Ergod. Th. & Dynam. Sys. (1999), **19**, 1343–1363
Printed in the United Kingdom

Hausdorff dimension for horseshoes in $\mathbb{R}^3$

KÁROLY SIMON† and BORIS SOLOMYAK‡

† *Institute of Mathematics, University of Miscolc, Miscolc-Egyetemváros, H-315, Hungary*
(e-mail: simon@gold.uni-miscolc.hu)
‡ *Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA*
(e-mail: solomyak@math.washington.edu)

(*Received* 30 *July* 1997 *and accepted in revised form* 26 *March* 1998)

Abstract. Using pressure formulas we compute the Hausdorff dimension of the basic set of 'almost every' $\mathcal{C}^{1+\alpha}$ horseshoe map in $\mathbb{R}^3$ of the form $F(x, y, z) = (\gamma(x, z), \tau(y, z), \psi(z))$, where $|\psi'| > 1$ and $0 < |\gamma'_x|, |\tau'_y| < \frac{1}{2}$ on the basic set. Similar results are obtained for attractors of nonlinear 'baker's maps' in $\mathbb{R}^3$.

1. *Introduction*

1.1. *Results.* McCluskey and Manning [**22**] have computed the Hausdorff dimension of the basic set of a $\mathcal{C}^1$ Axiom A surface diffeomorphism using pressure formulas. Since their work, it has been an open problem to compute the dimension of an Axiom A basic set in $\mathbb{R}^n$ for $n > 2$. In this paper we give a partial solution to this problem; namely, we compute the Hausdorff dimension of 'almost every' horseshoe map in $\mathbb{R}^3$ from the class defined below. The results of McMullen [**23**] on self-affine sets show that we cannot expect a similar result for all horseshoe maps in $\mathbb{R}^3$. Our methods extend to $n > 3$ but we only consider the case $n = 3$ for the sake of simplicity.

The *horseshoe map* in $\mathbb{R}^3$ is defined in a way similar to the classical Smale's horseshoe map in $\mathbb{R}^2$. Let D be the set shaped as a 'stadium' in the xz plane crossed with $[0, 1]$ (see Figure 1). We have $D = [0, 1]^3 \cup D_1 \cup D_2$, where D_1 (respectively D_2) is a semicircular region crossed with $[0, 1]$ and attached to the top (respectively the bottom) of the cube $[0, 1]^3$. For the rest of the paper, by a horseshoe map we mean a $\mathcal{C}^{1+\alpha}$ transformation $F : D \to D$ which satisfies the following conditions (H1)–(H3).

(H1) There exist disjoint closed intervals $I_1, \ldots, I_m$ whose union is a proper subset of $[0, 1]$ such that denoting $\Delta_k = [0, 1]^2 \times I_k$ we have

$$F(x, y, z) = (\gamma(x, z), \tau(y, z), \psi(z)) \quad \text{for } (x, y, z) \in \bigcup_1^m \Delta_k. \tag{1}$$

Further, $F(\bigcup_1^m \Delta_k) \subset (0, 1)^2 \times [0, 1]$, $F([0, 1]^3 \setminus \bigcup_1^m \Delta_k) \subset D_1 \cup D_2$, and $F(D_1 \cup D_2) \subset D_1$.

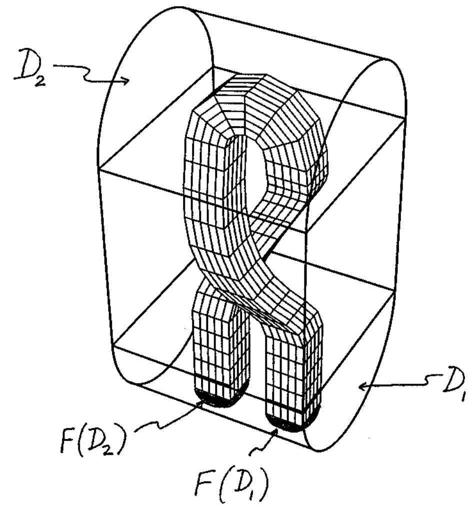

FIGURE 1. Horseshoe map.

(H2) $\lambda_1 < |\gamma_x'|, |\tau_y'| < \lambda_2$, where $0 < \lambda_1 < \lambda_2 < \frac{1}{2}$ are fixed for the rest of the paper.
(H3) $|\psi'| > 1$ on I_k and $\psi(I_k) = [0, 1]$ for each $k = 1, \ldots, m$.
We do not assume that F is one-to-one. The basic set $\Lambda = \Lambda(F) = \bigcap_{n\in\mathbb{Z}} F^n([0, 1]^3)$ is called a horseshoe. Note that the precise form of F outside $\bigcup_1^m \Delta_k$ does not affect this basic set. The two contracting directions are the x and y axes, and the expanding direction is the z axis. The $\mathbf{t}$-perturbation of F, denoted by $F^{\mathbf{t}}$, is a horseshoe map satisfying

$$F^{\mathbf{t}}(x, y, z) = (\gamma(x, z) + t_k^1, \tau(y, z) + t_k^2, \psi(z)) \quad \text{for } (x, y, z) \in \Delta_k,\ k \le m,$$

where $\mathbf{t} = ((t_1^1, t_1^2), \ldots, (t_m^1, t_m^2)) \in \mathbb{R}^{2m}$. We say that $\mathbf{t} \in \mathbb{R}^{2m}$ is F*-admissible* if $F^{\mathbf{t}}(\bigcup_1^m \Delta_k) \subset (0, 1)^2 \times [0, 1]$. Then $F^{\mathbf{t}}$ can be extended to a $\mathcal{C}^{1+\alpha}$ map on D satisfying (H1)–(H3). Clearly, the set of F-admissible $\mathbf{t}$ is a non-empty open set.

We are going to compute the Hausdorff dimension for almost all horseshoes in the following sense: a statement **S** *holds for almost all horseshoe maps* means that for every F, the statement **S** is true for $F^{\mathbf{t}}$ for Lebesgue-a.e. F-admissible $\mathbf{t} \in \mathbb{R}^{2m}$.

The Hausdorff dimension is computed using four pressure (Bowen) equations. The formulas below apply if F is invertible; in the non-invertible case we have to pass to the symbolic space, see §2 for details. We write $P_T(\phi)$ for the pressure of a function ϕ with respect to the transformation T.

The rates of contraction in the two stable directions are given by the following functions defined on Λ:

$$f_1 = \log|\gamma_x'| \quad \text{and} \quad f_2 = \log|\tau_y'|.$$

Denote by s_1, s_2, r_1, r_2 the (unique) solutions of the following four pressure equations:

$$P_{F^{-1}}(s_1 f_1) = 0, \quad P_{F^{-1}}(s_2 f_2) = 0,$$
$$P_{F^{-1}}(f_1 + r_1 f_2) = 0, \quad P_{F^{-1}}(r_2 f_1 + f_2) = 0.$$

Further, let

$$s = \max\{s_1, s_2\}, \quad r = \max\{r_1, r_2\}.$$

By the monotonicity of pressure, if $s = s_1 > 1$ then $f_1 < s_1 f_1$, so $P_{F^{-1}}(f_1) > 0$. Therefore, if $P_{F^{-1}}(f_1 + r_1 f_2) = 0$ then $r_1 > 0$, hence

$$s > 1 \Longrightarrow r > 0. \tag{2}$$

Finally, let δ_u be such that

$$P_F(-\delta_u \log |\psi'|) = 0. \tag{3}$$

We denote by $\dim_H(\Lambda)$ and $\dim_B(\Lambda)$, respectively, the Hausdorff and the box (Minkowski) dimension of Λ. The reader is referred to Falconer [**11**] for the definitions and basic facts of dimension theory.

THEOREM 1. *For almost every horseshoe map F with the basic set Λ:*
(i) *if $s \le 1$ then* $\dim_H(\Lambda) = \dim_B(\Lambda) = \delta_u + s$*;*
(ii) *if $s > 1$ then* $\dim_H(\Lambda) = \dim_B(\Lambda) = \delta_u + 1 + \min\{r, 1\}$.

A similar statement is proved for some 'baker's maps'. Let I_k be closed intervals with disjoint interiors such that $\bigcup_1^m I_k = [0, 1]$. A piecewise $\mathcal{C}^{1+\alpha}$ map $F : [0, 1]^3 \to [0, 1]^3$ is called a *nonlinear skinny baker's transformation* if it satisfies (1), (H2) and (H3). Here we do not assume that F is one-to-one either. The term 'skinny' is used by Chin *et al* [**7**] to emphasize that the contraction rates are less than $\frac{1}{2}$. We consider the attractor $\Lambda = \bigcap_{n=1}^{\infty} F^n([0, 1]^3)$. As in the horseshoe case, we define **t**-perturbations and F-admissible **t** which gives meaning to the words 'almost every baker's map'. We then proceed to define r and s via four pressure equations. The Lebesgue measure in $\mathbb{R}^k$ is denoted by $\mathcal{L}_k$.

THEOREM 2. *For almost every nonlinear skinny baker's transformation:*
(i) *if $s \le 1$ then* $\dim_H(\Lambda) = \dim_B(\Lambda) = 1 + s$*;*
(ii) *if $s > 1$ then* $\dim_H(\Lambda) = \dim_B(\Lambda) = 2 + \min\{r, 1\}$*;*
(iii) *if $r > 1$ then* $\mathcal{L}_3(\Lambda) > 0$.

Remarks. 1. The 'skinny' condition $|\gamma_x'|, |\tau_y'| < \frac{1}{2}$ in Theorems 1 and 2 is essential; the statements may fail if one of the contraction rates is bigger than $\frac{1}{2}$. The appropriate example is given in the next subsection. We should note, however, that all known examples are linear and have a rather special number-theoretic nature. It is believed that for a 'generic' horseshoe the dimension formulas hold assuming just $|\gamma_x'|, |\tau_y'| < 1$. This is one of the major unsettled problems in the field.

2. Perhaps one can replace the four pressure equations used in Theorems 1 and 2 by only one (but more complicated) equation, using the non-additive thermodynamic formalism introduced by Barreira [**1**].

3. In Theorems 1 and 2, the case $r > 1$ is only possible when F is non-invertible. This can be deduced, for example, from Corollary 3.2(i) below.

1.2. *Discussion.* Here we consider 'projections' and some special cases of our results restricting ourselves to horseshoe maps; baker's maps require obvious modifications. Then we mention some related results in the literature. Finally, a few comments on the proof of Theorem 1 are given.

1.2.1. Let F be a horseshoe map (1) satisfying (H1)–(H3). The projection $P_z\Lambda$ of the basic set onto the z-axis is the easiest to understand. It follows from the definition of F and Λ that

$$P_z\Lambda = \mathrm{Rep}(\psi) := \left\{\zeta \in [0,1] : \psi^l(\zeta) \in \bigcup_1^m I_k,\ \forall l \geq 0\right\}. \tag{4}$$

In other words, $P_z\Lambda$ is the repeller for ψ or, equivalently, the attractor of the iterated function system $\{\psi_k^{-1}\}_{k=1}^m$, where $\psi_k^{-1} : [0,1] \to I_k$ are the branches of ψ^{-1}. Such sets are sometimes called 'cookie-cutter sets'; see Bedford [**2**] and Falconer [**14**, Ch. 4]. Recall that $\psi \in \mathcal{C}^{1+\alpha}$. Let $g(z) = -\log|\psi'(z)|$ for $z \in \mathrm{Rep}(\psi)$. Further, denote $J_{i_1\ldots i_n} = \psi_{i_1}^{-1} \circ \cdots \circ \psi_{i_n}^{-1}([0,1])$. The next theorem is well known and has a long history starting with Bowen [**5**] and Ruelle [**28**]; part (v) is due to Falconer [**11**]. In fact, it holds for $\mathcal{C}^1$-maps, see Barreira [**1**] and Gatzouras and Peres [**16**].

THEOREM 3. (Classical) *If* $t = \dim_H(\mathrm{Rep}(\psi))$ *then:*
(i) $P_\psi(tg) = 0$*;*
(ii) $t = \sup(h_\nu/-\int g\,d\nu)$*, where the supremum is taken over all invariant measures* ν*, and* h_ν *is the entropy;*
(iii) $t = \lim_{n\to\infty} d_n$*, where* d_n *satisfies the equation* $\sum_{i_1\ldots i_n} |J_{i_1\ldots i_n}|^{d_n} = 1$*;*
(iv) $0 < \mathcal{H}^t(\mathrm{Rep}(\psi)) < \infty$*, where* $\mathcal{H}^t$ *is the Hausdorff measure;*
(v) $\dim_B(\mathrm{Rep}(\psi)) = t$*.*

1.2.2. Now consider the special case when $\gamma(x,z) = \gamma_k(x)$ and $\tau(y,z) = \tau_k(y)$ for $(x,y,z) \in \Delta_k$. Then $\Lambda = \Lambda' \times P_z\Lambda$, where Λ' is the projection of Λ onto the xy plane, so we may refer to Λ as a 'product' horseshoe. Observe that Λ' is the attractor of the iterated function system $\{(\gamma_k(x), \tau_k(y))\}_1^m$ on the square $[0,1]^2$. It is a nonlinear 'self-affine set', in the terminology of Bedford and Urbański [**3**]. The dimension of such sets is difficult to compute, even when they are linear, that is, when $\gamma_k(x) = \gamma_k x + a_k$ and $\tau_k(y) = \tau_k y + b_k$. McMullen [**23**] has shown that the Hausdorff dimension may be strictly smaller than the box dimension. However, Falconer [**10**] proved that for *almost all* linear self-affine sets, that is, for almost all translations $(a_k, b_k)_1^m$, the Hausdorff and box dimensions coincide, provided the contraction rates are sufficiently small. 'Sufficiently small' was less than $\frac{1}{3}$ in Falconer [**10**] and relaxed to less than $\frac{1}{2}$ in Solomyak [**31**]. If one of the contraction rates is greater than $\frac{1}{2}$, even the 'almost-all'-type results may fail. Indeed, let $m = 2$ and $\gamma_i = \gamma$, $\tau_i = \tau$ for $i = 1, 2$. As observed by Edgar [**8**], it follows from Przytycki and Urbański [**27**] that if $\gamma \in (\frac{1}{2}, 1)$ satisfies $1 = \sum_{i=1}^p \gamma^i$ and $\tau \in (0, \frac{1}{2})$, then the Hausdorff dimension of the self-affine set is strictly smaller than the box dimension, *for almost all translations.* This is the example we referred to in Remark 1 at the end of §1.1.

1.2.3. Now we go back to the horseshoe map $F(x,y,z) = (\gamma(x,z), \tau(y,z), \psi(z))$ and consider the projection onto the xz plane. Of course, the projection onto the yz plane can

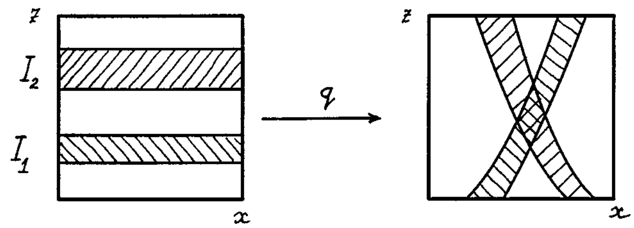

FIGURE 2. 'Projected' horseshoe map.

be treated similarly. We get the map

$$q(x, z) = (\gamma(x, z), \psi(z))$$

defined on $\bigcup_1^m([0, 1] \times I_k)$. Studying such transformations is an important ingredient of our proofs. An example of q is given in Figure 2; notice that the images of $[0, 1] \times I_k$ may overlap even if F is invertible. To our knowledge, similar maps were first studied by Yakobson [**32**] and then by Falconer [**9**]. Later, answering a question of Yakobson, Simon [**29**] proved that, in spite of the overlaps, the dimension of the attractor can be expressed using the pressure formula, provided that the overlapping strips intersect 'transversally'. In this paper no such assumption is needed but we obtain the dimension formula for almost all, rather than all, maps.

1.2.4. There is a large literature on the topics related to this paper. We mention some (but certainly not all) of the papers here without going into details. Extra bibliographic entries can be found in Pesin [**25**].

(Generalized) solenoids form another important class of Axiom A basic sets in $\mathbb{R}^3$. Again the difficulties arise when the two contraction rates are distinct. Bothe [**4**] studied a class of solenoids with two variable contraction rates and proved that a formula similar to that in Theorem 1(i) holds generically if the contraction rates are sufficiently small. Simon [**30**] used the methods from Simon [**29**] to obtain the dimension formula for solenoids with two distinct *constant* contraction rates.

The equality of the Hausdorff and box dimensions in the setting of McCluskey and Manning [**22**] (for $\mathcal{C}^1$ Axiom A surface diffeomorphisms) follows from the work of Palis and Viana [**24**].

Bedford and Urbański [**3**] studied nonlinear self-affine sets and obtained pressure formulas for the dimension assuming that $\min_{k,x} |\gamma_k'(x)| > \max_{k,y} |\tau_k'(y)|$ and the natural measure projected onto the x axis is absolutely continuous.

Falconer [**13**] estimated the dimension of some non-conformal repellers but he needed special geometric assumptions to guarantee that the estimates are sharp. He used a sub-additive thermodynamic formalism. Hu [**17**] computed the box dimension for a class of non-conformal repellers. Barreira [**1**] introduced a non-additive thermodynamic formalism and applied it to estimate the Hausdorff dimension of hyperbolic sets. Zhang [**33**] used another version of thermodynamic formalism to prove upper estimates for the Hausdorff dimension.

Chin *et al* [**7**] studied the correlation dimension (with respect to the natural measure) for almost all *linear* skinny baker's maps (and with γ and τ independent of z). Hunt [**19**] is investigating dimension characteristics of the natural measure on the attractor for nonlinear maps similar to those considered in Theorem 2.

In the survey of Gatzouras and Peres [**15**] the variational principle for dimension on repellers is discussed; their paper also contains a large bibliography on linear self-affine sets which we do not duplicate. Some self-affine sets in $\mathbb{R}^2$ and the corresponding 'product' horseshoes in $\mathbb{R}^3$, with one of the contraction rates greater than $\frac{1}{2}$, were considered by Pollicott and Weiss [**26**].

1.2.5. Now let us make some comments about the proof of Theorem 1. The upper estimates are proved for the upper box dimension; they are rather standard and hold for all (not just almost all) horseshoes under consideration. We use potential theoretic methods to get lower estimates of the Hausdorff dimension. 'Almost-all' type results follow from Fubini's theorem, so we have no way to check concrete cases. This method goes back to Kaufman [**20**]. Our proof relies on the scheme of Falconer [**10**] in several places. We use the tools of thermodynamic formalism, especially Gibbs (equilibrium) measures. It follows from (H1)–(H3) that the stable manifolds are horizontal planes and the unstable manifolds are smooth curves which can be parameterized by z. The z-slices of the basic set can be represented as attractors of iterated function systems but with a different family of contractions applied at each step. This is taken into account by the two-sided symbolic coding. There are two cases in Theorem 1, corresponding to parts (i) and (ii). They are referred to as the case of 'small contractions' and 'large contractions' respectively. In the case of small contractions the dimension of almost every Λ turns out to be the maximum of dimensions of its projections onto the xz and yz planes.

2. *Preliminaries*

First we give definitions of r and s valid in the general (possibly non-invertible) case. Then we consider 'projections' of our horseshoe maps and develop tools to study them.

Fix a horseshoe map F. Let $\Sigma = \{1, \dots, m\}^{\mathbb{Z}}$. We denote the natural projection from the symbolic space Σ to the basic set Λ by Π_F:

$$\begin{aligned}\Pi_F(\mathbf{i}) &= \Pi_F(\dots, i_{-n}, \dots, i_0, \dots, i_n, \dots)\\ &= \bigcap_{n=0}^{\infty}\left[\Delta(i_0, \dots, i_{-n}) \bigcap F^n(\Delta(i_n, \dots, i_1))\right],\end{aligned}$$

where $\Delta(j_0, \dots, j_k) := [0,1]^2 \times \{\zeta \in [0,1] : \psi^l(\zeta) \in I_{j_l}, l = 0, \dots, k\}$. Then $\Pi_F \circ \sigma^{-1} = F \circ \Pi_F$, where σ is the left shift on Σ. We write $\Pi_F = (\Pi_F^1, \Pi_F^2, \Pi_F^3)$. Consider the functions

$$\phi_1(\mathbf{i}) := \log|\gamma_x'(\Pi_F^1(\sigma\mathbf{i}), \Pi_F^3(\sigma\mathbf{i}))| \quad \text{and} \quad \phi_2(\mathbf{i}) := \log|\tau_y'(\Pi_F^2(\sigma\mathbf{i}), \Pi_F^3(\sigma\mathbf{i}))|. \tag{5}$$

Then s_1, s_2, r_1, r_2 are defined as the unique solutions of the equations

$$P(s_1\phi_1) = 0, \quad P(s_2\phi_2) = 0, \quad P(\phi_1 + r_1\phi_2) = 0, \quad P(r_2\phi_1 + \phi_2) = 0, \tag{6}$$

where P is the pressure with respect to σ. If F is invertible, the projection Π_F is one-to-one and this definition coincides with the one given in §1.1; see, for example, Bowen [**5**].

Recall (4) that $\mathrm{Rep}(\psi) \subset [0,1]$ is the repeller of ψ, that is, the set on which all forward iterates of ψ are defined. Since $\Lambda = \bigcap_{n\in\mathbb{Z}} F^n([0,1]^3)$ is a subset of $[0,1]^2 \times \mathrm{Rep}(\psi)$ we are only interested in F on this set. Observe that all $\mathbf{t}$-perturbations $F^{\mathbf{t}}$ have the same z component ψ; we fix ψ for the rest of the paper. Notice that

$$\Pi^3(\mathbf{i}) := \Pi^3_F(\mathbf{i}) = \lim_{n\to\infty} \psi^{-1}_{i_0} \circ \cdots \circ \psi^{-1}_{i_{-n}}(0)$$

depends only on ψ, so Π^3 will be the same for all maps under consideration. For $z \in [0,1]$ let $\Sigma_z = \{\mathbf{i} \in \Sigma : \Pi^3(\mathbf{i}) = z\}$. Clearly, $\Sigma_z \neq \emptyset$ if and only if $z \in \mathrm{Rep}(\psi)$. Let $\Sigma_+ := \{1,\ldots,m\}^{\mathbb{N}}$. It is easy to see that the projection $P_+ : \Sigma \to \Sigma_+$ restricted to Σ_z is a bijection, so we can identify Σ_z with Σ_+. This identification will be meant whenever we write $\Sigma_z \sim \Sigma_+$.

For every horseshoe map $F(x,y,z) = (\gamma(x,z), \tau(y,z), \psi(z))$ there are two associated 'projected' maps:

$$q(x,z) = (\gamma(x,z), \psi(z)) \quad \text{and} \quad r(y,z) = (\tau(y,z), \psi(z)).$$

It is convenient to consider these maps on their own, in such a way that F is not used explicitly. We shall work with q and γ in the xz plane; obvious modifications are needed for the yz plane. Clearly, $q : [0,1] \times \mathrm{Rep}(\psi) \to [0,1] \times \mathrm{Rep}(\psi)$. An example of the map q is given in Figure 2, see §1. For $\mathbf{i} \in \Sigma$ let

$$G_{\mathbf{i}}(x) := \gamma(x, \psi^{-1}_{i_1}(\Pi^3(\mathbf{i}))).$$

Then $G_{\mathbf{i}} : [0,1] \to [0,1]$ and $\lambda_1 < |G'_{\mathbf{i}}(x)| < \lambda_2$ for each $x \in [0,1]$. The collection of all $\mathcal{G} = \{G_{\mathbf{i}}\}_{\mathbf{i}\in\Sigma}$ obtained in this way (so ultimately arising from some horseshoe map) will be denoted Γ. The corresponding $\mathbf{t}$-perturbations are

$$G^{\mathbf{t}}_{\mathbf{i}}(x) = \gamma(x, \psi^{-1}_{i_1}(\Pi^3(\mathbf{i}))) + t_{i_1}. \tag{7}$$

We say that $\mathbf{t}$ is $\mathcal{G}$-admissible if $\mathcal{G}^{\mathbf{t}} \in \Gamma$. Since $\Pi^3(\mathbf{i})$ is uniquely determined by $(\ldots, i_{-2}, i_{-1}, i_0)$, it follows from the definition of $G_{\mathbf{i}}$ that

$$i_k = j_k \text{ for } -\infty < k \le 1 \Rightarrow G_{\mathbf{i}}(x) \equiv G_{\mathbf{j}}(x). \tag{8}$$

Denote

$$G_{\mathbf{i},n}(x) := G_{\mathbf{i}} \circ G_{\sigma\mathbf{i}} \circ \cdots \circ G_{\sigma^{n-1}\mathbf{i}}(x). \tag{9}$$

It follows from (8) that

$$i_k = j_k \text{ for } -\infty < k \le n \Rightarrow G_{\mathbf{i},n}(x) \equiv G_{\mathbf{j},n}(x). \tag{10}$$

We shall need cylinder sets $[i_1 \ldots i_n] = \{j \in \Sigma : j_k = i_k, 1 \le k \le n\}$. For $z \in \mathrm{Rep}(\psi)$ let $I_{z,i_1,\ldots,i_n} := G_{\mathbf{i},n}([0,1])$, where $\mathbf{i} \in [i_1 \ldots i_n]$ is such that $z = \Pi^3(\mathbf{i})$. Then $I_{z,i_1,\ldots,i_n} \subset I_{z,i_1,\ldots,i_{n-1}}$. For $\mathcal{G} \in \Gamma$ define

$$\Pi_{\mathcal{G}}(\mathbf{i}) := \bigcap_{n=1}^{\infty} I_{z,i_1,\ldots,i_n} = \lim_{n\to\infty} G_{\mathbf{i},n}(0), \quad \text{where } z = \Pi^3(\mathbf{i}). \tag{11}$$

It follows from the definitions of Π_F and $G_{\mathbf{i},n}$ that $\Pi_{\mathcal{G}} \equiv \Pi_F^1$. Set $\Lambda^{\mathcal{G}}(z) = \Pi_{\mathcal{G}}(\Sigma_z)$. Then $\Lambda^{\mathcal{G}}(z)$ is just the projection of the slice $\Lambda(z)$ onto the x axis. It follows from (9) that $\{G_{\mathbf{i},n}\}$ is a cocycle:

$$G_{\mathbf{i},n+m}(x) = G_{\mathbf{i},n}(G_{\sigma^n\mathbf{i},m}(x)),$$

therefore,

$$\Pi_{\mathcal{G}}(\mathbf{i}) = G_{\mathbf{i},n}(\Pi_{\mathcal{G}}(\sigma^n\mathbf{i})). \tag{12}$$

By the definition of $I_{z,i_1,\dots,i_n}$ and the mean value theorem, for any $n \geq 1$ and $\mathbf{i} \in \Sigma_z$, there exists $u \in [0, 1]$ such that

$$G'_{\mathbf{i},n}(u) = |I_{z,i_1,\dots,i_n}|. \tag{13}$$

For each $\mathcal{G} \in \Gamma$ define

$$\varphi_{\mathcal{G}}(\mathbf{i}) := \log|G'_{\mathbf{i}}(\Pi_{\mathcal{G}}(\sigma\mathbf{i}))|. \tag{14}$$

It is immediate from the definitions that $\varphi_{\mathcal{G}} \equiv \phi_1$ (see (5)). Since $\gamma \in \mathcal{C}^{1+\alpha}$, the function $\varphi_{\mathcal{G}}$ is Hölder continuous. One easily computes

$$\log|G'_{\mathbf{i},n}(x)| = \sum_{k=0}^{n-1} \varphi_{\mathcal{G}}(\sigma^k\mathbf{i}), \tag{15}$$

where $x = \Pi_{\mathcal{G}}(\sigma^n\mathbf{i})$.

Recall that for each continuous function $f : \Sigma \to \mathbb{R}$ the topological pressure of f (with respect to σ) is defined by

$$P(f) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} \exp\left[\sup_{\mathbf{i}\in[i_1\dots i_n]} \sum_{k=0}^{n-1} f(\sigma^k\mathbf{i})\right];$$

see Bowen [**5**].

3. *Beginning of the proof*

We continue to work with the class Γ introduced in the previous section. The Lipschitz constant for $\mathcal{G} = \{G_{\mathbf{i}}\}_{\mathbf{i}\in\Sigma} \in \Gamma$ is defined by

$$L(\mathcal{G}) := \sup_{x,y\in[0,1],\mathbf{i}\in\Sigma} \frac{|G'_{\mathbf{i}}(x) - G'_{\mathbf{i}}(y)|}{|x-y|^{\alpha}}.$$

The distance between two elements $\mathcal{T}, \mathcal{G}$ of Γ is

$$\varrho(\mathcal{G}, \mathcal{T}) := \sup_{\mathbf{i}\in\Sigma} \|G_{\mathbf{i}} - T_{\mathbf{i}}\|,$$

where

$$\|h\| := \|h\|_{\sup} + \|h'\|_{\sup} + \sup_{x,y\in[0,1]} \frac{|h'(x) - h'(y)|}{|x-y|^{\alpha}}.$$

Next we prove some distortion inequalities and collect their immediate consequences. There are many similar estimates in the literature, but since they are of crucial importance for us, complete proofs are given.

LEMMA 3.1. *There exists a constant $C_1 > 0$ such that for all $\mathcal{G}, \mathcal{T} \in \Gamma$, $n \in \mathbb{N}$ and for any $\mathbf{i} \in \Sigma$, $u, v \in [0,1]$,*

(i)

$$C_1^{-L(\mathcal{G})} < \frac{|G'_{\mathbf{i},n}(u)|}{|G'_{\mathbf{i},n}(v)|} < C_1^{L(\mathcal{G})}.$$

(ii) *For any $\mathcal{G} \in \Gamma$ there exists a constant $C_2 = C_2(\alpha, L(\mathcal{G}), \lambda_1, \lambda_2)$ such that for all $\mathcal{T} \in \Gamma$ satisfying $\varrho(\mathcal{G}, \mathcal{T}) \le 1$ and for any $\mathbf{i} \in \Sigma$, $n \in \mathbb{N}$,*

$$\exp[-n \cdot C_2 \cdot \varrho(\mathcal{G}, \mathcal{T})^\alpha] < \frac{|T'_{\mathbf{i},n}(0)|}{|G'_{\mathbf{i},n}(0)|} < \exp[n \cdot C_2 \cdot \varrho(\mathcal{G}, \mathcal{T})^\alpha].$$

Proof. (i) It is clearly enough to check one inequality. We have

$$\begin{aligned}
\log \frac{|G'_{\mathbf{i},n}(u)|}{|G'_{\mathbf{i},n}(v)|} &= \sum_{k=0}^{n-1} \log \left| \frac{G'_{\sigma^k \mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(u))}{G'_{\sigma^k \mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(v))} \right| \\
&\le \sum_{k=0}^{n-1} \left| \frac{G'_{\sigma^k \mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(u)) - G'_{\sigma^k \mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(v))}{G'_{\sigma^k \mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(u))} \right| \\
&\le \frac{L(\mathcal{G})}{\lambda_1} \sum_{k=0}^{n-1} |G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(u) - G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(v)|^\alpha \\
&\le \frac{L(\mathcal{G})}{\lambda_1} \sum_{k=0}^{n-1} \lambda_2^{(n-k)\alpha} \cdot |v-u|^\alpha < L(\mathcal{G}) \cdot \frac{1}{\lambda_1(1-\lambda_2{}^\alpha)},
\end{aligned}$$

and the desired statement follows. In the second displayed line above we used that $\log|x/y| \le |(x-y)/y|$, and in the last but one inequality we used that by the mean value theorem,

$$\left|G_{\mathbf{j},m}(u) - G_{\mathbf{j},m}(v)\right| \le \lambda_2 \cdot |G_{\sigma\mathbf{j},m-1}(u) - G_{\sigma\mathbf{j},m-1}(v)|.$$

(ii) Observe that

$$\begin{aligned}
|T_{\mathbf{j},m}(u) - G_{\mathbf{j},m}(u)| &\le |T_{\mathbf{j}}(T_{\sigma\mathbf{j},m-1}(u)) - G_{\mathbf{j}}(T_{\sigma\mathbf{j},m-1}(u))| \\
&\quad + |G_{\mathbf{j}}(T_{\sigma\mathbf{j},m-1}(u)) - G_{\mathbf{j}}(G_{\sigma\mathbf{j},m-1}(u))| \\
&\le \varrho(\mathcal{T}, \mathcal{G}) + \lambda_2 |T_{\sigma\mathbf{j},m-1}(u) - G_{\sigma\mathbf{j},m-1}(u)| \\
&< \varrho(\mathcal{T}, \mathcal{G}) \frac{1}{1-\lambda_2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|T'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0)) - G'_{\sigma^k\mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))| \\
&\quad < |T'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0)) - G'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))| \\
&\qquad + |G'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0)) - G'_{\sigma^k\mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))| \\
&\quad < \varrho(\mathcal{T}, \mathcal{G}) + L(\mathcal{G}) |T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0) - G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0)|^\alpha \\
&\quad \le \varrho(\mathcal{T}, \mathcal{G}) + L(\mathcal{G}) \varrho(\mathcal{T}, \mathcal{G})^\alpha (1-\lambda_2)^{-\alpha} \\
&\quad \le [1 + L(\mathcal{G})(1-\lambda_2)^{-\alpha}] \cdot \varrho(\mathcal{T}, \mathcal{G})^\alpha,
\end{aligned}$$

using that $\varrho(\mathcal{T},\mathcal{G}) \le 1$ in the last step.

Since $\log|x/y| \le |(x-y)/y|$ we obtain for $k \ge 0$ that

$$\log\left|\frac{T'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))}{G'_{\sigma^k\mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))}\right| \le \frac{[1+L(\mathcal{G})(1-\lambda_2)^{-\alpha}]\cdot\varrho(\mathcal{T},\mathcal{G})^\alpha}{\lambda_1}$$

whence

$$\frac{1}{n}\log\left|\frac{T'_{\mathbf{i},n}(0)}{G'_{\mathbf{i},n}(0)}\right| = \frac{1}{n}\sum_{k=0}^{n-1}\log\left|\frac{T'_{\sigma^k\mathbf{i}}(T_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))}{G'_{\sigma^k\mathbf{i}}(G_{\sigma^{k+1}\mathbf{i},(n-1-k)}(0))}\right|$$
$$\le \frac{[1+L(\mathcal{G})(1-\lambda_2)^{-\alpha}]\cdot\varrho(\mathcal{T},\mathcal{G})^\alpha}{\lambda_1}.$$

This implies the desired inequality and completes the proof of the lemma. □

COROLLARY 3.2. *For any $\mathcal{G} = \{G_\mathbf{i}\}$ and $\mathcal{T} = \{T_\mathbf{i}\}$ in Γ the following hold:*

(i)

$$P(a\varphi_\mathcal{G} + b\varphi_\mathcal{T}) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\dots i_n}|G'_{\mathbf{i},n}(0)|^a\cdot|T'_{\mathbf{i},n}(0)|^b;$$

(ii)

$$\exp(-n\cdot C_2\cdot\|\mathbf{t}\|^\alpha) < \frac{|(G^\mathbf{t}_{\mathbf{i},n})'(0)|}{|G'_{\mathbf{i},n}(0)|} < \exp(n\cdot C_2\cdot\|\mathbf{t}\|^\alpha),$$

where $\|t\| = \max_i |t_i|$ and C_2 is the constant from Lemma 3.1(ii).

(iii) *For $0 < \varepsilon < \min\{1, s(\mathcal{G})\}$ and $a > 0$ define*

$$\delta := \left(\frac{-\frac{1}{2}\varepsilon\log\lambda_2}{aC_2}\right)^{1/\alpha}.$$

Then for arbitrary $\mathbf{i}, n$,

$$|(G^\mathbf{t}_{\mathbf{i},n})'(0)|^{-a} \le |G'_{\mathbf{i},n}(0)|^{-a-\varepsilon/2} \quad \textit{for all } \mathbf{t}\in B_\delta(\mathbf{0}).$$

(iv) *For all z such that $\Sigma_z \ne \emptyset$ and any $\mathbf{i},\mathbf{j}\in\Sigma_z$ such that $n = \max\{l\in\mathbb{Z} : i_k = j_k$ for all $k \le l\}$,*

$$|\Pi_\mathcal{G}(\mathbf{i}) - \Pi_\mathcal{G}(\mathbf{j})| \ge C_1^{-L(\mathcal{G})}|G'_{\mathbf{i},n}(0)|\cdot|\Pi_\mathcal{G}(\sigma^n\mathbf{i}) - \Pi_\mathcal{G}(\sigma^n\mathbf{j})|.$$

Proof. (i) It follows from (15) that $f = a\varphi_\mathcal{G} + b\varphi_\mathcal{T}$ satisfies

$$\exp\left[\sup_{\mathbf{i}\in[i_1\dots i_n]}\sum_{k=0}^{n-1}f(\sigma^k\mathbf{i})\right] = \sup_{\mathbf{i}\in[i_1\dots i_n]}|G'_{\mathbf{i},n}(\Pi^1(\sigma^n\mathbf{i}))|^a\cdot|T'_{\mathbf{i},n}(\Pi^1(\sigma^n\mathbf{i}))|^b.$$

Then Lemma 3.1(i) implies the result.

(ii) follows from Lemma 3.1(ii), letting $\mathcal{T} = \mathcal{G}^\mathbf{t}$ and using the fact that $\varrho(\mathcal{G}^\mathbf{t},\mathcal{G}) = \|\mathbf{t}\|$.

(iii) is an easy calculation using part (ii) of this corollary and the fact that $|G'_{\mathbf{i},n}| \le \lambda_2^n$.

(iv) follows from (10), (12) and Lemma 3.1(i). □

Recall that for $\mathcal{G} \in \Gamma$ a vector $\mathbf{t} \in \mathbb{R}^m$ is $\mathcal{G}$-admissible if $\mathcal{G}^{\mathbf{t}} \in \Gamma$. The set of $\mathcal{G}$-admissible $\mathbf{t}$ is open and non-empty. The next lemma establishes, in some sense, 'transversality in the parameter space'. It is an adaptation of Lemma 3.1 in Falconer [**10**], with a modification similar to Proposition 3.1 in Solomyak [**31**].

LEMMA 3.3. *Let $\mathcal{G} \in \Gamma$ and let $B \subset \mathbb{R}^m$ be a convex set of $\mathcal{G}$-admissible vectors. Fix an arbitrary z with $\Sigma_z \neq \emptyset$. Then there exists a constant C_3 such that*

$$\forall\, \mathbf{i}, \mathbf{j} \in \Sigma_z \text{ with } i_1 \neq j_1,\ \mathcal{L}_m\{\mathbf{t} \in B : |\Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i}) - \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{j})| \leq \rho\} \leq C_3\rho \text{ for } \rho > 0. \tag{16}$$

Proof. We have by (7) and (11), for all $\mathbf{i} \in \Sigma_z$,

$$\Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i}) = \lim_{n\to\infty} G^{\mathbf{t}}_{\mathbf{i},n}(0) = t_{i_1} + G_{\mathbf{i}}(t_{i_2} + G_{\sigma\mathbf{i}}(t_{i_3} + G_{\sigma^2\mathbf{i}}(t_{i_4} + \cdots))).$$

Thus, for all $k \leq m$,

$$\frac{\partial \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i})}{\partial t_k} = \delta_{i_1,k} + G'_{\mathbf{i}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma\mathbf{i}))[\delta_{i_2,k} + G'_{\sigma\mathbf{i}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma^2\mathbf{i}))(\delta_{i_3,k} + \cdots)], \tag{17}$$

where $\delta_{i,k}$ is the Kronecker symbol. We need to analyse

$$f(\mathbf{t}) := \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i}) - \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{j}),$$

where $i_1 \neq j_1$ and $\mathbf{i}, \mathbf{j} \in \Sigma_z$ (consequently, $i_k = j_k$ for all $k \leq 0$).

To simplify notation, assume that $i_1 = 1$ and $j_1 = 2$. Then we have from (17) that

$$\begin{aligned}
&\frac{\partial f(\mathbf{t})}{\partial t_1} - \frac{\partial f(\mathbf{t})}{\partial t_2} \\
&\quad = 2 + G'_{\mathbf{i}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma\mathbf{i}))[(\delta_{i_2,1} - \delta_{i_2,2}) + G'_{\sigma\mathbf{i}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma^2\mathbf{i}))((\delta_{i_3,1} - \delta_{i_3,2}) + \cdots)] \\
&\qquad - G'_{\mathbf{j}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma\mathbf{j}))[(\delta_{j_2,1} - \delta_{j_2,2}) + G'_{\sigma\mathbf{j}}(\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma^2\mathbf{j}))((\delta_{j_3,1} - \delta_{j_3,2}) + \cdots)].
\end{aligned}$$

Since $|G'_{\mathbf{i}}(x)| \leq \lambda_2 < 1/2$ we have for $\mathbf{i} \in \Sigma$ that

$$\begin{aligned}
\frac{\partial f(\mathbf{t})}{\partial t_1} - \frac{\partial f(\mathbf{t})}{\partial t_2} &\geq 2 - 2\lambda_2(1 + \lambda_2(1 + \lambda_2(1 + \cdots))) \\
&= 2 - 2\sum_{n\geq 1}\lambda_2^n = \frac{2(1 - 2\lambda_2)}{1 - \lambda_2} > 0,
\end{aligned} \tag{18}$$

Consider the transformation $T : B \to \mathbb{R}^m$ defined by

$$T(\mathbf{t}) = \mathbf{y} = (y_1, \ldots y_m),$$

where

$$y_1 = f(\mathbf{t}), \quad y_2 = t_1 + t_2, \quad y_k = t_k \quad \text{for } k \neq 1, 2.$$

It is easy to see from (18) and the convexity of B that T is one-to-one on B. The Jacobian determinant $\mathcal{J}T$ satisfies

$$|\mathcal{J}T(\mathbf{t})| = \left|\frac{\partial f(\mathbf{t})}{\partial t_1} - \frac{\partial f(\mathbf{t})}{\partial t_2}\right| \geq \frac{2(1 - 2\lambda_2)}{1 - \lambda_2} > 0.$$

Let

$$A_\rho := \{\mathbf{t} \in B : |\Pi_{\mathcal{G}^\mathbf{t}}(\mathbf{i}) - \Pi_{\mathcal{G}^\mathbf{t}}(\mathbf{j})| \le \rho\}.$$

We have

$$\begin{aligned}\mathcal{L}_m(A_\rho) &\le \max_{\mathbf{y}\in\mathbb{R}^m} |\mathcal{J}T(\mathbf{y})|^{-1}\mathcal{L}_m(TA_\rho)\\ &\le \frac{1-\lambda_2}{2(1-2\lambda_2)} \cdot \mathcal{L}_m\{\mathbf{y} \in TB : |y_1| \le \rho\}\\ &\le \frac{1-\lambda_2}{2(1-2\lambda_2)} \cdot 2^{m+1}\rho.\end{aligned}$$

The last inequality holds since $\mathbf{t}$ is $\mathcal{G}$-admissible hence $|y_k| = |t_k| \le 1$ for $k \ne 1, 2$ and $|y_2| \le |t_1| + |t_2| \le 2$. The estimate (16) is proved. □

COROLLARY 3.4. *Let $\mathcal{G} \in \Gamma$. Fix arbitrary z with $\Sigma_z \ne \emptyset$. Then for any $0 < a < 1$ there exists a constant $C_4 = C_4(\mathcal{G}, a)$ (independent of z) such that for all $\mathbf{i}, \mathbf{j} \in \Sigma_z$ with $i_1 \ne j_1$,*

$$\int_{\mathbf{t}\in B} \frac{d\mathbf{t}}{|\Pi_{\mathcal{G}^\mathbf{t}}(\mathbf{i}) - \Pi_{\mathcal{G}^\mathbf{t}}(\mathbf{j})|^a} \le C_4,$$

where B is a convex set of $\mathcal{G}$-admissible $\mathbf{t}$.

Proof. It is immediate, writing the integral in terms of the distribution function and using (16). □

The following lemma will be used in the proof of Theorem 1(ii). For technical reasons, in this lemma we use $\mathbf{v}$ instead of $\mathbf{t}$.

LEMMA 3.5. *Let $\mathcal{G} \in \Gamma$. There exists a constant $C_5 > 0$ such that for each $\theta > 0$ one can choose $\delta > 0$ so that*

$$\mathcal{L}_m\{\mathbf{v} \in B_\delta(\mathbf{0}) \subset \mathbb{R}^m : |\Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{i}) - \Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{j})| \le \rho\} \le C_5 \cdot \min\left\{\frac{\rho}{|G'_{\mathbf{i},n}(0)|^{1+\theta}}, 1\right\}$$

holds if $i_k = j_k$ for all $-\infty < k \le n$.

Proof. It follows from (10) that $G^\mathbf{v}_{\mathbf{i},n} \equiv G^\mathbf{v}_{\mathbf{j},n}$. Using (12) we obtain that

$$\begin{aligned}|\Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{i}) - \Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{j})| &= |G^\mathbf{v}_{\mathbf{i},n}(\Pi_{\mathcal{G}^\mathbf{v}}(\sigma^n\mathbf{i})) - G^\mathbf{v}_{\mathbf{i},n}(\Pi_{\mathcal{G}^\mathbf{v}}(\sigma^n\mathbf{j}))|\\ &= |(G^\mathbf{v}_{\mathbf{i},n})'(x)| \cdot |\Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{i}') - \Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{j}')|,\end{aligned}$$

for $\mathbf{i}' = \sigma^n\mathbf{i}$, $\mathbf{j}' = \sigma^n\mathbf{j}$ and $x \in (0, 1)$. Next, by Lemma 3.1(i),

$$|(G^\mathbf{v}_{\mathbf{i},n})'(x)| \ge C_1^{-L(\mathcal{G})} \cdot |(G^\mathbf{v}_{\mathbf{i},n})'(0)| \ge C_1^{-L(\mathcal{G})} \cdot |G'_{\mathbf{i},n}(0)|^{1+\theta}$$

for each $\mathbf{v} \in B_\delta(\mathbf{0})$, where δ is chosen as in Corollary 3.2(iii) for $a = 1$ and $\frac{1}{2}\varepsilon = \theta$, and so that all $\mathbf{t} \in B_\delta(\mathbf{0})$ are $\mathcal{G}$-admissible. Thus,

$$\begin{aligned}&\mathcal{L}_m\{\mathbf{v} \in B_\delta(\mathbf{0}) : |\Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{i}) - \Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{j})| \le \rho\}\\ &\quad\le \mathcal{L}_m\left\{\mathbf{v} \in B_\delta(\mathbf{0}) : |\Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{i}') - \Pi_{\mathcal{G}^\mathbf{v}}(\mathbf{j}')| \le \frac{C_1^{L(\mathcal{G})}\rho}{|G'_{\mathbf{i},n}(0)|^{1+\theta}}\right\}\\ &\quad\le \min\left\{\frac{C_1^{L(\mathcal{G})} \cdot C_3\rho}{|G'_{\mathbf{i},n}(0)|^{1+\theta}}, \mathcal{L}_m(B_\delta(\mathbf{0}))\right\}.\end{aligned}$$

Here we applied Lemma 3.3. □

4. *The case of 'small contractions'*

Here, after some preparation, we prove lemmas that will be used in the proof of Theorem 1(i).

Consider $\mathcal{G} \in \Gamma$ and the corresponding function $\varphi_{\mathcal{G}}$ on Σ (see §2 for definitions). Define the function $\Psi_{\mathcal{G}}(a) := P(a\varphi_{\mathcal{G}})$ for $a > 0$. It follows from Corollary 3.2(i) and the inequalities $\lambda_1^n \le |G'_{\mathbf{i},n}| \le \lambda_2^n$ that

$$\log \lambda_1 \le \frac{\Psi_{\mathcal{G}}(a+u) - \Psi_{\mathcal{G}}(a)}{u} \le \log \lambda_2 \tag{19}$$

for all positive a and u. Since $\Psi_{\mathcal{G}}(0) = \log m > 0$, this implies that the function $\Psi_{\mathcal{G}}$ has a unique zero, which we denote $s(\mathcal{G})$. One can show that $s(\mathcal{G}) = \lim_{n\to\infty} d_n(z)$ where $\sum_{i_1,\dots,i_n} |I_{z,i_1,\dots,i_n}|^{d_n(z)} = 1$, for all z such that $\Sigma_z \ne \emptyset$. We omit the calculation since it is rather standard and we do not need it for the proof.

In this section we let $\nu = \nu_{\mathcal{G}}$ be the Gibbs measure for $s(\mathcal{G})\varphi_{\mathcal{G}}$ on Σ (recall that this function is Hölder continuous, so the Gibbs measure exists). Since $P(s\varphi_{\mathcal{G}}) = 0$, by the definition of a Gibbs measure,

$$\nu([i_1 \dots i_n]) \in [C_6^{-1}, C_6] \cdot \exp\left[s \cdot \sum_{k=0}^{n-1} \varphi_{\mathcal{G}}(\sigma^k \mathbf{i})\right], \tag{20}$$

where $C_6 > 0$ depends only on $\mathcal{G}$. By Lemma 3.1(i), (13) and (15),

$$C_1^{-L(\mathcal{G})} \le \frac{\exp\left[\sum_{k=0}^{n-1} \varphi_{\mathcal{G}}(\sigma^k \mathbf{i})\right]}{|I_{z,i_1,\dots,i_n}|} \le C_1^{L(\mathcal{G})}.$$

Thus, there is a constant $C_7 = C_7(\mathcal{G}) > 0$, such that for each $u, v \in [0, 1]$ and $i_1 \dots i_n$,

$$\frac{|I_{u,i_1\dots i_n}|}{|I_{v,i_1\dots i_n}|} \in [C_7^{-1}, C_7], \quad \frac{|I_{u,i_1\dots i_n}|}{|G'_{\mathbf{i},n}(0)|} \in [C_7^{-1}, C_7] \quad \text{and} \quad \frac{\nu([i_1 \dots i_n])}{|I_{u,i_1\dots i_n}|^s} \in [C_7^{-1}, C_7]. \tag{21}$$

LEMMA 4.1. *Suppose that* $\mathcal{G} \in \Gamma$. *Then*

(i) $\overline{\dim}_B(\Lambda^{\mathcal{G}}(z)) \le s(\mathcal{G})$ *for all* z *such that* $\Sigma_z \ne \emptyset$;

(ii) *the function* $s(\mathcal{G})$ *is continuous on* Γ.

Proof. (i) We say that $I_{z,i_1,\dots,i_n}$ is an ε-interval if $|I_{z,i_1,\dots,i_n}| \le \varepsilon$ but $|I_{z,i_1,\dots,i_{n-1}}| > \varepsilon$. All ε-intervals (for fixed z) form a cover of $\Lambda^{\mathcal{G}}(z)$. It follows from (20) and (21) that the number of ε-intervals is of the order ε^{-s}, up to a multiplicative constant; see, for example, Proposition 2.1 in Hueter and Lalley [**18**] for details. By the definition of the upper box dimension, the desired inequality follows.

(ii) Recall that $\Psi_{\mathcal{G}}(a) = P(a\varphi_{\mathcal{G}})$. We have for $\mathcal{T} \in \Gamma$,

$$\begin{aligned}
|\Psi_{\mathcal{G}}(s(\mathcal{T}))| = |P(s(\mathcal{T})\varphi_{\mathcal{G}})| &= |P(s(\mathcal{T})\varphi_{\mathcal{G}}) - P(s(\mathcal{T})\varphi_{\mathcal{T}})| \\
&= \lim_{n\to\infty} \frac{1}{n} \left| \log \frac{\sum_{i_1\dots i_n} |G'_{\mathbf{i},n}(0)|^{s(\mathcal{T})}}{\sum_{i_1\dots i_n} |T'_{\mathbf{i},n}(0)|^{s(\mathcal{T})}} \right| \\
&< C_2 \cdot s(\mathcal{T}) \cdot \varrho(\mathcal{G}, \mathcal{T})^{\alpha}
\end{aligned}$$

by Lemma 3.1(ii). Therefore, by (19),

$$|s(\mathcal{T}) - s(\mathcal{G})| \le \frac{|\Psi_{\mathcal{G}}(s(\mathcal{T})) - \Psi_{\mathcal{G}}(s(\mathcal{G}))|}{|\log \lambda_2|} = \frac{|\Psi_{\mathcal{G}}(s(\mathcal{T}))|}{|\log \lambda_2|} \le \frac{C_2 \cdot s(\mathcal{T}) \cdot \varrho(\mathcal{G}, \mathcal{T})^{\alpha}}{|\log \lambda_2|}$$

which implies that the function s is continuous at $\mathcal{T}$. □

The next lemma contains the major part of the proof of Theorem 1(i).

LEMMA 4.2. *Let $\mathcal{G} \in \Gamma$ be such that $s(\mathcal{G}) \le 1$. Further, let z be arbitrary with $\Sigma_z \ne \emptyset$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\dim_H(\Lambda^{\mathcal{G}^{\mathbf{t}}}(z)) > s(\mathcal{G}) - \varepsilon$$

for Lebesgue-a.e. $\mathbf{t} \in B_\delta(\mathbf{0})$.

Proof. Let $s = s(\mathcal{G})$. Recall that ν is the Gibbs measure for the function $s\varphi_{\mathcal{G}}$. Consider the projected measure on Σ_+. It induces a measure μ on Σ_z since $\Sigma_z \sim \Sigma_+$. Then (20) and (21) imply

$$\mu([i_1, \dots, i_n]) \in (C_8^{-1}, C_8) \cdot |G'_{\mathbf{i},n}(0)|^s, \tag{22}$$

where the constant C_8 depends on $\mathcal{G}$. Denote the product measure $\mu \times \mu$ by μ_2. By the potential-theoretic characterization of the Hausdorff dimension (see Falconer [**12**, p. 64]), it is enough to show that

$$R(\mathbf{t}) = \iint_{\Sigma_z \times \Sigma_z} \frac{d\mu_2(\mathbf{i}, \mathbf{j})}{|\Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i}) - \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{j})|^{s-\varepsilon}} < \infty$$

for a.e. $\mathbf{t} \in B_\delta(\mathbf{0}) \subset \mathbb{R}^m$ for some $\delta > 0$. Indeed, this means that the $(s-\varepsilon)$-energy of the 'push-down' measure $\mu \circ \left(\Pi_{\mathcal{G}^{\mathbf{t}}}\right)^{-1}$ on $\Lambda^{\mathcal{G}^{\mathbf{t}}}(z)$ is finite. Following the scheme of Kaufman [**20**], we shall prove that

$$\int_{\mathbf{t} \in B_\delta(\mathbf{0})} R(\mathbf{t})\, d\mathbf{t} < \infty \quad \text{where } \delta = \left(\frac{-\frac{1}{2}\varepsilon \log \lambda_2}{(s-\varepsilon) C_2} \right)^{1/\alpha}$$

and C_2 comes from Lemma 3.1(ii) and Corollary 3.2(ii).

Let $\mathbf{i}, \mathbf{j} \in \Sigma_z$. Below we write $\mathbf{i} \wedge \mathbf{j} = \tau$ for a finite word $\tau = (\tau_1, \dots, \tau_n)$ if $i_k = j_k = \tau_k$ for all $1 \le k \le n$ and $i_{n+1} \ne j_{n+1}$. We have by Fubini's theorem that

$$\begin{aligned}
\int_{\mathbf{t} \in B_\delta(\mathbf{0})} R(\mathbf{t})\, d\mathbf{t} &= \sum_{n \ge 0} \sum_{\tau = (i_1 \dots i_n)} \iint_{\mathbf{i} \wedge \mathbf{j} = \tau} \left(\int_{\mathbf{t} \in B_\delta(\mathbf{0})} \frac{d\mathbf{t}}{|\Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{i}) - \Pi_{\mathcal{G}^{\mathbf{t}}}(\mathbf{j})|^{s-\varepsilon}} \right) d\mu_2(\mathbf{i}, \mathbf{j}) \\
&< \text{constant} \cdot \sum_{n \ge 0} \sum_{\tau = (i_1 \dots i_n)} \iint_{\mathbf{i} \wedge \mathbf{j} = \tau} \Bigg[|G'_{\mathbf{i},n}(0)|^{-(s-\varepsilon/2)} \\
&\qquad \times \int_{\mathbf{t} \in B_\delta(\mathbf{0})} \frac{d\mathbf{t}}{|\Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma^n \mathbf{i}) - \Pi_{\mathcal{G}^{\mathbf{t}}}(\sigma^n \mathbf{j})|^{s-\varepsilon}} \Bigg] d\mu_2(\mathbf{i}, \mathbf{j}) \\
&< \text{constant} \cdot \sum_{n \ge 0} \sum_{\tau = (i_1 \dots i_n)} \iint_{\mathbf{i} \wedge \mathbf{j} = \tau} |G'_{\mathbf{i},n}(0)|^{-(s-\varepsilon/2)}\, d\mu_2(\mathbf{i}, \mathbf{j})
\end{aligned}$$

$$
\begin{aligned}
&< \text{constant} \cdot \sum_{n\geq 0} \lambda_2^{n\varepsilon/2} \sum_{\tau=(i_1\ldots i_n)} \iint_{\mathbf{i}\wedge\mathbf{j}=\tau} \frac{1}{\mu([\tau])} \cdot d\mu_2(\mathbf{i},\mathbf{j}) \\
&= \text{constant} \cdot \sum_{n\geq 0} \lambda_2^{n\varepsilon/2} < \infty,
\end{aligned}
$$

where the first inequality follows from Corollary 3.2(iii) and (iv). The second one is just Corollary 3.4 and the third inequality comes from (22). This concludes the proof of the lemma. □

COROLLARY 4.3. *Let* $\mathcal{G} \in \Gamma$. *Fix any* z *such that* $\Sigma_z \neq \emptyset$. *Then*

$$\dim_H(\Lambda^{\mathcal{G}^{\mathbf{t}}}(z)) = s(\mathcal{G}^{\mathbf{t}}) \quad \text{for a.e. } \mathcal{G}\text{-admissible } \mathbf{t} \text{ such that } s(\mathcal{G}^{\mathbf{t}}) \leq 1.$$

Proof. The upper estimate for dimension follows from Lemma 4.1(i) since $\dim_H \leq \overline{\dim}_B$; it holds for all $\mathcal{G}$-admissible $\mathbf{t}$.

Suppose that the statement of the corollary is false. Then for some $\varepsilon > 0$ there exists a density point $\mathbf{t}_0$ of those $\mathbf{t}$ for which $\dim_H(\Lambda^{\mathcal{G}^{\mathbf{t}}}(z)) < s(\mathcal{G}^{\mathbf{t}}) - \varepsilon \leq 1 - \varepsilon$. Since $s(\mathcal{G}^{\mathbf{t}})$ is continuous by Lemma 4.1(ii) and the $\mathbf{t}$-perturbation of $\mathcal{G}^{\mathbf{t}_0}$ coincides with the $(\mathbf{t}_0 + \mathbf{t})$-perturbation of $\mathcal{G}$, this contradicts Lemma 4.2. □

5. *The case of 'large contractions'*

In the previous section we considered the case $s \leq 1$. Here we assume that $s > 1$ preparing for the proof of Theorem 1(ii). As we saw in (2) this implies that $r > 0$. Therefore, without loss of generality we may assume throughout this section that $r = r_1 > 0$ (see (6) for the definition of r_1).

We introduce some notation used throughout §5. Fix an arbitrary horseshoe map $F(x, y, z) = (\gamma(x, z), \tau(y, z), \psi(z))$ satisfying (H1)–(H3) and z such that $\Sigma_z \neq \emptyset$. Put $\Sigma_z^2 := \Sigma_z \times \Sigma_z$. Let $\mathcal{G}, \mathcal{T}$ be the elements of Γ defined by the first two component functions of F, that is

$$G_{\mathbf{i}}(x) := \gamma(x, \psi_{i_1}^{-1}(\Pi^3(\mathbf{i}))) \quad \text{and} \quad T_{\mathbf{i}}(y) := \tau(y, \psi_{i_1}^{-1}(\Pi^3(\mathbf{i}))). \tag{23}$$

The functions associated with them, as in (14), will be denoted $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{T}}$. In this section we let ν be the Gibbs measure of the function $\varphi_{\mathcal{G}} + r\varphi_{\mathcal{T}}$. The measure μ is the induced measure on Σ_z, that is, we first consider the projected measure on Σ_+ and then the measure on Σ_z using the identification $\Sigma_z \sim \Sigma_+$. (*Warning:* μ and ν in this section are different from μ and ν in the previous section.) By the definition of Gibbs measure and Lemma 3.1(i) there exists $C_9 > 0$ such that

$$\mu([i_1, \ldots, i_n]) \in (C_9^{-1}, C_9) \cdot |G'_{\mathbf{i},n}(0)| \cdot |T'_{\mathbf{i},n}(0)|^r. \tag{24}$$

In this section $\mathbf{t}_1, \mathbf{t}_2$ are always vectors from $\mathbb{R}^m$, and whenever we write $\mathbf{t}$ it is always a vector from $\mathbb{R}^m \times \mathbb{R}^m$ with $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$. We use the l^∞ norm on $\mathbb{R}^{2m}$. Put $\Pi^{\mathbf{t}}(\mathbf{i}) := (\Pi_{\mathcal{G}^{\mathbf{t}_1}}(\mathbf{i}), \Pi_{\mathcal{T}^{\mathbf{t}_2}}(\mathbf{i}))$ for $\mathbf{i} \in \Sigma$. We write $\Lambda^{\mathbf{t}}$ for the basic set of $F^{\mathbf{t}}$. Further, $\Lambda^{\mathbf{t}}(z)$ denotes the z-horizontal section of $\Lambda^{\mathbf{t}}$, so that $\Lambda^{\mathbf{t}}(z) = \Pi^{\mathbf{t}}(\Sigma_z)$. For $\mathbf{t} \in \mathbb{R}^{2m}$ we denote by $\mathcal{G}^{\mathbf{t}}$ and $\mathcal{T}^{\mathbf{t}}$ the $\mathbf{t}$-perturbations of $\mathcal{G}$ and $\mathcal{T}$ and write $\varphi_{\mathcal{G}}^{\mathbf{t}}, \varphi_{\mathcal{T}}^{\mathbf{t}}$ for the corresponding functions on Σ. Let $r_1(\mathbf{t})$ be the solution of the equation $P(\varphi_{\mathcal{G}} + r_1(\mathbf{t})\varphi_{\mathcal{T}}) = 0$.

LEMMA 5.1. *The function* $\mathbf{t} \mapsto r_1(\mathbf{t})$ *is continuous.*

Proof. As in Lemma 4.1(ii), we prove a more general statement. The calculations are similar to those of Lemma 4.1(ii), so we shall be brief.

Let $\mathcal{G}, \mathcal{T} \in \Gamma$. Denote by $r_1(\mathcal{G}, \mathcal{T})$ the unique solution of the equation

$$P(\varphi_{\mathcal{G}} + r_1(\mathcal{G}, \mathcal{T})\varphi_{\mathcal{T}}) = 0.$$

We shall prove that $(\mathcal{G}, \mathcal{T}) \mapsto r_1(\mathcal{G}, \mathcal{T})$ is continuous in the metric induced by $\varrho(\mathcal{G}, \mathcal{H})$. Let $\mathcal{G}, \mathcal{T}, \mathcal{H}, \mathcal{U} \in \Gamma$. Using Corollary 3.2(i), it is easy to see that

$$\begin{aligned} |P(\varphi_{\mathcal{G}} + r_1(\mathcal{H}, \mathcal{U})\varphi_{\mathcal{T}})| &= |P(\varphi_{\mathcal{G}} + r_1(\mathcal{G}, \mathcal{T})\varphi_{\mathcal{T}}) - P(\varphi_{\mathcal{G}} + r_1(\mathcal{H}, \mathcal{U})\varphi_{\mathcal{T}})| \\ &\geq |\log \lambda_2||r_1(\mathcal{G}, \mathcal{T}) - r_1(\mathcal{H}, \mathcal{U})|. \end{aligned}$$

On the other hand, again using Corollary 3.2(i) and Lemma 3.1(ii), we obtain

$$\begin{aligned} |P(\varphi_{\mathcal{G}} + r_1(\mathcal{H}, \mathcal{U})\varphi_{\mathcal{T}})| &= |P(\varphi_{\mathcal{G}} + r_1(\mathcal{H}, \mathcal{U})\varphi_{\mathcal{T}}) - P(\varphi_{\mathcal{H}} + r_1(\mathcal{H}, \mathcal{U})\varphi_{\mathcal{U}})| \\ &< C_2 r_1(\mathcal{H}, \mathcal{U})[\varrho(\mathcal{G}, \mathcal{H})^{\alpha} + r_1(\mathcal{H}, \mathcal{U})\varrho(\mathcal{T}, \mathcal{U})^{\alpha}]. \end{aligned}$$

Combining the inequalities yields the desired continuity. □

Below we denote $B_\delta^2(\mathbf{0}) := B_\delta(\mathbf{0}) \times B_\delta(\mathbf{0})$.

LEMMA 5.2. *Fix* $\theta > 0$ *and let* $\delta > 0$ *be as in Lemma 3.5. Further, fix* $\mathbf{i}, \mathbf{j} \in \Sigma_z$ *such that* $\mathbf{i} \wedge \mathbf{j} = (i_1, \ldots, i_n)$. *Let* $K := |G'_{\mathbf{i},n}(0)|^{1+\theta}$ *and* $L := |T'_{\mathbf{i},n}(0)|^{1+\theta}$. *Then for each* $1 < a < 2$ *there is a positive constant* $C_1' = C_1'(a)$ *(independent of* $\mathbf{i}, \mathbf{j}, n$*) such that*

$$\mathcal{I} := \int_{\mathbf{t} \in B_\delta^2(\mathbf{0})} \frac{d\mathbf{t}}{\|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\|^a} \leq C_1' \min\left\{\frac{1}{K \cdot L^{a-1}}, \frac{1}{L \cdot K^{a-1}}\right\}.$$

Proof. Without loss of generality, we may assume $K \geq L$. Using Lemma 3.5 twice (for $\mathbf{v} = \mathbf{t}_1$ and $\mathbf{v} = \mathbf{t}_2$ and the families $\mathcal{G}, \mathcal{T}$ respectively), one can see that

$$\begin{aligned} \mathcal{I} &= a \int_{\rho=0}^{\infty} \mathcal{L}_{2m}\{\mathbf{t} \in B_\delta^2(\mathbf{0}) : \|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\| \leq \rho\}\rho^{-1-a}\, d\rho \\ &\leq C_5^2 \cdot a \cdot \int_{\rho=0}^{\infty} \min\left\{\frac{\rho}{K}, 1\right\} \cdot \min\left\{\frac{\rho}{L}, 1\right\} \cdot \rho^{-1-a}\, d\rho \\ &\leq C_5^2 \cdot a \cdot \left[\int_0^L \frac{\rho^2}{KL}\rho^{-1-a}\, d\rho + \int_L^{\infty} \frac{\rho}{K}\rho^{-1-a}\, d\rho\right] \\ &= C_1'(a) \cdot \frac{1}{KL^{a-1}} \leq C_1'(a) \cdot \frac{1}{LK^{a-1}}. \end{aligned}$$

□

Now we are ready for the main lemma in the proof of Theorem 1(ii).

LEMMA 5.3.

(i) *Suppose that* $0 < r \leq 1$. *Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\dim_H(\Lambda^{\mathbf{t}}(z)) \geq 1 + r - \varepsilon \quad \text{for a.e. } \mathbf{t} \in B_\delta^2(\mathbf{0}).$$

(ii) *If* $r > 1$ *then*

$$\mathcal{L}_2(\Lambda^{\mathbf{t}}(z)) > 0 \quad \text{for a.e. } \mathbf{t} \in B_\delta^2(\mathbf{0}).$$

Proof. (i) Fix ε such that $1 \geq r = r_1 > \varepsilon > 0$ and then choose θ satisfying

$$0 < \theta < \frac{\varepsilon \log \lambda_2}{2 \log \lambda_1}. \tag{25}$$

Determine δ from θ as in Lemma 3.5. The desired estimate follows from the potential-theoretic characterization of the Hausdorff dimension if we show that

$$\mathcal{I} := \int_{\mathbf{t} \in B_\delta^2(\mathbf{0})} \iint_{\Sigma_z^2} \frac{d\mu_2(\mathbf{i}, \mathbf{j})}{\|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\|^{1+r_1-\varepsilon}} \, d\mathbf{t} < \infty.$$

Using Fubini's theorem we obtain

$$\begin{aligned}
\mathcal{I} &= \sum_{n \geq 0} \sum_{\tau=(i_1 \ldots i_n)} \iint_{\mathbf{i} \wedge \mathbf{j} = \tau} \left(\int_{\mathbf{t} \in B_\delta^2(\mathbf{0})} \frac{d\mathbf{t}}{\|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\|^{1+r_1-\varepsilon}} \right) d\mu_2(\mathbf{i}, \mathbf{j}) \\
&< \text{constant} \cdot \sum_{n \geq 0} \sum_{\tau=(i_1 \ldots i_n)} \iint_{\mathbf{i} \wedge \mathbf{j} = \tau} \frac{d\mu_2(\mathbf{i}, \mathbf{j})}{|G'_{\mathbf{i},n}(0)|^{1+\theta} \cdot |T'_{\mathbf{i},n}(0)|^{(1+\theta)(r_1-\varepsilon)}},
\end{aligned}$$

by Lemma 5.2 with $a = 1 + r_1 - \varepsilon \in (1, 2)$. It follows from (24) that

$$\mu_2(\{(\mathbf{i}, \mathbf{j}) \in \Sigma_z^2 : \mathbf{i} \wedge \mathbf{j} = (i_1, \ldots, i_n)\}) \leq C_9 |G'_{\mathbf{i},n}(0)| \cdot |T'_{\mathbf{i},n}(0)|^{r_1} \cdot \mu([i_1, \ldots, i_n]).$$

Thus,

$$\mathcal{I} \leq \text{constant} \cdot \sum_{n \geq 0} \sum_{(i_1 \ldots i_n)} \frac{|G'_{\mathbf{i},n}(0)| \cdot |T'_{\mathbf{i},n}(0)|^{r_1} \cdot \mu([i_1, \ldots, i_n])}{|G'_{\mathbf{i},n}(0)|^{1+\theta} \cdot |T'_{\mathbf{i},n}(0)|^{(r_1-\varepsilon)(1+\theta)}}. \tag{26}$$

Since $\theta < \varepsilon/2$ and $r \leq 1$ we have $r_1 - (1+\theta)(r_1 - \varepsilon) = \varepsilon + \varepsilon\theta - \theta r_1 \geq \varepsilon/2$. Therefore,

$$\frac{|G'_{\mathbf{i},n}(0)| \cdot |T'_{\mathbf{i},n}(0)|^{r_1}}{|G'_{\mathbf{i},n}(0)|^{1+\theta} \cdot |T'_{\mathbf{i},n}(0)|^{(r_1-\varepsilon)(1+\theta)}} \leq \lambda_1^{-n\theta} \lambda_2^{n\varepsilon/2} < \lambda_3^n$$

for some $\lambda_3 < 1$, since $\lambda_2^{\varepsilon/2} < \lambda_1^\theta$ by (25). Now (26) implies

$$\mathcal{I} \leq \text{constant} \sum_{n \geq 0} \lambda_3^n \sum_{(i_1 \ldots i_n)} \mu([i_1, \ldots, i_n]) = \text{constant} \sum_{n \geq 0} \lambda_3^n < \infty$$

and the proof of part (i) is complete.

(ii) The proof of this part follows the scheme of Mattila [**21**, p. 130]. As above, we assume that $r = r_1$ so $r_1 > 1$. Let $\mu_{\mathbf{t}} := \mu \circ (\Pi^{\mathbf{t}})^{-1}$ be the push-down measure supported on $\Lambda^{\mathbf{t}}(z)$ and set

$$\underline{D}(\mu_{\mathbf{t}}, \mathbf{u}) := \liminf_{\rho \to 0} \frac{\mu_{\mathbf{t}}(B_\rho(\mathbf{u}))}{\mathcal{L}_2(B_\rho(\mathbf{u})))},$$

where $\mathbf{u} \in \mathbb{R}^2$ and $B_\rho(\mathbf{u})$ is the ball in $\mathbb{R}^2$ in the l^∞-metric. To prove that $\mathcal{L}_2(\Lambda^{\mathbf{t}}(z)) > 0$ for a.e. $\mathbf{t} \in B_\delta^2(\mathbf{0})$ it is enough to show that $\mu_{\mathbf{t}}$ is absolutely continuous with respect to $\mathcal{L}_2$ for a.e. $\mathbf{t} \in B_\delta^2(\mathbf{0})$. However, to see this it is enough to prove the following statement (see Mattila [**21**, p. 36]):

$$\mathcal{I} := \int_{\mathbf{t} \in B_\delta^2(\mathbf{0})} \int_{\Lambda^{\mathbf{t}}(z)} \underline{D}(\mu_{\mathbf{t}}, \mathbf{u}) \, d\mu_{\mathbf{t}} \, d\mathbf{t} < \infty.$$

We fix θ so that

$$0 < \theta < (r_1 - 1)\frac{\log \lambda_2}{\log \lambda_1 + \log \lambda_2}$$

and choose δ for this θ as in Lemma 3.5. Then we apply Fatou's lemma and make a change of variable to get

$$\mathcal{I} \le \liminf_{\rho \to 0} \frac{1}{4\rho^2} \iint_{\Sigma_z^2} \mathcal{L}_{2m}\{\mathbf{t} \in B_\delta^2(\mathbf{0}) : \|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\| \le \rho\}\, d\mu_2(\mathbf{i}, \mathbf{j}). \tag{27}$$

For any $\mathbf{i}, \mathbf{j} \in \Sigma_z^2$ with $\mathbf{i} \wedge \mathbf{j} = (i_1, \dots, i_n)$, applying Lemma 3.5 twice we obtain

$$\mathcal{L}_{2m}\{\mathbf{t} \in B_\delta^2(\mathbf{0}) : \|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})\| \le \rho\} \le C_5^2 \frac{\rho^2}{|G'_{\mathbf{i},n}(0)|^{1+\theta} |T'_{\mathbf{i},n}(0)|^{1+\theta}}.$$

This and (27) yield

$$\begin{aligned}
\mathcal{I} &\le \text{constant} \cdot \sum_{n \ge 0} \sum_{(i_1 \dots i_n)} \frac{\mu([i_1, \dots i_n])}{|G'_{\mathbf{i},n}(0)|^{1+\theta} |T'_{\mathbf{i},n}(0)|^{1+\theta}} \mu([i_1, \dots i_n]) \\
&\le \text{constant} \cdot \sum_{n \ge 0} \sum_{(i_1 \dots i_n)} \frac{|G'_{\mathbf{i},n}(0)||T'_{\mathbf{i},n}(0)|^{r_1} \mu([i_1, \dots i_n])}{|G'_{\mathbf{i},n}(0)|^{1+\theta} |T'_{\mathbf{i},n}(0)|^{1+\theta}} \\
&\le \text{constant} \cdot \sum_{n \ge 0} \lambda_1^{-n\theta} \lambda_2^{(r_1 - 1 - \theta)n} \sum_{(i_1 \dots i_n)} \mu([i_1, \dots i_n]) < \infty,
\end{aligned}$$

since $\lambda_2^{(r_1-1-\theta)} < \lambda_1^\theta$ by the choice of θ. This completes the proof of part (ii). □

COROLLARY 5.4. *Let $z \in [0, 1]$ be such that $\Sigma_z \neq \emptyset$ and let Q be the set of F-admissible* $\mathbf{t}$. *Then:*

(i) $\dim_H(\Lambda^{\mathcal{G}^{\mathbf{t}}}(z)) \ge 1 + \min\{r(\mathbf{t}), 1\}$ *for a.e.* $\mathbf{t} \in Q$ *such that* $r(\mathbf{t}) > 0$*;*

(ii) $\mathcal{L}_2(\Lambda^{\mathbf{t}}(z)) > 0$ *for a.e.* $\mathbf{t} \in Q$ *such that* $r(\mathbf{t}) > 1$.

Proof. (i) follows from Lemma 5.3(i) and Lemma 5.1 (the continuity of $r(\mathbf{t})$) similar to the proof of Corollary 4.3.

(ii) Suppose that the statement is false. Then there is an admissible point $\mathbf{t}_0$ such that $r(\mathbf{t}_0) > 1$ but in any ball around $\mathbf{t}_0$ there is a set of $\mathbf{t}$ having positive $\mathcal{L}_2$-measure for which $\mathcal{L}_2\left(\Lambda^{\mathbf{t}}(z)\right) = 0$. Considering $\Lambda^{\mathbf{t}_0}$ instead of Λ leads to a contradiction with Lemma 5.3(ii). □

6. *Conclusion of the proof*

In the following lemma we collect estimates above for the upper box dimension. They are rather standard and hold for all, not just almost all, horseshoes.

LEMMA 6.1. *Let Λ be a horseshoe and $z \in [0, 1]$ is such that $\Sigma_z \neq 0$. Then:*

(i) $\overline{\dim}_B(\Lambda(z)) \le s = \max\{s_1, s_2\}$*;*

(ii) $\overline{\dim}_B(\Lambda(z)) \le 1 + r = 1 + \max\{r_1, r_2\}$ *provided* $s > 1$ *(and hence* $r > 0$*);*

(iii) $\overline{\dim}_B(\Lambda) \le \delta_u + \begin{cases} s \\ 1 + r & \text{if } s > 1. \end{cases}$

Here r_i, s_i are defined in (6) and δ_u is from (3).

Proof. (i) Let $I^1_{z,i_1,\dots,i_n} := G_{\mathbf{i},n}([0,1])$ and $I^2_{z,i_1,\dots,i_n} := T_{\mathbf{i},n}([0,1])$, where $z = \Pi^3(\mathbf{i})$. Then

$$\bigcup_{i_1,\dots,i_n} (I^1_{z,i_1,\dots,i_n} \times I^2_{z,i_1,\dots,i_n}) \supset \Lambda(z).$$

We say that $I^j_{z,i_1,\dots,i_n}$ is an ε-interval if $|I^j_{z,i_1,\dots,i_{n-1}}| \le \varepsilon$ but $|I^j_{z,i_1,\dots,i_{n-1}}| > \varepsilon$ (here j can be 1 or 2). The number of ε-intervals I^j is not greater than constant$\cdot\varepsilon^{-s_j}$, similar to the proof of Lemma 4.1(i). It follows that $\Lambda(z)$ can be covered by constant $\cdot\,(\varepsilon^{-s_1} + \varepsilon^{-s_2})$ squares of side ε which implies the desired estimate.

(ii) Let $\mathcal{A}_1$ be the set of finite sequences $u = (i_1,\dots,i_n)$ (for any n) such that $I^1_{z,i_1,\dots,i_n}$ is an ε-interval and $|I^2_{z,i_1,\dots,i_n}| \ge |I^1_{z,i_1,\dots,i_n}|$. Let $\mathcal{A}_2$ be the set of finite sequences $u = (i_1,\dots,i_n)$ (for any n) such that $I^2_{z,i_1,\dots,i_n}$ is an ε-interval and $|I^1_{z,i_1,\dots,i_n}| \ge |I^2_{z,i_1,\dots,i_n}|$. It is easy to see that any $\mathbf{i} \in \Sigma_z$ belongs to a cylinder set $[u]$, with $u \in \mathcal{A}_1 \cup \mathcal{A}_2$. Thus, $\Lambda(z)$ is covered by

$$\bigcup_{u\in\mathcal{A}_1} (I^1_{z,u} \times I^2_{z,u}) \cup \bigcup_{u\in\mathcal{A}_2} (I^1_{z,u} \times I^2_{z,u}).$$

The rectangles $I^1_{z,u} \times I^2_{z,u}$ for $u \in \mathcal{A}_1 \cup \mathcal{A}_2$ have the shorter side of length between $\lambda_1\varepsilon$ and ε. If $u \in \mathcal{A}_1$ the rectangle is stretched in the y direction, and if $u \in \mathcal{A}_2$ the rectangle is stretched in the x direction. Let $N(\varepsilon)$ be the minimal number of squares of side ε needed to cover $\Lambda(z)$. Then

$$\begin{aligned} N(\varepsilon) &\le \sum_{u\in\mathcal{A}_1} \left(\frac{|I^2_{z,u}|}{|I^1_{z,u}|} + 1 \right) + \sum_{u\in\mathcal{A}_2} \left(\frac{|I^1_{z,u}|}{|I^2_{z,u}|} + 1 \right) \\ &\le \frac{2}{\lambda_1\varepsilon} \left[\sum_{u\in\mathcal{A}_1} |I^2_{z,u}| + \sum_{u\in\mathcal{A}_2} |I^1_{z,u}| \right]. \end{aligned} \tag{28}$$

Let μ be the measure on Σ_z as in §5. Then

$$\mu([u]) \ge C_2' |I^1_{z,u}| \cdot |I^2_{z,u}|^{r_1}$$

for some $C_2' > 0$ by (21). Therefore,

$$1 \ge \sum_{u\in\mathcal{A}_2} \mu([u]) \ge C_2' \sum_{u\in\mathcal{A}_2} |I^1_{z,u}| \cdot |I^2_{z,u}|^{r_1} \ge C_2'(\varepsilon\lambda_1)^{r_1} \sum_{u\in\mathcal{A}_2} |I^1_{z,u}|,$$

hence $\sum_{u\in\mathcal{A}_2} |I^1_{z,u}| \le$ constant $\cdot\, \lambda_1^{-r_1}\varepsilon^{-r_1}$. Similarly, using the Gibbs measure for $r_2\varphi_{\mathcal{G}} + \varphi_{\mathcal{T}}$ we obtain that $\sum_{u\in\mathcal{A}_1} |I^2_{z,u}| \le$ constant$\cdot\lambda_1^{-r_2}\varepsilon^{-r_2}$. Combining these inequalities with (28) yields

$$N(\varepsilon) \le \text{constant} \cdot \varepsilon^{-(1+r)},$$

where $r = \max\{r_1, r_2\}$. Since $\overline{\dim}_B(\Lambda(z)) = \limsup_{\varepsilon\to 0} \log N(\varepsilon)/\log 1/\varepsilon$, the statement is proved.

(iii) A straightforward calculation based on (1), (H2) and (H3) shows that the unstable manifolds for our horseshoe map are smooth curves which form angles bounded away from zero with horizontal planes (the stable manifolds). Then the statement routinely follows from parts (i) and (ii) and Theorem 3. $\square$

Now we are ready to prove our theorems which follow by Fubini's Theorem from Lemma 6.1, Corollary 4.3, and Corollary 5.4.

Proof of Theorem 1. The upper estimates are obtained in Lemma 6.1 so we only have to show the lower estimates. Fix a horseshoe map F and let Q be the set of F-admissible $\mathbf{t}$. Then Q is a non-empty bounded open set. For z fixed, we define $\mathcal{G}$ and $\mathcal{T}$ as in (23). Further, we write $s(\mathbf{t}) := \max\{s(\mathcal{G}^{\mathbf{t}_1}), s(\mathcal{T}^{\mathbf{t}_2})\}$ for $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2) \in \mathbb{R}^{2m}$. Then Corollary 4.3 implies that $\dim_H(\Lambda^{\mathbf{t}}(z)) \geq s(\mathbf{t})$ for a.e. $\mathbf{t}$ such that $s(\mathbf{t}) \leq 1$ since $\Lambda^{\mathcal{G}^{\mathbf{t}_1}}(z)$ is the projection of $\Lambda(z)$ onto the x axis and $\Lambda^{\mathcal{G}^{\mathbf{t}_2}}(z)$ is the projection of $\Lambda(z)$ onto the y axis. Thus, we have, together with Corollary 5.4(i), that for every $z \in \mathrm{Rep}(\psi)$, for Lebesgue almost every $\mathbf{t} \in Q$,

$$\dim_H(\Lambda^{\mathbf{t}}(z)) \geq \begin{cases} s(\mathbf{t}) & \text{if } s(\mathbf{t}) \leq 1 \\ 1 + r(\mathbf{t}) & \text{if } s(\mathbf{t}) > 1. \end{cases} \tag{29}$$

Recall (Theorem 3) that $0 < \mathcal{H}^{\delta_u}(\mathrm{Rep}(\psi)) < \infty$. Define $A := Q \times \mathrm{Rep}(\psi)$. Then $(\mathcal{L}_{2m} \times \mathcal{H}^{\delta_u})(A) > 0$. We call a pair $(\mathbf{t}, z) \in A$ 'good' if (29) holds. By Fubini's theorem, for $\mathcal{L}_{2m}$-almost every $\mathbf{t} \in Q$, for $\mathcal{H}^{\delta_u}$-a.e. $z \in \mathrm{Rep}(\psi)$, the pair $(\mathbf{t}, z)$ is 'good.' Then by the 'generalized slicing theorem' (see Corollary 7.12 in Falconer [**12**]) we get the desired lower bound. The proof of Theorem 1 is complete. □

Proof of Theorem 2. The proof is the same as for Theorem 1, except that here $\delta_u = 1$. Part (iii) follows from Fubini's theorem and Corollary 5.4(ii). □

Acknowledgements. We are grateful to the referee for the careful reading of the manuscript and many helpful comments. KS was supported by OTKA Foundation grant F019099. BS was supported by NSF Grant 9500744.

References

[1] L. M. Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergod. Th. & Dynam. Sys.* **16** (1996), 871–927.

[2] T. Bedford. Applications of dynamical systems theory to fractals—a study of cookie-cutter Cantor sets. *Fractal Geometry and Analysis*. Eds J. Bélair and S. Dubuc. Kluwer, Dordrecht, 1991, pp. 1–44.

[3] T. Bedford and M. Urbański. The box and Hausdorff dimension of self-affine sets. *Ergod. Th. & Dynam. Sys.* **10** (1990), 627–644.

[4] H. G. Bothe. The Hausdorff dimension of certain solenoids. *Ergod. Th. & Dynam. Sys.* **15** (1995), 449–474.

[5] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, New York, 1975.

[6] R. Bowen. Hausdorff dimension of quasi-circles. *Publ. Math. I.H.E.S.* **50** (1979), 11–26.

[7] W. Chin, B. Hunt and J. A. Yorke. Correlation dimension for iterated function systems. *Trans. Amer. Math. Soc.* **349** (1997), 1783–1796.

[8] G. A. Edgar. Fractal dimension of self-similar sets: some examples. *Suppl. Rend. Circ. Mat. Palermo, Ser. II* **28** (1992), 341–358.

[9] K. Falconer. The Hausdorff dimension of some fractals and attractors of overlapping construction. *J. Stat. Phys.* **47**(1/2) (1987).

[10] K. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Camb. Phil. Soc.* **103** (1988), 339–350.

[11] K. Falconer. Dimensions and measures of quasi self-similar sets. *Proc. Amer. Math. Soc.* **106** (1989), 543–554.

[12] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, Chichester, 1990.

[13] K. Falconer. Bounded distortion and dimension for non-conformal repellers. *Math. Proc. Camb. Phil. Soc.* **115** (1994), 315–334.

[14] K. Falconer. *Techniques in Fractal Geometry*. Wiley, Chichester, 1997.

[15] D. Gatzouras and Y. Peres. The variational principle for Hausdorff dimension. *Ergodic Theory of* $\mathbb{Z}^d$*-Actions (LMS Lecture Notes Series, 228).* Cambridge University Press, 1996, pp. 259–272.

[16] D. Gatzouras and Y. Peres. Invariant measures of full dimension for some expanding maps. *Ergod. Th. & Dynam. Sys.* **17** (1997), 147–167.

[17] H. Hu. Box dimension and topological pressure for some expanding maps. *Comm. Math. Phys.* **191**(2) (1998), 397–407.

[18] I. Hueter and S. P. Lalley. Falconer's formula for the Hausdorff dimension of a self-affine set in $\mathbb{R}^2$. *Ergod. Th. & Dynam. Sys.* **15** (1995), 77–97.

[19] B. Hunt. Pointwise dimension of attractors of iterated function systems with overlap. *Preprint*, 1999.

[20] R. Kaufman. On Hausdorff dimension of projections. *Mathematika* **15** (1968), 153–155.

[21] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, 1995.

[22] H. McCluskey and A. K. Manning. Hausdorff dimension for horseshoes. *Ergod. Th. & Dynam. Sys.* **3** (1983), 251–260.

[23] K. McMullen. The Hausdorff dimension of general Sierpinski gasket. *Nagoya Math. J.* **96** (1984), 1–9.

[24] J. Palis and M. Viana. On the continuity of Hausdorff dimension and limit capacity of horseshoe. *Dynamical Systems, Valparaiso 1986 (Lecture Notes in Mathematics, 1331)*. Springer, 1988, pp. 150–160.

[25] Ya. Pesin. *Dimension Theory in Dynamical Systems*. The University of Chicago Press, 1997.

[26] M. Pollicott and H. Weiss. The dimensions of some self affine limit sets in the plane. *J. Stat. Phys.* **77** (1994), 841–866.

[27] F. Przytycki and M. Urbański. On Hausdorff dimension of some fractal sets. *Studia Mathematica* **93** (1989), 155–186.

[28] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.* **2** (1982), 99–107.

[29] K. Simon. Hausdorff dimension for non-invertible maps. *Ergod. Th. & Dynam. Sys.* **13** (1993), 199–212.

[30] K. Simon. The Hausdorff dimension of the general Smale–Williams solenoid. *Proc. Amer. Math. Soc.* **125:4** (1997), 1221–1228.

[31] B. Solomyak. Measure and dimension for some fractal families. *Math. Proc. Cambridge Phil. Soc.* **124**(3) (1998), 531–546.

[32] M. V. Yakobson. Invariant measures for some one dimensional attractors. *Ergod. Th. & Dynam. Sys.* **2** (1982), 317–337.

[33] Y. Zhang. Dynamical upper bounds for Hausdorff dimension of invariant sets. *Ergod. Th. & Dynam. Sys.* **17** (1997), 739–756.