_1$ has positive Lebesgue measure. On the other hand, as it was noted in* [4, p. 215], *the algebraic mean of γ_i's is less than 1, then* $\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 < 1$ *so,* $\mathcal{L}\mathrm{eb}(F_2 - F_1) = 0$.

Remark 3. *Dekking and Grimmett investigated a related problem in* [3]. *Namely, they considered a higher dimensional random Cantor set and studied the Lebesgue measure of its orthogonal projections. They worked with the generated branching process in random and varying environment. From this respect we use the same method, however, in our case we use a 45° projection which implies that we have two different types of individuals (the* *left triangles and right triangles see Figure 3) and we need to take care of the independence of their line of inheritance. This is one of the reasons that the implementation of the method introduced in* [3] *becomes much more complicated in our proof.*

It follows from the main result of [4] and our theorem together that the Palis Conjecture (Problem 1) mentioned above does not hold in our case.

Corollary 1. *Let* $M = 3$ *and*

$$(p_0, p_1, p_2) = (0.52, 0.5, 0.72).$$

In this case we have

$$\gamma_0 = p_0^2 + p_1^2 + p_2^2 = 1.0388, \quad \gamma_1 = \gamma_2 = p_0p_1 + p_1p_2 + p_2p_0 = 0.9944,$$

*This implies that the difference of random Cantor sets almost surely contains no interval (by Theorem 1 part **(a)**). On the other hand, the product*

$$\gamma_0\gamma_1\gamma_2 = 1.0272$$

is greater than 1*. Thus it follows from the main result of the paper that this difference of random Cantor sets almost surely has positive Lebesgue measure, conditioned on non-extinction.*

Remark 4. *Conditioned on* $F \neq \emptyset$*, we have (see* [5], [6]*)*

$$\dim_{\mathrm{H}} F = \log\left(\sum_{i=0}^{M-1} p_i\right) / \log M$$

almost surely. The condition (**A1**) *implies that* $\sum_{i=0}^{M-1} p_i > \sqrt{M}$*, thus conditioned on non-extinction* $\dim_{\mathrm{H}} F > 1/2$ *almost surely.*

Remark 5 (The deterministic case)**.** *We use the same construction as before but we assume that all the probabilities* p_i *are either zero or one. The Cantor set obtained in this way is denoted by* F*. This situation was settled (essentially completely) in* [4, Section 8]*. However, it was not remarked there that the proof of* [4, Theorem 2] *implies that the Palis conjecture holds in this case. That is either* $\mathcal{L}\mathrm{eb}(F - F) = 0$ *or* $F - F$ *contains an interval.*

2.3. The case of $F - F$ type random Cantor sets.

Theorem 3. *If both conditions* (**A1**) *and* (**A2**) *of Theorem 2 hold, then conditional on* $F_1 \neq \emptyset$*, we have*

$$\mathcal{L}\mathrm{eb}(F_1 - F_1) > 0$$

almost surely.

This result is a consequence of Theorem 2. We prove it in Section 6.

2.4. **A generalization.** Here we consider the same problem as in Theorem 2 but we assume that the random Cantor sets are constructed with different probabilities: $\mathbf{p} = (p_0, \dots, p_{M-1})$ and $\mathbf{q} = (q_0, \dots, q_{M-1})$. That is, the probability that $I_{i_1 \dots i_k}$ is selected given that $I_{i_1 \dots i_{k-1}}$ was selected, is equal to p_{i_k} for F_1 and (independently) q_{i_k} for F_2. Following the notation of [4, Section 4.4] let

$$\widetilde{\gamma}_k := \sum_{j=0}^{M-1} q_j p_{j+k \ (\bmod M)}.$$

Then the conclusion of Theorem 2 remains valid under the following assumptions:

Theorem 4. *Let F_1, F_2 be independent random Cantor sets constructed as above. We assume that the following hold:*

$(\widetilde{\mathbf{A1}})$ $$\widetilde{\Gamma} := \widetilde{\gamma}_0 \cdots \widetilde{\gamma}_{M-1} > 1,$$

$(\widetilde{\mathbf{A2}})$ *for every $0 \leq k \leq M-1$ we have $0 < p_k$ and $0 < q_k$.*
Then conditional on $F_1, F_2 \neq \emptyset$,

$$\mathcal{L}\mathrm{eb}(F_2 - F_1) > 0$$

holds almost surely.

The proof of this theorem is the same as the proof of Theorem 2 with obvious modifications.

The paper is organized as follows: For the convenience of the reader in Section 3 we repeat the notation of [4]. In Section 4 and 5 we prove our Main result. In the last section we prove our results about the $F - F$ type random Cantor sets.

3. Notation

We can visualize the difference of two points x and y on the line as follows: Take the point $A = (x, y) \in \mathbb{R}^2$. Then $y - x$ is the $45°$ projection of A to the y-axis. Let us denote the $45°$ projection to the line $\{(x, y) : x + y = 1\}$ by π. (See Figure 1.) That is,

$$\pi(x, y) := \left(\frac{1 - (y - x)}{2}, \frac{1 + (y - x)}{2} \right).$$

Then we have

$$\mathcal{L}\mathrm{eb}_1(F_2 - F_1) = \frac{2}{\sqrt{2}} \mathcal{L}\mathrm{eb}_1(\pi(F_1 \times F_2)).$$

Therefore, to decide if $F_1 - F_2$ is a set of positive Lebesgue measure it is enough to consider the same problem for the set $\pi(F_1 \times F_2)$. Since it is more convenient to study the $90°$ projection to the first axis than $45°$ projection, we rotate the square $[0, 1]^2$ in the positive direction and we move it in such

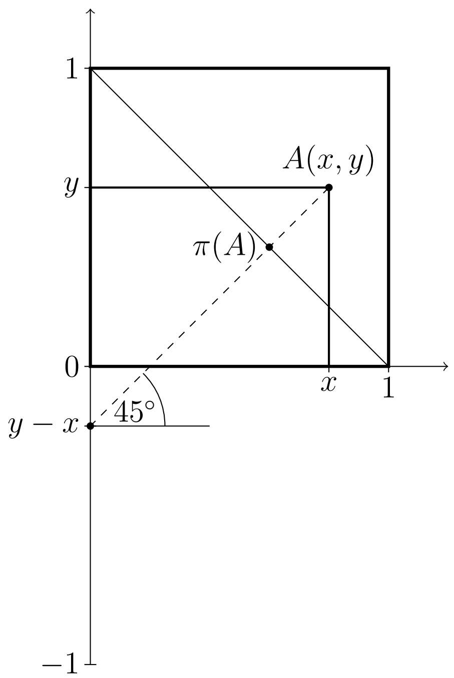

FIGURE 1. The definition of π.

a way that its center lies in the origin of the coordinate axis. Let us call this transformation φ. We call the rotated square $\widetilde{Q} = \varphi[0,1]^2$. (See Figure 2.)

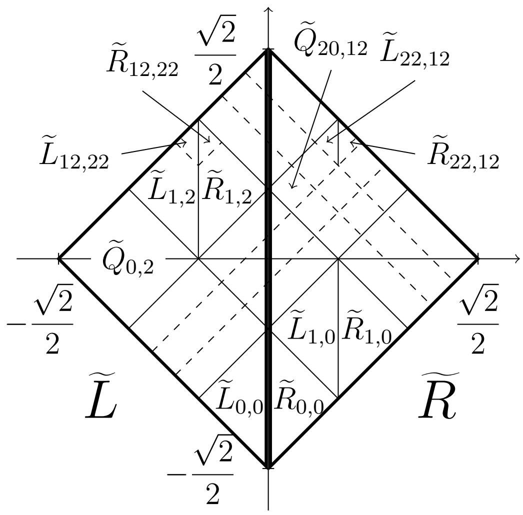

FIGURE 2. The definition of the $\widetilde{Q}$ and the left and right triangles.

The y-axis divides $\widetilde{Q}$ into two triangles: $\widetilde{L}$ and $\widetilde{R}$. Similarly, put $\widetilde{Q}_{\ell,k} := \varphi(I_\ell \times I_k)$. Then the vertical diagonal divides $\widetilde{Q}_{\ell,k}$ into the triangles $\widetilde{L}_{\ell,k}$ and $\widetilde{R}_{\ell,k}$. (See Figure 2.)

Now we introduce the transformation $\psi : \widetilde{Q} \to \mathbb{R}^2$ as follows: $\psi|_{\widetilde{R}} :=$ identity. Further, ψ moves the left half $\widetilde{L}$ exactly to the "top" of $\widetilde{R}$ (as shown in Figure 3) so that the image $L := \psi(\widetilde{L})$ has the same projection to x axis as $R := \widetilde{R}$ and they are adjacent to each other.
That is, $\mathrm{Proj}(L) = \mathrm{Proj}(R)$, where we write Proj for the $90°$ projection to the x-axis.

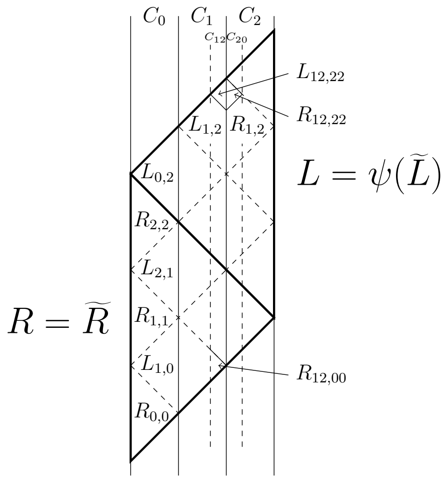

FIGURE 3. The vertical columns and higher level triangles

Put

$$\Lambda := \psi(\varphi(F_1 \times F_2)), \text{ and } \Lambda^n := \psi(\varphi(F_1^n \times F_2^n)).$$

We call L and R level 0 triangles. The collections of the triangles

$$\{L_{k,\ell} : 0 \le k, \ell \le M-1\}, \qquad \{R_{k,\ell} : 0 \le k, \ell \le M-1\}$$

are called the level 1 left triangles and level 1 right triangles respectively. The vertical sides of the level 1 left and right triangles naturally define M vertical columns. Namely, we partition the interval $[0, \frac{\sqrt{2}}{2}]$ into $J(k) := \frac{\sqrt{2}}{2} \cdot \left[\frac{k}{M}, \frac{k+1}{M}\right]$, $0 \le k \le M-1$ and we define the k-th level 1 column

$$C(k) := \{(x, y) : x \in J(k)\}.$$

Analogously, for every $n > 1$ the n-th iterate of the system naturally defines the level n left and right triangles and level n columns. Namely, for every $n \ge 1$ and for every $\underline{k}_n := (k_1, \dots, k_n) \in \{0, \dots, M-1\}^n$ first we define the interval

$$J(\underline{k}_n) := \frac{\sqrt{2}}{2} \cdot \left[k_1 \cdot M^{-1} + \cdots + k_n \cdot M^{-n}, k_1 \cdot M^{-1} + \cdots + k_n \cdot M^{-n} + M^{-n}\right].$$

Now the level n column corresponding to $\underline{k}_n := (k_1, \dots, k_n) \in \{0, \dots, M-1\}^n$ is defined as

$$C(\underline{k}_n) := \{(x, y) : x \in J(\underline{k}_n)\}.$$

It follows naturally from the definition of the level n approximations F_1^n, F_2^n of our random Cantor sets F_1, F_2 that we have to divide $[0,1]^2$ into level n squares of the form $I_{\underline{k}_n} \times I_{\underline{\ell}_n}$. The corresponding level n squares of $\widetilde{Q}$ are

$$\widetilde{Q}_{\underline{k}_n,\underline{\ell}_n} := \varphi(I_{\underline{k}_n} \times I_{\underline{\ell}_n}), \text{ where } \underline{k}_n, \underline{\ell}_n \in \{0,\dots,M-1\}^n .$$

The rotated square $\widetilde{Q}_{\underline{k}_n,\underline{\ell}_n}$ is divided by its vertical diagonal into the triangles $\widetilde{L}_{\underline{k}_n,\underline{\ell}_n}, \widetilde{R}_{\underline{k}_n,\underline{\ell}_n}$. We obtain the level n left and right triangles as

$$L_{\underline{k}_n,\underline{\ell}_n} := \psi\left(\widetilde{L}_{\underline{k}_n,\underline{\ell}_n}\right), R_{\underline{k}_n,\underline{\ell}_n} := \psi\left(\widetilde{R}_{\underline{k}_n,\underline{\ell}_n}\right),$$

where $\underline{k}_n, \underline{\ell}_n \in \{0,\dots,M-1\}^n$. When we want to state assertions which are valid for both $L_{\underline{k}_n,\underline{\ell}_n}$ and $R_{\underline{k}_n,\underline{\ell}_n}$ then we use the following notation: we write $V \in \{L,R\}$ and define $V_{\underline{k}_n,\underline{\ell}_n}$ as $L_{\underline{k}_n,\underline{\ell}_n}$ if $V = L$ and $R_{\underline{k}_n,\underline{\ell}_n}$ otherwise. From the geometry of the construction it is immediate that the following fact holds:

Fact 1. *Let $V \in \{L,R\}$. We pick some level n V-triangles $V_{\underline{i}_n^1,\underline{j}_n^1},\dots,V_{\underline{i}_n^\ell,\underline{j}_n^\ell}$ from the level n approximation Λ^n of Λ which are in the same column $C(\underline{k}_n)$. That is,*

$$V_{\underline{i}_n^1,\underline{j}_n^1},\dots,V_{\underline{i}_n^\ell,\underline{j}_n^\ell} \subset C(\underline{k}_n) \cap \Lambda^n.$$

Then the random Cantor sets

$$\left\{V_{\underline{i}_n^m,\underline{j}_n^m} \cap \Lambda\right\}_{m=1}^{\ell}$$

are independent.

For $\underline{k}_n := (k_1,\dots,k_n) \in \{0,\dots M-1\}^n$ and for $U, V \in \{L,R\}$ we define random variable $Z^{UV}(\underline{k}_n)$ as the number of the level n V-triangles in the intersection of the level 0 ("big") U-triangle with the column $C(\underline{k}_n) \cap \Lambda^n$. (See Figure 4.)
As in [4], the mean matrices are

$$\mathcal{M}(\underline{k}_n) := \begin{bmatrix} \mathbb{E}Z^{LL}(\underline{k}_n) & \mathbb{E}Z^{LR}(\underline{k}_n) \\ \mathbb{E}Z^{RL}(\underline{k}_n) & \mathbb{E}Z^{RR}(\underline{k}_n) \end{bmatrix}.$$

Then from the definition one can easily check that

$$\mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1)\cdots\mathcal{M}(k_n).$$

Further, an immediate calculation yields that

$$\mathcal{M}(k) = \begin{bmatrix} u_{M-1-k} & u_{M-k} \\ u_{k+1} & u_k \end{bmatrix}, \tag{3.1}$$

where u_k was introduced in (2.1). The first (second) column sum of $\mathcal{M}(k)$ shows the expected number of the left (right) level 1 triangles in the column $C(k)$ respectively. They can be expressed as

$$\mathbb{E}Z^{LL}(k) + \mathbb{E}Z^{RL}(k) = \gamma_{k+1}, \qquad \mathbb{E}Z^{LR}(k) + \mathbb{E}Z^{RR}(k) = \gamma_k. \tag{3.2}$$

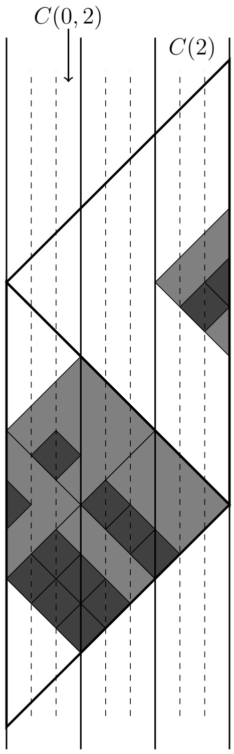

FIGURE 4. For this realization: $M = 3$, $Z^{LL}(2) = 1$, $Z^{LR}(2) = 0$, $Z^{RL}(2) = 0$, $Z^{RR}(2) = 1$. Further, $Z^{LL}(0,2) = Z^{LR}(0,2) = 0$ and $Z^{RL}(0,2) = 2$, $Z^{RR}(0,2) = 3$.

4. The proof of the main result

This Section is organized as follows: First we state a Proposition which carries the main part of Theorem 2. Then we prove Theorem 2 using this proposition. In the next Section we verify our Proposition.

Proposition 1. *We assume that both of the hypotheses of Theorem 2 hold. Then for Lebesgue almost all* $x \in \left[0, \frac{\sqrt{2}}{2}\right]$ *we have*

$$\mathbb{P}\left\{x \in \mathrm{Proj}(\Lambda)\right\} > 0.$$

Proof of Theorem 2. We remind the reader that the probability space of the pairs of independent random Cantor sets was denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. First we show that

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\Lambda)\right) > 0)\right\} > 0. \tag{4.1}$$

This is equivalent to the following inequality:

$$\mathbb{E}\left(\mathcal{L}\mathrm{eb}(\mathrm{Proj}(\Lambda))\right) > 0. \tag{4.2}$$

We define the function $\chi : \Omega \times [0, \sqrt{2}/2] \to \{0, 1\}$ by

$$\chi(\omega, x) := \begin{cases} 1, & \text{if } x \in \mathrm{Proj}(\Lambda(\omega)); \\ 0, & \text{otherwise.} \end{cases}$$

Then using Proposition 1 we obtain that (4.2) holds as follows:

$$\mathbb{E}\left(\mathcal{L}\mathrm{eb}(\mathrm{Proj}(\Lambda))\right) = \iint \chi(\omega, x)\, \mathrm{d}x\, \mathrm{d}\mathbb{P}(\omega) =$$

$$\iint \chi(\omega, x)\, \mathrm{d}\mathbb{P}(\omega)\, \mathrm{d}x = \int_{x=0}^{\sqrt{2}/2} \mathbb{P}\left\{x \in \mathrm{Proj}(\Lambda(\omega))\right\}\, \mathrm{d}x > 0.$$

The rest of the proof is a standard argument showing that $\{\mathcal{L}\mathrm{eb}\,(\mathrm{Proj}(\Lambda)) > 0\}$ is a $0-1$ event.
Using (4.1) and the definition of ψ and φ we obtain that

$$c := \mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\varphi(F_1 \times F_2)) > 0\right)\right\} > 0. \tag{4.3}$$

Now we show that

$$\mathcal{L}\mathrm{eb}(\mathrm{Proj}(\varphi(F_1 \times F_2))) > 0 \quad \text{ if } F_1, F_2 \neq \emptyset$$

holds ($\mathbb{P}$) almost surely.
Let A_n (B_n) be the number of level n intervals in the level n approximation F_1^n (F_2^n) respectively. We assume that $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$. Then it follows from Remark 4 and the definition of Hausdorff dimension that

$$0 < \mathrm{d}im_H F_1 \leq \lim_{n\to\infty} \frac{\log A_n}{\log M^n} \qquad \text{and} \qquad 0 < \mathrm{d}im_H F_2 \leq \lim_{n\to\infty} \frac{\log B_n}{\log M^n}$$

hold almost surely. Thus A_n and B_n tends to infinity almost surely. We fix an integer $K > 0$ and choose $N > 0$ such that $A_N, B_N \geq K$ holds. Therefore we can choose the words

$$\underline{k}^1, \dots, \underline{k}^K \in \{0, \dots, M-1\}^N \text{ and } \underline{\ell}^1, \dots, \underline{\ell}^K \in \{0, \dots, M-1\}^N$$

such that $\underline{k}^i \neq \underline{k}^j$ and $\underline{\ell}^i \neq \underline{\ell}^j$ for all $i \neq j$ and $I_{\underline{k}^i}$ ($I_{\underline{\ell}^i}$) is contained in F_1^n (F_2^n) respectively. The K random Cantor sets

$$\left\{\widetilde{Q}_{\underline{k}^i, \underline{\ell}^i} \cap \varphi(F_1 \times F_2)\right\}_{i=1}^K$$

are independent realizations of scaled copies of $\varphi(F_1 \times F_2)$ type Cantor sets. Thus

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\varphi(F_1 \times F_2)) > 0\right) \mid F_1, F_2 \neq \emptyset\right\} \geq$$

$$1 - \prod_{i=1}^K \left(1 - \mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\widetilde{Q}_{\underline{k}^i, \underline{l}^i} \cap \varphi(F_1 \times F_2)) > 0\right)\right\}\right) =$$

$$1 - (1-c)^K.$$

Since K was arbitrary we have

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\varphi(F_1 \times F_2)) > 0 \mid F_1, F_2 \neq \emptyset\right)\right\} = 1,$$

which is equivalent to

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}(F_1 - F_2) > 0 \mid F_1, F_2 \neq \emptyset\right\} = 1.$$

$\square$

5. The proof of Proposition 1

5.1. **A Branching process with random environment.** We define a random variable $\mathcal{U} \in \text{Uniform}\left[0, \frac{\sqrt{2}}{2}\right]$ and let $\mathcal{P}$ be the distribution of $\mathcal{U}$ and we define $\mathcal{E}$ as the corresponding expectation. In order to prove Proposition 1 it is enough to show that

$$\mathcal{P}\left(\mathbb{P}\left\{\mathcal{U} \in \text{Proj}(\Lambda)\right\} > 0\right) = 1. \tag{5.1}$$

We recall that the measure $\mathbb{P}$ refers to the construction of the pair of random Cantor sets F_1, F_2. The base M expansion of $\sqrt{2} \cdot \mathcal{U}$ naturally defines a random infinite sequence $(i_1, i_2, \dots) \in \{0, \dots, M-1\}^{\mathbb{N}}$

$$\mathcal{U} = \frac{\sqrt{2}}{2} \cdot \left(\frac{i_1}{M} + \frac{i_2}{M^2} + \cdots\right). \tag{5.2}$$

In order to check (5.1) we define a branching process with random environment

$$\mathcal{Z}_n(\overline{\theta}) = \mathcal{Z}_n(\underbrace{i_1, \dots, i_N}_{\theta_0}, \dots, \underbrace{i_{(n-1)N+1}, \dots, i_{nN}}_{\theta_{n-1}}, \dots),$$

where N is a large integer defined in (5.7) below. The environment is $\overline{\theta} = (\theta_0, \theta_1, \dots)$, where $\theta_k = (i_{kN+1}, \dots, i_{(k+1)N})$. To verify that (5.1) holds we want to define $\mathcal{Z}_n(\overline{\theta})$ in such a way that

(C1): If $\left\{\mathcal{Z}_n(\overline{\theta})\right\}_{n \geq 0}$ does not die out then

$$\frac{\sqrt{2}}{2} \cdot \left(\frac{i_1}{M} + \frac{i_2}{M^2} + \cdots\right) \in \text{Proj}(\Lambda),$$

(C2): $\mathcal{P}$ almost surely: $\left\{\mathcal{Z}_n(\overline{\theta})\right\}_{n \geq 0}$ does not die out with positive $\mathbb{P}$ probability.

The definition of the branching process $\mathcal{Z}_n(\overline{\theta})$ is somewhat involved. It is a random number of some carefully chosen pairs of left and right triangles of level nN. For each such pair, we will choose some descendants, or successors, of level $(n+1)N$, and their total number will be $\mathcal{Z}_{n+1}(\overline{\theta})$. Figure 5 illustrates this procedure: the pair $(L_j^{(k-1)}, R_j^{(k-1)})$ in the left column, which counts towards $\mathcal{Z}_{k-1}(\overline{\theta})$, has the pairs (A, B) and (C, D) as its descendants, which count towards $\mathcal{Z}_k(\overline{\theta})$.

To define $\mathcal{Z}_n(\overline{\theta})$ precisely, we need some notation. For $k \in \{0, \dots, M-1\}$ and $V \in \{L, R\}$ let $q(k, V)$ be the probability ($\mathbb{P}$) that $C(k) \cap V \cap \Lambda^1$ contains both level 1 left and level 1 right triangles. Now we define

$$q := \min_{k=0,\dots,M-1} \max_{V \in \{L,R\}} q(k, V) \tag{5.3}$$

It follows from the condition (**A2**) that $q > 0$. To see, this we fix $k \in \{0, \dots, M-1\}$. Then using (3.1) by condition (**A2**) the matrix $\mathcal{M}(k)$ has a strictly positive row. Let us say that the first row has this property. We prove that in this case

$$q(k,L)>0. \tag{5.4}$$

Namely, the expected values of both of the level 1 left and right triangles in $C(k)\cap L\cap\Lambda^1$ are positive. It is immediate from this and from the way we constructed our random sets F_1,F_2 that we have both level 1 left and level 1 right triangles in $C(k)\cap L\cap\Lambda^1$ with positive probability which is exactly (5.4).
For $k\in\{0,\dots,M-1\}$ we define $U(k)\in\{L,R\}$ as follows:

$$U(k)=\begin{cases} L, & \text{if } q(k,L)\geq q;\\ R, & \text{otherwise.}\end{cases} \tag{5.5}$$

It follows from (5.3) that

$$q(k,U(k))\geq q. \tag{5.6}$$

Finally, we fix a large natural number N which satisfies

$$(N-1)\frac{1}{M}\log\Gamma+\log q>0. \tag{5.7}$$

We recall that it was the assumption (**A1**) of Theorem 2 that $\Gamma>1$. Now we specify an algorithm with which we select some of the pairs of the level kN triangles contained in $C(i_1,\dots,i_{kN})\cap\Lambda^{kN}$. See Figure 5 for the visualization of some of the key steps of the construction.
We define Pair_k by induction as follows:

$$\mathrm{Pair}_0:=\{(L,R)\}.$$

Assume that we have already defined

$$\mathrm{Pair}_{k-1}=\left\{\left(L_1^{(k-1)},R_1^{(k-1)}\right),\dots,\left(L_{z_{k-1}}^{(k-1)},R_{z_{k-1}}^{(k-1)}\right)\right\},$$

with the following properties:

(P1): $L_i^{(k-1)}$ is a left triangle and $R_i^{(k-1)}$ is a right triangle, both of them of level $(k-1)N$ and both of them are contained in

$$C(i_1,\dots,i_{(k-1)N})\cap\Lambda^{(k-1)N}.$$

(P2): The z_{k-1} events

$$\left\{\left(L_i^{(k-1)}\cup R_i^{(k-1)}\right)\cap\Lambda\right\}_{i=1}^{z_{k-1}}$$

are independent. Namely, the boundary of the sets

$$\left\{\left(L_i^{(k-1)}\cup R_i^{(k-1)}\right)\right\}_{i=1}^{z_{k-1}}$$

can intersect at most in a single point.

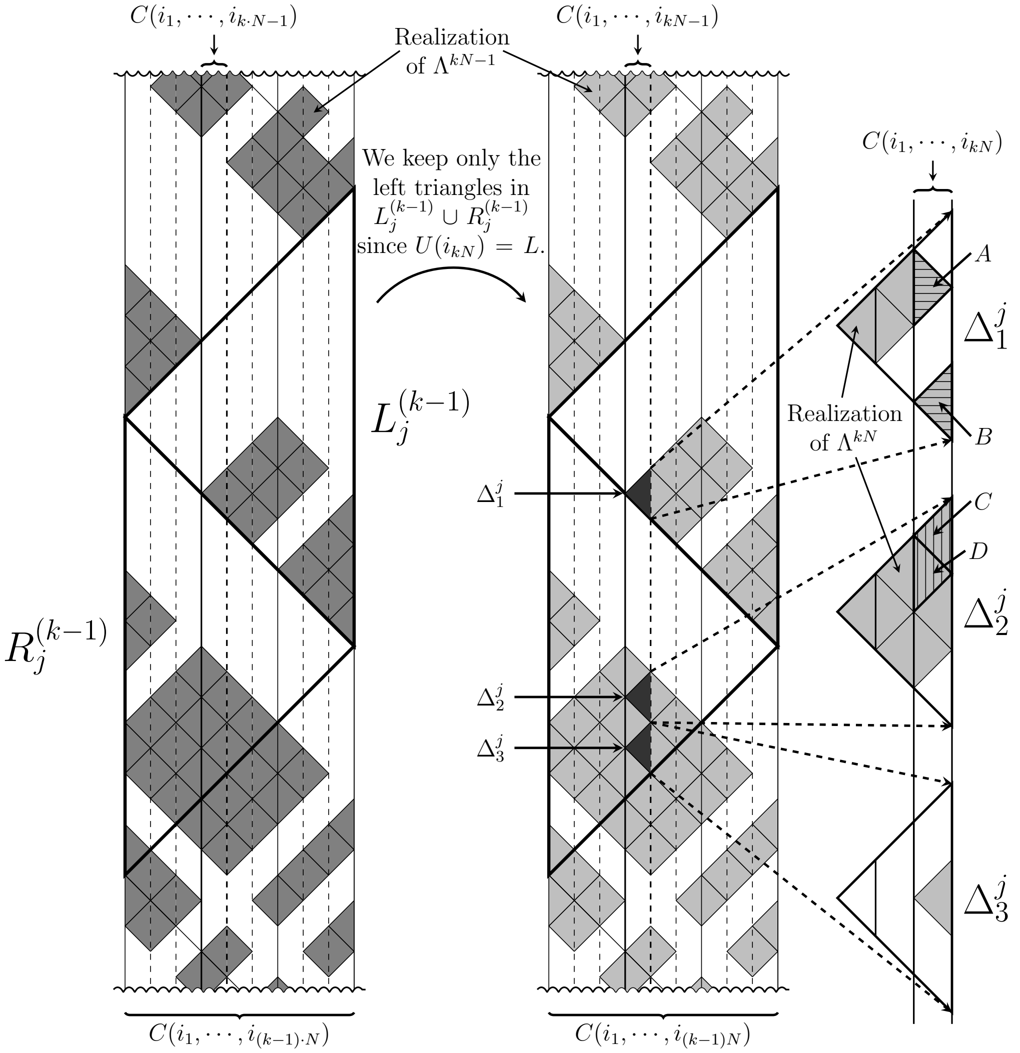

FIGURE 5. The level $(k-1)N$ pair $(L_j^{(k-1)}, R_j^{(k-1)})$ gives birth to the level kN pairs (A, B) and (C, D). That is, $\mathrm{Desc}_j^{(k-1)} = \{(A, B), (C, D)\}$.

For all $1 \le j \le z_{k-1}$ the set of (N step) descendants of $(L_j^{(k-1)}, R_j^{(k-1)})$ (denoted by $\mathrm{Desc}_j^{(k-1)}$) will be defined as a set of some of the level kN pairs of left and right triangles contained in $C(i_1, \dots, i_{kN}) \cap \Lambda^{kN}$.

Definition of the set $\mathrm{Desc}_j^{(k-1)}$: First we consider all of the level $kN - 1$ triangles contained in

$$C(i_1, \dots, i_{kN-1}) \cap \Lambda^{kN-1} \cap \left(L_j^{(k-1)} \cup R_j^{(k-1)} \right).$$

In Figure 5 these are 3 left and 3 right triangles. Among these, we keep only the left triangles if $U(i_{kN}) = L$, otherwise, we keep the right ones. The collection of the level $kN-1$ triangles obtained in this way is denoted

$$\Delta_1^j, \dots, \Delta_{K_j}^j. \tag{5.8}$$

In Figure 5 we kept the left triangles $\Delta_1^j, \Delta_2^j, \Delta_3^j$. For each $1 \leq \ell \leq K_j$ we select (if we can) exactly one level kN left and exactly one level kN right triangle which are contained in the following intersection:

$$C(i_1, \dots, i_{kN}) \cap \Delta_\ell^j \cap \Lambda^{kN}.$$

It follows from the definition of $U(i_{kN})$ and (5.6) that the probability ($\mathbb{P}$) that we can make such a selection is at least $q > 0$. That is,

$$\begin{aligned} \mathbb{P}\{\exists \text{ both level } kN \text{ left and right triangles} & \\ \text{in } C(i_1, \dots, i_{kN}) \cap \Delta_l^j \cap \Lambda^{kN}\big\} \geq q. & \end{aligned} \tag{5.9}$$

The set $\mathrm{Desc}_j^{(k-1)}$ consists of those pairs of level kN left and right triangles which were selected for some $1 \leq \ell \leq K_j$. In our example only Δ_1^j and Δ_2^j contain pairs (these are (A, B) and (C, D)), so the pair $(L_j^{(k-1)}, R_j^{(k-1)})$ has exactly two descendants.

Now we can define

$$\mathrm{Pair}_k := \bigcup_{j=1}^{z_{k-1}} \mathrm{Desc}_j^{(k-1)}.$$

It is immediate form the construction that Pair_k satisfies property **(P1)** with k instead of $k-1$. To see that property **(P2)** also holds first we write

$$\mathrm{Pair}_k = \left\{ \left(L_1^{(k)}, R_1^{(k)}\right), \dots, \left(L_{z_k}^{(k)}, R_{z_k}^{(k)}\right) \right\}.$$

It follows from the construction that all of the triangles Δ_ℓ^j, $1 \leq j \leq z_{k-1}, 1 \leq \ell \leq K_j$, are of the same type. Namely, either all of them are left or all of them are right triangles. Further they are in the same $kN-1$ column. It follows from Fact 1 that the random Cantor sets

$$\left\{\Delta_\ell^j \cap \Lambda\right\}_{1 \leq j \leq z_{k-1}, 1 \leq \ell \leq K_j}$$

are independent. Since all elements of Pair_k are in different

$$\left\{\Delta_\ell^j\right\}_{1 \leq j \leq z_{k-1}, 1 \leq \ell \leq K_j}$$

therefore, the random Cantor sets

$$\left\{ \left(L_i^{(k)} \cup R_i^{(k)}\right) \cap \Lambda \right\}_{i=1}^{z_k}$$

are independent.

Now we let

$$\mathcal{Z}_0(\overline{\theta}) := 1, \qquad \mathcal{Z}_k(\overline{\theta}) := \#\mathrm{Pair}_k.$$

Then $\left\{\mathcal{Z}_k(\overline{\theta})\right\}_{k \geq 0}$ is a branching process with random environment since Pair_k satisfies properties **(P1)**, **(P2)** above. Now we prove that $\left\{\mathcal{Z}_k(\overline{\theta})\right\}_{k \geq 0}$

satisfies Conditions **(C1)** and **(C2)**. It is obvious that **(P1)** implies that **(C1)** holds. We obtain that **(C2)** holds as a corollary of [1, Theorem 3].

Corollary 2 (Corollary of [1, Theorem 3])**.** *Suppose that*

(a): *There exists $c > 0$ such that for all $\overline{\theta}$ we have $\mathbb{P}(\mathcal{Z}_1(\overline{\theta}) > 0) > c$.*
(b): $D := \frac{1}{M^n} \sum\limits_{(j_1,\dots,j_N)\in\{0,\dots,M-1\}^N} \log \mathbb{E}\left(\mathcal{Z}_1(\overline{\theta} \mid \theta_0 = (j_1,\dots,j_N))\right) > 0.$

*Then **(C2)** holds. That is, $\mathcal{P}$ almost surely: $\left\{\mathcal{Z}_n(\overline{\theta})\right\}_{n\geq 1}$ does not die out with positive $\mathbb{P}$ probability.*

It is easy to see that condition **(a)** holds with the choice of $c = q^N$. The fact that condition **(b)** holds is an immediate corollary of the following lemma:

Lemma 1. *Let $(i_1, i_2, \dots)$ be the random infinite sequence defined in (5.2). The Assumptions of Theorem 2 imply that*

$$D = \mathcal{E}\left\{\log \mathbb{E}\left(\mathcal{Z}_1(\overline{\theta}) \mid \theta_0 = (i_1, i_2, \dots, i_N)\right)\right\} > 0. \tag{5.10}$$

We remind the reader that $\mathcal{E}$ was defined at the beginning of Section 5.1 and that $\mathbb{E}$ denotes the expectation on the probability space which corresponds to the construction of our random Cantor sets.

Proof. We introduce the random variables

$$X_n := \mathbb{E}Z^{LL}(i_1,\dots,i_n) + \mathbb{E}Z^{RL}(i_1,\dots,i_n)$$

and

$$Y_n := \mathbb{E}Z^{LR}(i_1,\dots,i_n) + \mathbb{E}Z^{RR}(i_1,\dots,i_n).$$

Note that X_n (Y_n) is the first (second) column sum of $\mathcal{M}(i_1,\dots,i_n)$. Although we do not use it in the proof but we remark that by the special choice of our matrices $\mathcal{M}(k), k = 0,\dots,M-1$, the random variables X_n and Y_n have the same distribution. We will show that for every $n > 0$ we have

$$\mathcal{E}(\log\ X_n) \geq n \cdot \log\ \sqrt[M]{\Gamma} \tag{5.11}$$

and

$$\mathcal{E}(\log\ Y_n) \geq n \cdot \log\ \sqrt[M]{\Gamma}. \tag{5.12}$$

First we prove (5.10) assuming (5.11) and (5.12), then we verify (5.11) and (5.12). The expected values of the number of the left (right) level $N-1$ triangles in

$$C(i_1,\dots,i_{N-1}) \cap \Lambda^{N-1} \cap (L \cup R)$$

are X_{N-1} (Y_{N-1}), respectively. Recall the construction of $\mathrm{Pair}_1 = \mathrm{Desc}_1^{(0)}$, the descendants of the pair (L, R). We defined $U(k)$ in (5.5). If $U(i_N) = L$ then $K_1 = K_1(i_1,\dots,i_{N-1})$ was defined in (5.8) as the number of level $N-1$ left triangles in $C(i_1,\dots,i_{N-1}) \cap \Lambda^{N-1}$. That is, $\mathbb{E}(K_1) = X_{N-1}$. On the other hand, if $U(i_N) = R$ then K_1 is the number of level $N-1$ right triangles in the same column. So in this case $\mathbb{E}(K_1) = Y_{N-1}$. Since $U(i_N)$

is independent of the random sequence $(i_1,\dots,i_{N-1})$, it follows from (5.11) and (5.12) that we have

$$\mathcal{E}(\log\ \mathbb{E}(K_1)) \geq (N-1)\cdot \log\ \sqrt[M]{\Gamma}, \tag{5.13}$$

Using (5.9) for every $1\leq \ell \leq K_1$ we obtain a (N step) descendant of (L,R) in $C(i_1,\dots,i_n)\cap \Delta_l^1$ with at least probability $q>0$, thus

$$\mathbb{E}\left(\mathcal{Z}_1(\overline{\theta}) \mid \theta_0=(i_1,i_2,\dots,i_N)\right) \geq \mathbb{E}(K_1)\cdot q.$$

Taking the logarithm and the expected value $\mathcal{E}$ on both sides and applying (5.13) and (5.7) we obtain that the assertion of our Lemma holds. Namely,

$$D=\mathcal{E}\left(\log \mathbb{E}\left(\mathcal{Z}_1(\overline{\theta}) \mid \theta_0=(i_1,i_2,\dots,i_N)\right)\right) \geq$$
$$\mathcal{E}\left(\log \mathbb{E}(K_1)\right)+\log q \geq (N-1)\cdot \log\ \sqrt[M]{\Gamma}+\log q>0.$$

Now we prove (5.11) and (5.12) by induction. Using (3.2) and the definition of X_n, Y_n for $n=1$ we obtain that:

$$\mathcal{E}(\log X_1)=\mathcal{E}(\log \gamma_{i_1+1})=\frac{\log \Gamma}{M}$$

and

$$\mathcal{E}(\log Y_1)=\mathcal{E}(\log \gamma_{i_1})=\frac{\log \Gamma}{M}.$$

We assume that both of the inequalities (5.11) and (5.12) hold for $n-1$, that is, we assume that

$$\min\left\{\mathcal{E}(\log X_{n-1}),\mathcal{E}(\log Y_{n-1})\right\} \geq (n-1)\cdot \frac{\log \Gamma}{M}. \tag{5.14}$$

The induction step from $n-1$ to n is analogous for X_n and for Y_n, therefore, we present the proof only for X_n. We denote the elements of the matrix $\mathcal{M}(k)$ as follows:

$$\mathcal{M}(k)=\begin{bmatrix} e_k & f_k \\ g_k & h_k \end{bmatrix}.$$

So we have

$$(X_n,Y_n)=(X_{n-1}\cdot e_{i_n}+Y_{n-1}\cdot g_{i_n}, X_{n-1}\cdot f_{i_n}+Y_{n-1}\cdot h_{i_n}).$$

Thus, from

$$X_n=\left(\frac{e_{i_n}}{\gamma_{i_n+1}}X_{n-1}+\frac{g_{i_n}}{\gamma_{i_n+1}}Y_{n-1}\right)\gamma_{i_n+1}$$

we obtain that

$$\mathcal{E}(\log X_n)=\mathcal{E}\left(\log\left(\frac{e_{i_n}}{\gamma_{i_n+1}}X_{n-1}+\frac{g_{i_n}}{\gamma_{i_n+1}}Y_{n-1}\right)\right)+\mathcal{E}(\log \gamma_{i_n+1}).$$

By the concavity of the logarithm function we get

$$\mathcal{E}(\log X_n)\geq \mathcal{E}\left(\frac{e_{i_n}}{\gamma_{i_n+1}}\log X_{n-1}+\frac{g_{i_n}}{\gamma_{i_n+1}}\log Y_{n-1}\right)+\underbrace{\mathcal{E}(\log \gamma_{i_n+1})}_{(\log\Gamma)/M}.$$

Using the induction hypothesis (5.14), and the fact that the random variables e_{i_n}/γ_{i_n+1} and X_{n-1}, similarly g_{i_n}/γ_{i_n+1} and Y_{n-1}, are independent, we get that

$$\mathcal{E}(\log X_n) \geq \frac{\log \Gamma}{M} \cdot \left(1 + (n-1) \cdot \mathcal{E}\Big(\underbrace{\frac{e_{i_n} + g_{i_n}}{\gamma_{i_n+1}}}_{1}\Big)\right) = n \cdot \frac{\log \Gamma}{M},$$

which yields (5.11). □

6. The proof of the result about the $C - C$ type random Cantor sets

Proof of Theorem 3. Let C_n be the number of level n intervals in the level n approximation F^n. We assume that $F \neq \emptyset$. Then it follows from Remark 4 that

$$0 < \mathrm{d}im_H F \leq \lim_{n\to\infty} \frac{\log C_n}{\log M^n}$$

almost surely. Thus C_n tends to infinity almost surely. We fix an integer $K > 0$ integer and choose $N > 0$ such that $C_N \geq 2K$ holds. Therefore we can choose the words

$$\underline{k}^1, \ldots, \underline{k}^{2K} \in \{0, \ldots, M-1\}^N$$

such that $\underline{k}^i \neq \underline{k}^j$ for all $i \neq j$ and $I_{\underline{k}^i}$ is contained in F^n. The K random Cantor sets

$$\left\{ \widetilde{Q}_{\underline{k}^{2i-1}, \underline{k}^{2i}} \cap \varphi(F \times F) \right\}_{i=1}^{K}$$

are scaled images of independent $\varphi(F_1 \times F_2)$ type sets. Thus

$$\begin{aligned} &\mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\varphi(F \times F)) > 0\right) \mid F \neq \emptyset\right\} \geq \\ &\qquad 1 - \prod_{i=1}^{K} \left(1 - \mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\widetilde{Q}_{\underline{k}^{2i-1}, \underline{k}^{2i}} \cap \varphi(F \times F)) > 0\right)\right\}\right) = \\ &\qquad\qquad 1 - (1-c)^K, \end{aligned}$$

where c was defined in (4.3). Since K was arbitrary, we have

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}\left(\mathrm{Proj}(\varphi(F \times F)) > 0 \mid F \neq \emptyset)\right\} = 1,$$

which is equivalent to

$$\mathbb{P}\left\{\mathcal{L}\mathrm{eb}(F - F) > 0 \mid F \neq \emptyset\right\} = 1.$$

□

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