# MULTIFRACTAL ANALYSIS OF BIRKHOFF AVERAGES FOR SOME SELF-AFFINE IFS

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ABSTRACT. In this paper we consider self-affine IFS $\{S_i\}_{i=1}^{m_0}$ on the plane of the form $S_i(x_1, x_2) = (\lambda_i x_1 + t_i^{(1)}, \xi_i x_2 + t_i^{(2)})$, where $0 < |\lambda_i|, |\xi_i| < \frac{1}{2}$. We describe the multifractal analysis of Birkhoff averages of the continuous functions. In Section 6 we compute it numerically in a special case (see Figure 1).

## 1. INTRODUCTION

It is in general an open problem to find the dimension spectrum for the Birkhoff averages of a Hölder continuous function on a nonconformal repeller. Namely, let $M \subset \mathbb{R}^2$ be open. Let $\Lambda$ be the nonconformal repeller of the $\mathcal{C}^{1+\alpha}$ map $F : M \to M$. For a Hölder continuous function $f : M \to \mathbb{R}$ and for a $\beta \in \mathbb{R}$ we define the set of those $x$ where the Birkhoff averages are equal to $\beta$:

$$K_\beta := \left\{ x \in \Lambda : \lim_{n\to\infty} \sum_{k=0}^{n-1} f(F^k x) = \beta \right\}. \tag{1}$$

The function

$$\beta \to \dim_{\mathrm{H}} K_\beta$$

is called the dimension spectrum of the Birkhoff averages of $f$. The goal of our paper is to answer this problem in some special cases. The dimension theory of non-conformal repellers is very difficult. We mostly have almost all type results in the self-affine case (i.e. when the local inverses are affine maps) in the sense of Falconer's paper [6] and upper estimates for the dimension of nonconformal repellers [7] ,[1]. For the multifractal case there has been work on a class of examples relating

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to Sierpiński carpets in $\mathbb{R}^2$, [10] and $\mathbb{R}^d$, [13]. In [8] Falconer looks at the $L^q$ spectrum for self-affine measures in the same almost all sense as [6] . Our research was motivated by a recent preprint of Barreira and Radu [2] in which the authors gave an almost all type lower bound on $\dim_{\mathrm{H}} K_\beta$ in the sense introduced by Falconer [6] assuming the following:

**A1:** $\|(DF_x)^{-1}\| < \frac{1}{2}$.

**A2:** The local inverses of $F$ can be presented in the form

$$S_j(x_1,x_2) = (\gamma_j(x_1), \tau_j(x_2)),\ j = 1,\dots,m_0,$$

**A3:** Let $r_\gamma$, and $r_\tau$ be the roots of the appropriate pressure formulae determined by the iterated function systems $\{\gamma_j\}_{j=1}^{m_0}$ and $\{\tau_j\}_{j=1}^{m_0}$. Then

$$r_\gamma, r_\tau < 1. \tag{2}$$

We remark that Assumption A3 can hold only if

$$\dim_{\mathrm{H}} \Lambda < 1 \tag{3}$$

that is A3 requires that $\Lambda$ is "small". Under these assumptions Barreira, Radu gave natural lower bound on the Hausdorff dimension of $K_\beta$ which is analogous to the one which holds in the conformal case and which holds at least almost surely in Falconer's sense mentioned above. The difficulty of the Barreira Radu paper is that they do not assume that the maps $\gamma_j(x_1), \tau_j(x_2)$ are affine. On the other hand their result does not settle the self affine case completely. In this note using completely different methods we restrict ourselves to the self-affine case. More precisely, throughout this paper we always assume that $\{S_j\}_{j=1}^{m_0}$ is a self-affine IFS on $\mathbb{R}^2$ and we also assume that A1 holds. Our result does not simply give a lower bound for the dimension of $K_\beta$ but gives an equality in Falconer's almost all sense. Note that while we state our results and present the proofs in $\mathbb{R}^2$ the results generalise to $\mathbb{R}^d$ without difficulty. Our proof consists of two parts. When giving the lower bound on the dimension of $K_\beta$ we do not need any additional assumption and the method of the proof is that we use our joint result [9] with M. Pollicott about the dimension of self-affine measures. To obtain the upper bound we need to assume that A2 holds and we use the methods of L. Olsen [12] combined with a method that we learnt from Barreira Saussol [3].

1.1. **Statement of our Main result.** Given the contractive self affine IFS $\{S_i\}_{i=1}^{m_0}$ on $\mathbb{R}^2$. For every $i = 1,\dots,m_0$ we can write $S_i$ in the form

$$S_i(x) = A_i \cdot x + t_i,$$

where $A_i$ is a $2 \times 2$ non-singular matrix. We will always assume that

$$\|A_i\| < \frac{1}{2} \quad \forall i = 1, \dots, m_0. \tag{4}$$

As usual we write $\Pi$ for the natural projection from the symbolic space $\Sigma = \{1, \dots, m_0\}^{m_0}$ to the attractor $\Lambda$ (which is the only non-empty compact set satisfying: $\Lambda = \cup_{i=1}^{m_0} S_i \Lambda$). That is

$$\Pi(\mathbf{i}) := \lim_{n \to \infty} S_{i_0 \dots i_n}(0),$$

where $S_{i_0 \dots i_n} := S_{i_0} \circ \cdots \circ S_{i_n}$. Since we use the $2 \cdot m_0$ dimensional vector

$$\mathbf{t} := (t_1, \dots, t_{m_0}) \in \underbrace{\mathbb{R}^2 \times \cdots \times \mathbb{R}^2}_{m_0}$$

as a parameter we use the notation $\Lambda^{\mathbf{t}}, \{S_i^{\mathbf{t}}\}_{i=1}^{m_0}$ and $\Pi^{\mathbf{t}}$.

**The Lyapunov dimension of a measure:** (see [9]) Let $\nu$ be an ergodic invariant measure on $\Sigma$ and let $A : \Sigma \to \mathbf{M}$, where $\mathbf{M}$ denotes the set of $d \times d$ matrices with elements from $\mathbb{R}$, be defined by

$$A(\mathbf{i}) := A_{i_0}.$$

Then for the stationary process given by the measure $\nu$ and

$$\left\{ P_n(A, \mathbf{i}) := A^*_{i_{n-1}} \cdots A^*_{i_0} \right\}_{n=0}^{\infty} \tag{5}$$

we denote the Lyaponov exponents [11, Theorem 5.7] by

$$\lambda_1(\nu) \geq \lambda_2(\nu). \tag{6}$$

**Definition 1** (Definition of the Lyapunov dimension $D(\nu)$)**.**

**(i):** *If*

$$k := k(\nu) = \max\{i : 0 < h_\nu + \lambda_1(\nu) + \cdots + \lambda_i(\nu)\} < d, \tag{7}$$

*then we define the Lyapunov dimension $D(\nu)$ by*

$$D(\nu) := k + \frac{h_\nu + \lambda_1(\nu) + \cdots + \lambda_k(n)}{-\lambda_{k+1}(\nu)}. \tag{8}$$

**(ii):** *If $h_\nu + \lambda_1 + \cdots + \lambda_d > 0$ then we define*

$$D(\nu) := d \cdot \frac{h_\nu}{-(\lambda_1 + \cdots + \lambda_d)}.$$

If we have $2 \times 2$ diagonal matrices of the form $A_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \xi_i \end{bmatrix}$ then we can define the functions $g_1, g_2 : \Sigma \to \mathbb{R}$ by $g_1(\mathbf{i}) = \log \lambda_{i_1}$ and

$g_2(\mathbf{i}) = \log \xi_{i_1}$. We then have that
(9)

$$\lambda_1(\nu) = \max\left\{\int g_1 \mathrm{d}\nu, \int g_2 \mathrm{d}\nu\right\} \text{ and } \lambda_2(\nu) = \min\left\{\int g_1 \mathrm{d}\nu, \int g_2 \mathrm{d}\nu\right\}.$$

Thus in this case we can extend the definition of Lyapunov exponents to invariant measures by using the integrals in (9).

We are also given a Hölder continuous function $f : \Sigma \to \mathbb{R}$. Let

$$\beta_{\max} = \max_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{\int f \mathrm{d}\mu\right\} \text{ and } \beta_{\min} = \min_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{\int f \mathrm{d}\mu\right\}$$

For a given $\beta \in (\beta_{\min}, \beta_{\max})$ we would like to find the dimension of the set

$$K_\beta^{\mathbf{t}} := \Pi^{\mathbf{t}} \left\{\mathbf{i} \in \Sigma : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) = \beta\right\},$$

at least for $\mathcal{L}eb_{d \cdot m_0}$ almost all $\mathbf{t}$, where $\sigma$ is the left shift on $\Sigma$ as usual. We write $\mathcal{M}_\sigma(\Sigma)$ for the set of $\sigma$-invariant measures on $\Sigma$ and $\mathcal{E}_\sigma(\Sigma)$ for the set of ergodic measures on $\Sigma$.

**Proposition 1.** *We assume that (4) holds. Let $\beta \in (\beta_{\min}, \beta_{\max})$ arbitrary. Then for almost all $\mathbf{t} \in \mathbf{R}^{d \cdot m_0}$ we have*

$$\dim_{\mathrm{H}} K_\beta^{\mathbf{t}} \geq \min\left\{\sup\left\{D(\mu) : \mu \in \mathcal{E}_\sigma(\Sigma), \int f d\mu = \beta\right\}, d\right\}.$$

To give an upper bound we need to restrict ourselves to $\mathbb{R}^2$ and we also have to assume that all matrices $A_i$ are diagonal.

**Proposition 2.** *We assume that $d = 2$ and that for all $i = 1, \ldots, m_0$ the matrix $A_i$ is diagonal. Let $\beta \in (\beta_{\min}, \beta_{\max})$ be arbitrary. Then for **all** $\mathbf{t} \in \mathbf{R}^{2 \cdot m_0}$ we have*

$$\dim_{\mathrm{H}} K_\beta^{\mathbf{t}} \leq \min\left\{\sup\left\{D(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), \int f d\mu = \beta\right\}, 2\right\}.$$

Finally by combining these results and using the variational principle combined with other properties of the pressure function we obtain the following main result.

**Theorem 1.** *Assume that the matrices $A_i$ are $2 \times 2$, diagonal and (4) holds. Then for almost all $\mathbf{t} \in \mathbf{R}^{2 \cdot m_0}$ we have that for any $\beta \in (\beta_{\min}, \beta_{\max})$,*

$$\dim_{\mathrm{H}} K_\beta^{\mathbf{t}} = \min\left\{\sup\left\{D(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), \int f d\mu = \beta\right\}, 2\right\}. \tag{10}$$

## 2. The lower estimate

Here we prove Proposition 1. Let

$$\Delta(\beta) := \left\{ \mathbf{i} \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) = \beta \right\}. \tag{11}$$

Then by definition $K_\beta^{\mathbf{t}} = \Pi^{\mathbf{t}}(\Delta(\beta))$. It follows from the Birkhoff Ergodic Theorem that for every $\mu \in \mathcal{E}_\sigma(\Sigma)$ with $\int f d\mu = \beta$ we have $\mu(\Delta(\beta)) = 1$. Thus we obtain that for such a measure $\mu$ and for each $\mathbf{t}$ we have

$$\dim_{\mathrm{H}}(K_\beta^{\mathbf{t}}) \geq \dim_{\mathrm{H}}(\mu \circ (\Pi^{\mathbf{t}})^{-1}). \tag{12}$$

On the other hand it was proved in [9, Theorem 4 (a)] that for $\mathcal{L}eb_{d\cdot m_0}$ almost all $\mathbf{t}$ we have

$$\dim_{\mathrm{H}}(\mu \circ (\Pi^{\mathbf{t}})^{-1}) = \min\left\{D(\mu), d\right\}. \tag{13}$$

This completes the proof of Proposition 1.

## 3. Notation and large deviation results

To prove Proposition 2 we will use results from the theory of large deviations. This is based on the methods used in [12]. In this section we will introduce our notation and state the necessary large deviation results.

**Definition 2.** *Let $X$ be a complete metric space and let $\{P_n\}_n$ be a sequence of probability measures on $X$. Let $\{a_n\}_n$ be a sequence of positive numbers with $\lim_{n\to\infty} a_n = \infty$ and let $I : X \to [0,\infty]$ be a lower semi continuous function with compact level sets. The sequence $\{P_n\}_n$ is said to* ***have the large deviation property*** *with constants $\{a_n\}_n$ and rate function $I$ if the following holds:*

(1) *For each closed $K \subset X$*
$$\limsup_n \frac{1}{a_n} \log P_n(K) \leq -\inf_{x\in K} I(x);$$
(2) *For each open $G \subset X$*
$$\liminf_n \frac{1}{a_n} \log P_n(G) \geq -\inf_{x\in G} I(x).$$

**Theorem 2.** *Let $X$ and $\{P_n\}$ be like above. Assume that the sequence $\{P_n\}_n$ has the large deviation property with constants $\{a_n\}_n$ and rate function $I$. Let $F : X \to \mathbb{R}$ be a continuous and bounded function. Then*

(1) $\lim\limits_{n\to\infty} \frac{1}{a_n} \log \int \exp(a_n F) dP_n = -\inf_{x\in X}(I(x) - F(x))$
(2) *For each $n$ we define the probability measure $Q_n$ on $X$ by*

$$Q_n(E) := \frac{\int_E \exp(a_n F) dP_n}{\int \exp(a_n F) dP_n}.$$

*Then the sequence of measures $\{Q_n\}_n$ has the large deviation property with constants $\{a_n\}_n$ and rate function $(I-F)-\inf_{x\in X}(I(x)-F(x))$.*

Proved in [4, Theorem II.7.1-2].

For a compact metric space $X$ we denote the set of probability measures on $X$ by $\mathcal{M}(X)$. We write $\mathcal{E}$ for the set of ergodic measures on $\Sigma$. Put $\sigma$ for the left shift on $\Sigma$. Following Olsen's paper [12] for every $\omega \in \Sigma$ we define $L_n(\omega) \in \mathcal{M}(\Sigma)$

$$L_n(\omega)(E) := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k\omega}(E) = \frac{1}{n}\#\left\{0 \le k \le n-1 : \sigma^k\omega \in E\right\}, \text{ for } E \subset \Sigma.$$

Furthermore we fix a Hölder continuous function $f : \Sigma \to \mathbb{R}$ and define $\Xi : \mathcal{M}(\Sigma) \to \mathbb{R}$

$$\Xi : \mu \to \int f d\mu.$$

Then naturally $\Xi \circ L_n : \Sigma \to \mathcal{M}(\Sigma) \to \mathbb{R}$:

$$\Xi L_n(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k\omega).$$

Our aim is to give an upper bound on the Hausdorff dimension of the natural projection of the set

$$\Delta(\beta) = \left\{\omega \in \Sigma : \lim_{n\to\infty} \Xi L_n(\omega) = \beta\right\},$$

for a $\beta \in \mathbb{R}$ since we have that $K_\beta^{\mathbf{t}} = \Pi^{\mathbf{t}}\Delta(\beta)$. To do that we need to estimate the dimension of the projection of the set

$$\Delta_n\left(\beta, \frac{1}{k}\right) := \left\{\omega \in \Sigma : \Xi L_m(\omega) \in B\left(\beta, \frac{1}{k}\right) \text{ holds for all } m \ge n\right\}$$

where $B\left(\beta, \frac{1}{k}\right)$ refers to the closed ball of radius $\frac{1}{k}$. Then obviously

$$\Delta(\beta) = \bigcap_k \bigcup_n \Delta_n\left(\beta, \frac{1}{k}\right) \tag{14}$$

Since we are assuming that our matrices are diagonal we will denote $A_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \xi_i \end{bmatrix}$ where $\lambda_i, \xi_i < 1$ for all $1 \leq i \leq m$. Thus we can determine the Lyapunov exponents with respect to a measure in terms of integrals. To do this we define functions $\Phi_1, \Phi_2 : \Sigma \to \mathbb{R}$ by $\Phi_1(\omega) = \log \lambda_{\omega_1}$ and $\Phi_2(\omega) = \log \xi_{\omega_1}$. We can now compute the Lyapunov exponents of an invariant measure $\mu$ to be

$$\lambda(\mu) = \int \Phi_1(\omega) \mathrm{d}\mu(\omega) \text{ and } \xi(\mu) = \int \Phi_2(\omega) \mathrm{d}\mu(\omega).$$

The Lyapunov dimension of an invariant measure $\mu$ is given by

$$\dim_L(\mu) = \min \left\{ -\frac{h(\mu)}{\max\{\lambda(\mu), \xi(\mu)\}}, 1 - \frac{h(\mu) + \max\{\lambda(\mu), \xi(\mu)\}}{\min\{\lambda(\mu), \xi(\mu)\}} \right\}.$$

We remark that if the measure $\mu$ is not only invariant but ergodic then this definition gives the same as Definition 1.

To use the decomposition (14) to help compute the dimension of $K_\beta$ we need the following three Lemmas.

**Lemma 1.** *The function $g : \mathcal{M}_\sigma(\Sigma) \to \mathbb{R}$, defined by $g(\mu) = \dim_L(\mu)$, is upper semi-continuous.*

*Proof.* This follows from the upper semi-continuity of entropy (see [17, Theorem 8.2]) and the continuity of the Lyapunov exponents $\xi(\mu)$ and $\lambda(\mu)$. $\square$

**Lemma 2** (Olsen)**.** *Let $X$ be a metric space and let $f : X \to \mathbb{R}$ be an upper semi-continuous function. Let $L_1, L_2, \ldots \subseteq X$ be non-empty compact subsets of $X$ with $L_1 \supseteq L_2 \supseteq \cdots$. Then*

$$\inf_k \sup_{t \in L_k} f(t) = \sup_{t \in \cap_k L_k} f(t)$$

*Proof.* See the proof of Lemma 3.2 in [12]. $\square$

Finally we need a Lemma which is similar to Lemma 3.3 in [12]. Note that we use the metric on $\Sigma$ defined by $d(\omega, \tau) = \frac{1}{2}^{|\omega \wedge \tau|}$ and $d_{\mathcal{M}}$ will denote the standard metric for the weak$^*$ topology defined on $\mathcal{M}(\Sigma)$.

**Lemma 3.** *Let $\omega, \tau \in \Sigma$. Then for every $r > 0$ $\exists M$ such that for $m \geq M$ if $|\omega \wedge \tau| = m$, then for all $C \in \mathbb{R}$*

$$\Xi L_m(\omega) \in B(C, r) \Longrightarrow \Xi L_m(\tau) \in B(C, 2r). \tag{15}$$

*Proof.* Since $\mathcal{M}(\Sigma)$ is compact w.r.t. the weak$^*$ topology and $\Xi : \mathcal{M}(\Sigma) \to \mathbb{R}$ is continuous therefore $\Xi$ is uniformly continuous. If we let

$m = |\omega \wedge \tau|$ then we have that

$$d_{\mathcal{M}}(L_m(\omega) - L_m(\tau)) \leq \left(\frac{1}{2}\right)^m .$$

The result now follows from the uniform continuity of $\Xi$. □

It immediately follows from this lemma that for

$$\overline{\omega|n} := (\omega_1, \dots, \omega_n, \omega_1, \dots, \omega_n, \dots)$$

and for all $n \geq M = M(r)$ and for all $C$ we have

$$\Xi L_n(\omega) \subset B(C, r) \Longrightarrow \Xi L_n(\overline{\omega|n}) \subset B(C, 2r). \tag{16}$$

## 4. Proof of Proposition 2

We need the following definitions: For a $0 \leq t \leq 2$ we define $F^t : \mathcal{M}_\sigma(\Sigma) \to \mathbb{R}$

$$F^t(\mu) := \begin{cases} t \max\{\lambda(\mu), \xi(\mu)\} & \text{for} \quad t \leq 1 \\ \max\{\lambda(\mu), \xi(\mu)\} + (t-1)\min\{\lambda(\mu), \xi(\mu)\} & \text{for} \quad 1 < t \leq 2 \end{cases}$$

It is important to note that these functions are continuous with respect to the weak-* topology. We wish to apply Theorem 2 to the space $\mathcal{M}_\sigma(\Sigma)$. Furthermore we need to define a sequence of measures on $\mathcal{M}_\sigma(\Sigma)$ satisfying the large deviation property. Namely, let $P_n \in \mathcal{M}(\mathcal{M}_\sigma(\Sigma))$

$$P_n(E) := \frac{1}{m_0^n} \sum_{|\omega|=n} \delta_{L_n(\overline{\omega})}(E),$$

where $E \subset \mathcal{M}(\Sigma)$ and $\overline{\omega} := (\omega, \omega, \dots)$. Observe that $\delta_{L_n(\overline{\omega})} \in \mathcal{M}_\sigma(\Sigma)$.

**Lemma 4** (Eizenberger-Kifer-Weiss)**.** *The sequence of measures $P_n$ satisfies the large deviation property with constants $\{a_n\}_n = n$ and rate function,*

$$I(\mu) = -\log m_0 - h(\mu). \tag{17}$$

*Proof.* See [5]. □

Moreover let

$$Q_n^t(E) := \frac{\int_E \exp(nF^t)\, dP_n}{\int \exp(nF^t)\, dP_n}.$$

To evaluate $Q_n^t(E)$ we observe that for a continuous $G : \mathcal{M}_\sigma(\Sigma) \to \mathbb{R}$ we have.

$$\int_E G(\mu)dP_n(\mu) = \frac{1}{m_0^n} \sum_{\substack{|\omega|=n \\ L_n(\overline{\omega})\in E}} G(L_n(\overline{\omega})), E \subset \mathcal{M}(\Sigma). \tag{18}$$

We use this to define when $t \leq 1$

$$G_n^t(\mu) := \exp\left(n \cdot F^t(\mu)\right) = \max\left\{\prod_{k=1}^m \lambda_k^{n\cdot t\cdot\mu([k])}, \prod_{k=1}^m \xi_k^{n\cdot t\cdot\mu([k])}\right\}$$

and for $1 < t \leq 2$

$$G_n^t(\mu) := \exp\left(n \cdot F^t(\mu)\right) = \max\left\{\prod_{k=1}^m \lambda_k^{n\cdot\mu([k])}\xi_k^{n\cdot(t-1)\cdot\mu([k])}, \prod_{k=1}^m \xi_k^{n\cdot\mu([k])}\lambda_k^{n\cdot(t-1)\cdot\mu([k])}\right\}.$$

Using this and the fact that $L_n(\overline{\omega})([k])$ is $\frac{1}{n}$ times the number of indices $1 \leq \ell \leq n$ for which $\omega_\ell = k$ we obtain that for an $|\omega| = n$:

$$G_n^t(L_n(\overline{\omega})) = \begin{cases} \max\{\lambda_\omega^t, \xi_\omega^t\} & \text{for} \quad t \leq 1 \\ \max\{\lambda_\omega \xi_\omega^{t-1}, \xi_\omega \lambda_\omega^{t-1}\} & \text{for} \quad 1 < t \leq 2 \end{cases}, \tag{19}$$

where $\lambda_\omega = \lambda_{\omega_1}\cdots\lambda_{\omega_n}$ and $\xi_\omega = \xi_{\omega_1}\cdots\xi_{\omega_n}$. From now on let

$$E := \left\{\mu \in \mathcal{M}_\sigma(\Sigma) : \Xi\mu \in B\left(\beta, \frac{2}{k}\right)\right\}.$$

We obtain from (18) and (19) that

$$Q_n^t(E) := \begin{cases} \dfrac{\sum_{\substack{|\omega|=n \\ \Xi L_n(\overline{\omega})\in B(\beta,\frac{2}{k})}} \max\{\lambda_\omega^t,\xi_\omega^t\}}{\sum_{|\omega|=n} \max\{\lambda_\omega^t,\xi_\omega^t\}} & \text{for} \quad t \leq 1 \\ \dfrac{\sum_{\substack{|\omega|=n \\ \Xi L_n(\overline{\omega})\in B(\beta,\frac{2}{k})}} \max\{\lambda_\omega\xi_\omega^{t-1},\xi_\omega\lambda_\omega^{t-1}\}}{\sum_{|\omega|=n} \max\{\lambda_\omega\xi_\omega^{t-1},\xi_\omega\lambda_\omega^{t-1}\}} & \text{for} \quad 1 < t \leq 2 \end{cases}. \tag{20}$$

We first consider the case where $t \leq 1$. Let $n \geq M(\frac{1}{k})$ which was defined in Lemma 3. Then it follows from (16) that for $\delta_j := |\Lambda| \max_{1\leq i\leq m_0} \sqrt{2}||A_i||^j$ and $j \geq n \geq M(\frac{1}{k})$ we have

$$\begin{aligned} \mathcal{H}_{\delta_j}^t\left(\Pi\left(\Delta_n(\beta, \frac{1}{k})\right)\right) &\leq |\mathrm{diam}\Lambda|^t \sum_{\substack{|\omega|=j \\ \Xi L_j(\overline{\omega})\in B(\beta,\frac{2}{k})}} \max\left\{\lambda_\omega^t, \xi_\omega^t\right\} \\ &= |\mathrm{diam}\Lambda|^t m_0^j Q_j^t(E) \int G_j^t(\mu)dP_j(\mu). \end{aligned} \tag{21}$$

For $1 < t \leq 2$ using the above definitions we get

$$\begin{aligned}\mathcal{H}^t_{\delta_j}\left(\Pi\left(\Delta_n(\beta,\frac{1}{k})\right)\right) &\leq |\mathrm{diam}\Lambda|^t \sum_{\substack{|\omega|=j \\ \Xi L_j(\overline{\omega})\in B(\beta,\frac{2}{k})}} \left(\left[\frac{\max\{\lambda_\omega,\xi_\omega\}}{\min\{\lambda_\omega,\xi_\omega\}}\right]+1\right)\min\{\lambda_\omega^{t-1},\xi_\omega^{t-1}\} \\ &\leq C|\mathrm{diam}\Lambda|^t m_0^j Q_j^t(E)\int G_j^t(\mu)dP_j(\mu)\end{aligned}$$

Where $C$ is chosen such that $\left(\left[\frac{\max\{\lambda_\omega,\xi_\omega\}}{\min\{\lambda_\omega,\xi_\omega\}}\right]+1\right) \leq C\frac{\max\{\lambda_\omega,\xi_\omega\}}{\min\{\lambda_\omega,\xi_\omega\}}$ (For example $C=2$ would suffice).

From Lemma 4 we have that $\{P_n\}\in\mathcal{M}(\mathcal{M}_\sigma(\Sigma))$ satisfies the large deviation property, with constants $\{n\}_n$ and rate function $I(\mu) := \log m - h(\mu)$. Thus from Theorem 2 (2) we have

(22)

$$\limsup_j \frac{1}{j}\log Q_j^t(E) = -\inf_{\substack{\mu\in\mathcal{M}_\sigma(\Sigma)\\ \Xi\mu\in B(\beta,\frac{2}{k})}}\left(\left(I(\mu)-F^t_\mu\right) - \inf_{\nu\in\mathcal{M}_\sigma(\Sigma)}\left(I(\nu)-F^t(\nu)\right)\right).$$

Furthermore, it follows from the definition of $G_j^t$ and Theorem 2 (1) that

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{j}\log\int G_j^t(\mu)dP_j(\mu) &= \lim_{j\to\infty}\frac{1}{j}\log\int\exp\left(j\cdot F^t(\mu)\right)dP_j(\mu) \\ &= -\inf_{\nu\in\mathcal{M}_\sigma(\Sigma)}\left(I(\nu)-F^t(\nu)\right) \qquad (23)\end{aligned}$$

The last two formulae with (21) gives that for $t\leq 1$

$$\lim_{j\to\infty}\log\mathcal{H}^t_{\delta_j}\left(\Pi\left(\Delta_n(C,\frac{1}{k})\right)\right) \leq \sup_{\substack{\mu\in\mathcal{M}_\sigma(\Sigma)\\ \Xi\mu\in B(\beta,\frac{2}{k})}}\left\{t\max\left\{\int\Phi_1\mathrm{d}\mu,\int\Phi_2\mathrm{d}\mu\right\}+h(\mu)\right\}$$

and for $1 < t \leq 2$

(24)

$$\begin{aligned}\lim_{j\to\infty}\log\mathcal{H}^t_{\delta_j}\left(\Pi\left(\Delta_n(\beta,\frac{1}{k})\right)\right) \leq \sup_{\substack{\mu\in\mathcal{M}_\sigma(\Sigma)\\ \Xi\mu\in B(\beta,\frac{2}{k})}}&\left\{\max\left\{\int\Phi_1\mathrm{d}\mu,\int\Phi_2\mathrm{d}\mu\right\}\right. \\ &\left.+(t-1)\min\left\{\int\Phi_1\mathrm{d}\mu,\int\Phi_2\mathrm{d}\mu\right\}+h(\mu)\right\}.\end{aligned}$$

For both expressions the right hand side is less than 0 if and only if

$$t > \sup_{\substack{\mu\in\mathcal{M}_\sigma(\Sigma)\\ \Xi\mu\in B(\beta,\frac{2}{k})}}\dim_L(\mu).$$

To complete the proof of Proposition 2 we simply apply Lemmas 1 and 2 for the sets

$$L_k := \left\{ \mu \in \mathcal{M}_\sigma(\Sigma) : \Xi\mu \in B\left(\beta, \frac{1}{k}\right) \right\}.$$

## 5. Proof of Theorem 1

In this section we assume that all the matrices $A_i$ satisfy $||A_i|| < \frac{1}{2}$ and are diagonal. By Proposition 1 we have that for almost all $\mathbf{t}$

$$\dim_{\mathrm{H}} K_\beta^{\mathbf{t}} \geq \min\left\{ \sup\left\{ D(\mu) : \mu \in \mathcal{E}_\sigma(\Sigma), \int f d\mu = \beta \right\}, 2 \right\}.$$

Note that the supremum is taken over ergodic measures. The remaining step to complete the proof of Theorem 1 is to show that this is equivalent to taking the supremum over invariant measures. To do this we use the pressure function ([17]) in a similar way to [3]. A similar approach when maximising related quantities in different settings were used in [15] and [16]. Let $\Phi_1, \Phi_2 : \Sigma \to \mathbb{R}$ be defined as before. For $0 \leq s \leq 1$ let

$$\Psi_1^s(\mathbf{i}) = s\Phi_1(\mathbf{i}) \text{ and } \Psi_2^s(\mathbf{i}) = s\Phi_2(\mathbf{i})$$

and for $1 < s \leq 2$ let

$$\Psi_1^s(\mathbf{i}) = \Phi_1(\mathbf{i}) + (s-1)\Phi_2(\mathbf{i}) \text{ and } \Psi_2^s(\mathbf{i}) = \Phi_2(\mathbf{i}) + (s-1)\Phi_1(\mathbf{i}).$$

Note that these functions are Hölder continuous. For an ergodic measure $\mu$ on $\Sigma$ with $D(\mu) \leq 2$ it follows that $D(\mu) = s$ where $s$ satisfies

$$h(\mu) + \max\left( \int \Psi_1^s \mathrm{d}\mu, \int \Psi_2^s \mathrm{d}\mu \right) = 0.$$

Thus it follows that if for $i = 1, 2$

$$h(\mu) + \int \Psi_i^s \mathrm{d}\mu > 0$$

then $D(\mu) \geq s$.

For $q \in \mathbb{R}$ and $0 \leq s \leq 2$ we look at the function

$$l(q, s) = qf - q\beta + \Psi_1^s.$$

Recall that

$$P(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \int f \mathrm{d}\mu + h(\mu) \right\}$$

and it follows from the variational principle that for Hölder functions there exists an ergodic equilibrium measure $\mu_{q,s}$ such that

$$P(qf - q\beta + \Psi_1^s) = h(\mu_{q,s}) + \int qf - q\beta + \Psi_1^s d\mu_{q,s}.$$

It also follows from standard properties of pressure that the pressure of this function will be continuous with respect to $s$ and $q$.

**Lemma 5.** *For $0 \leq s \leq 2$ and $\beta \in (\beta_{\min}, \beta_{\max})$ there exists $q$, which depends continuously on $s$ and $\beta$, such that*

$$\int (qf - q\beta)\, d\mu_{q,s} = 0.$$

*Proof.* Recall (see for example Proposition 4.10 in [14]) that

$$\frac{d}{dq} P(l(q,s)) = \int (qf - q\beta) \mathrm{d}\mu_{q,s}.$$

Thus it follows from that properties of the pressure function that

$$\frac{d}{dq} P(l(q,s)) = \int (qf - q\beta) \mathrm{d}\mu_{q,s}$$

behaves continuously with both $q$ and $s$. Thus it is sufficient to show that for any $0 \leq s \leq 2$ we can find $q_1, q_2$ such that

$$\int q_2 f - q_2 \beta \mathrm{d}\mu_{q_2,s} \leq 0 \text{ and } \int q_1 f - q_1 \beta \mathrm{d}\mu_{q_1,s} \geq 0.$$

Note that we can find $k_1, k_2$ such that

$$k_1 \leq \int \Psi_1^s \mathrm{d}\mu + h(\mu) \leq k_2$$

for all invariant measures. If we take $q_1 > \frac{|k_1|+|k_2|}{\beta_{\max}-\beta}$ and choose $\nu$ to be an invariant measure with maximum integral (that is $\int f \mathrm{d}\nu = \beta_{\max}$) then we have that

$$P(l(q_1, s)) > k_2.$$

Thus for the equilibrium measure $\mu_{q_1,s}$ we have that

$$\int q(f - \beta) \mathrm{d}\mu_{q,s} + \int \Psi_1^s \mathrm{d}\mu_{q,s} + h(\mu_{q_1,s}) \geq k_2$$

and hence $\int q(f - \beta) \mathrm{d}\mu_{q,s} \geq 0$. We find $q_2$ similarly. $\square$

We will denote this value of $q$ by $q(s)$.

**Lemma 6.** *If we let*

$$s = \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), \int f d\mu = \beta \right\} \tag{25}$$

*and assume that $0 \leq s \leq 2$ then we have that $P(q(s)f - q(s)\beta + \Psi_i) = 0$ for either $i = 1$ or $i = 2$ and the supremum in (25) is achieved by the ergodic equilibrium state for $q(s)f - q(s)\beta + \Psi_i$.*

*Proof.* We will assume without loss of generality that the supremum in (25) is the same if we just consider measures where $\int \Phi_1(\mathbf{i})\mathrm{d}\mu \geq \int \Phi_2(\mathbf{i})\mathrm{d}\mu$. In this proof we write simply $q$ instead of $q(s)$. We can find arbitrarily small $\varepsilon \geq 0$ for which there exists an invariant measure $\mu$ such that

$$h(\mu) + \int \Psi_1^{(s-\varepsilon)} d\mu = 0 \text{ and } \int f \mathrm{d}\mu = \beta.$$

Thus we have that

$$h(\mu) + \int (qf - \beta q + \Psi_1^{(s-\varepsilon)})\mathrm{d}\mu = 0$$

and hence $P(qf - \beta q + \Psi_1^{(s-\varepsilon)}) \geq 0$. From the continuity of the pressure function it follows that

$$P(qf - \beta q + \Psi_1^s) \geq 0.$$

Let $p = P(qf - \beta q + \Psi_1^s) \geq 0$ and let $\mu_{q,s}$ be the associated equilibrium measure. If we assume that $p > 0$ it follows that

$$0 < \int \Psi_1^s \mathrm{d}\mu_{q,s} + h(\mu_{q,s})$$

and thus

$$D(\mu_{q,s}) > s$$

which is a contradiction. Thus p=0 and hence

$$D(\mu_{q,s}) = s.$$

□

Theorem 1 immediately follows from this Lemma together with Propositions 1 and 2.

## 6. Example

Consider the special case when for an $\mathbf{i} = (i_0, i_1, \dots) \in \Sigma$

$$f(\mathbf{i}) := \log p_{i_0},$$

for a given probability vector $(p_1, \dots, p_m)$ and we have

$$A_i := \begin{bmatrix} \lambda & 0 \\ 0 & \xi \end{bmatrix} \text{ for all } i = 1, \dots, m,$$

moreover $0 < \xi < \lambda < \frac{1}{2}$. In this case we can explicitly calculate the 'almost all' formula for the dimension of $K_\beta$ given in Theorem 1.

This simple calculation and the statement of Theorem 1 gives that for almost all $\mathbf{t}$ we have

$$\dim_{\mathrm{H}} K_{\beta}^{\mathbf{t}} = \min\left\{\frac{\log\left(\sum\limits_{k=1}^{m} p_k^{t_0}\right) - \beta t_0}{-\log\lambda}, 1 + \frac{\log\left(\sum\limits_{k=1}^{m} p_k^{t_0}\right) - \beta t_0 + \log\lambda}{-\log\xi}\right\},$$

where we obtain $t_0 = t_0(\beta)$ as the solution of the equation

$$\sum_{k=1}^{m} p_k^{t_0}\log p_k = \beta\cdot\sum_{k=1}^{m} p_k^{t_0}.$$

This is so because it is easy to verify that the supremum in (10) is attained for the Bernoulli measure

$$\mu = \left\{\frac{p_1^{t_0}}{\sum\limits_{k=1}^{m} p_k^{t_0}}, \dots, \frac{p_m^{t_0}}{\sum\limits_{k=1}^{m} p_k^{t_0}}\right\}^{\mathbb{N}}.$$

We made a computation when $m = 4$ and $(p_1, \dots, p_4) = (1/2, 1/4, 1/8, 1/8)$ and $\lambda = 0.4, \xi = 1/6$. We computed the right hand side of formulae (10) and this is shown by Figure 1. Note that in this case

$$\beta_{\min} = \log\frac{1}{8}, \qquad \beta_{\max} = \log\frac{1}{2}$$

and for a typical $\mathbf{t}$ we have

$$\dim_{\mathrm{H}} K_{\beta_{\min}}^{\mathbf{t}} = \frac{\log 2}{-\log(2/5)} = 0.7564707974$$

as is shown on Figure 1. Note that the curve is smooth except at the values where $\dim K_\beta = 1$.

## 7. Final Comments

(1) Generalising the results to higher dimensions than 2 is not a problem. The proofs were given in 2-dimensions to simplify the presentation.

(2) It is possible to get an upper bound in the non-diagonal case as well. For a matrix $A$ we define the singular values $\alpha_1(A) \geq \alpha_2(A)$ to be the eigenvalues of $A^*A$. We can define the singular value function for $0 \leq s \leq 2$ by

$$\begin{aligned} \phi^s(A) &= \alpha_1(A)^s \text{ for } s \leq 1 \\ \phi^s(A) &= \alpha_1(A)\alpha_2(A)^{s-1} \text{ for } 1 < s \leq 2. \end{aligned}$$

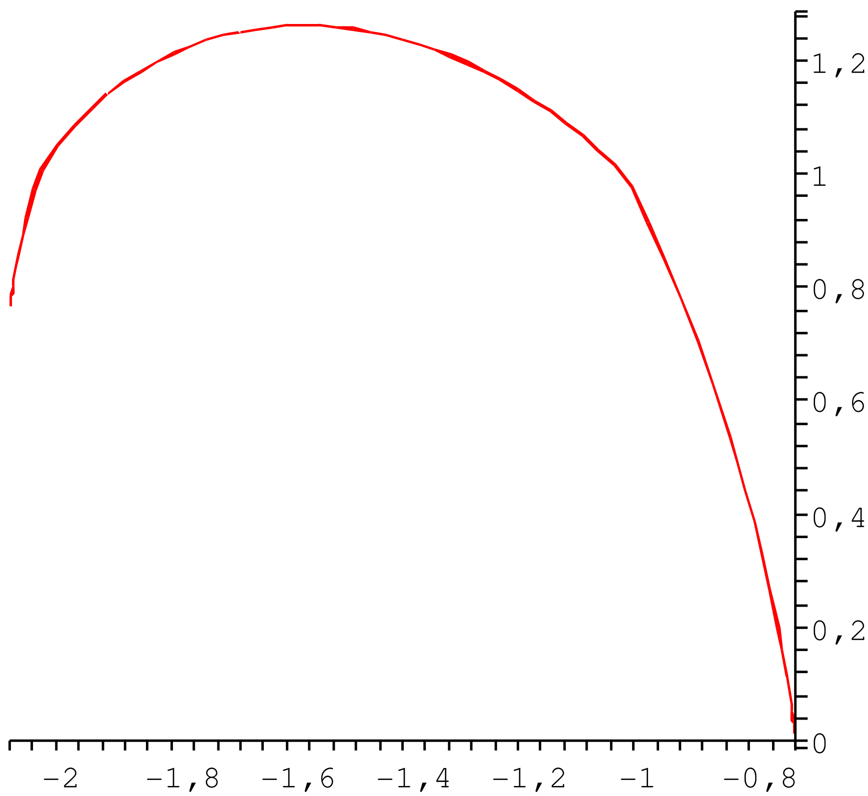


FIGURE 1. The dimension spectrum in a special case

Note that this function is submultiplicative (see [6] Lemma 2.1). We define $F^t : \mathcal{M}_\sigma(\Sigma) \to \mathbb{R}$ by

$$F^t(\mu) = \int \log \phi^t(A_{i_1}) \mathrm{d}\mu(\mathbf{i}).$$

We define $P_n$ and $Q_n$ as in the proof of Proposition 2. We define the sets $\Delta_n(\beta, \frac{1}{k})$ as before and let $\delta_j = \max_{1\leq i\leq m} |\Lambda|||A_i||^j$. If we let $G_n^t(\mu) = \exp(nF^t(\mu))$ then

$$G_n^t(L_n\overline{\omega}) = \phi^t(\omega_1)\cdots\phi^t(\omega_n)$$

for $\omega = (\omega_1, \ldots, \omega_n)$. Thus we get

$$\begin{aligned}
\mathcal{H}^t_{\delta_j}\left(\Pi\left(\Delta_n(\beta, \frac{1}{k})\right)\right) &\leq |\mathrm{diam}\Lambda|^t \sum_{\substack{|\omega|=j \\ \Xi L_j(\overline{\omega}) \in B(C, \frac{2}{k})}} \phi^s(A_{\omega_1}) \cdots \phi^s(A_{\omega_n}) \\
&= |\mathrm{diam}\Lambda|^t m^j Q^t_j(E) \int G^t_j(\mu) dP_m(\mu).
\end{aligned}$$

Using an identical method to Proposition 2 we can then show that

$$\dim K_\beta \leq \sup\left\{ s(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), \int f \mathrm{d}\mu = \beta \right\}$$

where $s(\mu)$ is the solution to $h(\mu) + F^t(\mu) = 0$. This estimate can be improved by applying the same method to the iterated function system generated by looking at $n$-fold compositions of the original system. This will give a better estimate because of the submultipilicity of the singular value function.

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