TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

THE ALGEBRAIC DIFFERENCE OF TWO RANDOM CANTOR SETS: THE LARSSON FAMILY

MICHEL DEKKING, KÁROLY SIMON, AND BALÁZS SZÉKELY

ABSTRACT. In this paper we consider a family of random Cantor sets on the line and consider the question whether the condition that the sum of the Hausdorff dimensions is larger than one implies the existence of interior points in the difference set of two independent copies. We give a new and complete proof that this is the case for the random Cantor sets introduced by Per Larsson.

1. Introduction

Algebraic differences of Cantor sets occur naturally in the context of the dynamical behavior of diffeomorphisms. From these studies a conjecture by Palis and Takens ([7]) originated, relating the size of the arithmetic difference

$$C_2 - C_1 = \{y - x : x \in C_1, y \in C_2\}$$

to the Hausdorff dimensions of the two Cantor sets C_1 and C_2: if

$$\dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2 > 1$$

then *generically* it should be true that

$$C_2 - C_1 \text{ contains an interval.}$$

For generic dynamically generated *non-linear* Cantor sets this was proved in 2001 by de Moreira and Yoccoz ([1]). The problem is open for generic linear Cantor sets. The problem was put into a probabilistic context by Per Larsson in his thesis [5], (see also [6]). He considers a two parameter family of random Cantor sets $C_{a,b}$, and claims to prove that the Palis conjecture holds for all relevant choices of the parameters a and b. Although the main idea of Larsson's argument is brilliant, unfortunately the proof contains significant gaps and incorrect reasonings. The aim of the present paper is to give a correct proof of this

2000 *Mathematics Subject Classification.* Primary 28A80 Secondary 60J80, 60J85

Key words and phrases. Random fractals, difference of Cantor sets, Palis conjecture, multitype branching processes.

The research of Dekking was partially supported by the NWO-OTKA. The research of Simon was supported by OTKA Foundation #K71693 and the NWO-OTKA common project. The research of Székely was supported by HSN Lab.

theorem. The most important error made by Larsson is as follows: during the construction a multitype branching process with uncountably many types appears naturally. The number of individuals in the n-th generation having types which fall into the set A is denoted $\mathcal{Z}_n(A)$ and the probability measure describing the branching process starting with a single type x individual is denoted $\mathbb{P}_x$. The argument presented in Larsson's paper requires that for some positive δ, q and $\rho > 1$ we have that **uniformly** both in x and in n the following holds:

$$\mathbb{P}_x(\mathcal{Z}_n(A) > \delta \cdot \rho^n) > q. \tag{1}$$

However, the main result in the theory of general multitype branching processes [4, Th.14.1] invoked by Larsson implies (1) without any uniformity.
Further, (as shown in [3]) the idea presented in Larsson's paper works only in the region (see also Figure 1) where

$$1 - 4a - 2b + 3a^2 - 6ab > 0. \tag{2}$$

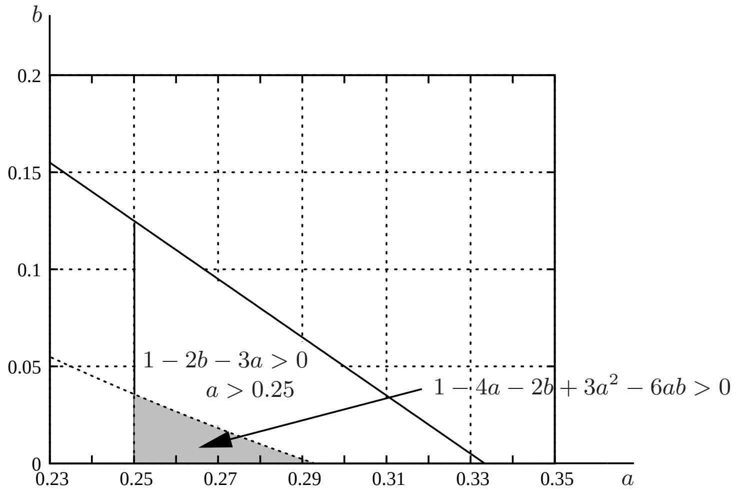

FIGURE 1. Regions described by equations (2) and (3).

Although we use a different setup, the main idea presented here follows the stream of Larsson's proof. We remark that for linear Cantor sets of a different nature the first two authors investigated the same problem in [2].

1.1. **Larsson's random Cantor sets.** It is assumed throughout this paper that

$$a > \frac{1}{4} \quad \text{and} \quad 3a + 2b < 1. \tag{3}$$

The first condition is a growth condition and we will see its importance in Section 7. The second condition is a geometric condition: Larsson's Cantor set is a natural randomization of the classical Cantor set; see

Figure 2: in the first step of the construction intervals of length a are put into the intervals $[b, \frac{1}{2}-\frac{a}{2}]$ and $[\frac{1}{2}+\frac{a}{2}, 1-b]$. This obviously requires $3a+2b<1$. We remark that it is useful to force a gap of length at least a in the middle, since otherwise the Newhouse thickness of the Cantor set would be larger than 1, which yields an interval in the difference set by Newhouse's theorem (see [7, p. 63]).
The construction is as follows: first remove the middle a part, then the b parts from both the beginning and the end of the unit interval. Then put intervals of length a according to a uniform distribution in the remaining two gaps $\left[b, \frac{1}{2}-\frac{a}{2}\right]$ and $\left[\frac{1}{2}+\frac{a}{2}, 1-b\right]$. These two randomly chosen intervals of length a are called the level one intervals of the random Cantor set $C_{a,b}$. We write $C_{a,b}^1$ for their union. In both of the two level one intervals we repeat the same construction independently of each other and of the previous step. In this way we obtain four disjoint intervals of length a^2. We emphasize that, because of independence, the relative positions of these second level intervals in the first level ones are in general completely different. Similarly, we construct the 2^n level n intervals of length a^n. We call their union $C_{a,b}^n$. Then Larsson's random Cantor set is defined by

$$C_{a,b} := \bigcap_{n=1}^{\infty} C_{a,b}^n.$$

See Figure 2.

FIGURE 2. The construction of the Cantor set $C_{a,b}$. The figure shows $C_{a,b}^1, \dots, C_{a,b}^4$.

The next theorem was stated by P. Larsson.

Theorem 1. *Let C_1, C_2 be independent random Cantor sets having the same distribution as $C_{a,b}$ defined above. Then the algebraic difference $C_2 - C_1$ almost surely contains an interval.*

Our paper is organized as follows: In the next section we give an elementary proof of the fact that the probability that $C_2 - C_1$ contains an interval is either 0 or 1. For the main part of the proof our starting point is the observation that $C_2 - C_1$ can be viewed as a 45° projection of the product set $C_1 \times C_2$. This leads in Section 3.1 to the introduction of the level n squares formed as the product of level n intervals of the Cantor sets C_1, C_2. We remark that Larsson does not use these squares at all. Then based on the family of these squares we will build up the intrinsic branching process, and we state our Main Lemma which will replace (1). In Section 4 we prove Theorem 1 assuming the Main Lemma. In Sections 5-10 we give a proof of the Main Lemma.

2. A 0-1–law

In the following we will use that the property of containing an interval is invariant for translations and scalings, and we will write "$C_2 - C_1$ contains an interval" also equivalently as "$C_2 - C_1$ has non-empty interior". It follows from translation invariance and the statistical self similarity of the Cantor set construction that

$$\mathbb{P}\left(\mathrm{Int}(C_2 - C_1) \neq \emptyset\right) = \mathbb{P}\left(\mathrm{Int}(C_2^{1,1} - C_1^{1,1}) \neq \emptyset\right),$$

where $C_i^{1,1} = C_i \cap [0, \frac{1}{2}]$, and $C_i^{1,2} = C_i \cap [\frac{1}{2}, 1]$. This observation is the basis for the following simple proof of the 0-1-law of the 'interval property'.

Proposition 1. $\mathbb{P}\left(C_2 - C_1 \supset I\right) = 0$ *or* 1.

Proof. Note that

$$\begin{aligned} p &:= \mathbb{P}\left(C_2 - C_1 \supset I\right) = 1 - \mathbb{P}\left(\mathrm{Int}(C_2 - C_1) = \emptyset\right) \\ &\geq 1 - \mathbb{P}\left(\mathrm{Int}(C_2^{1,1} - C_1^{1,1}) = \emptyset, \mathrm{Int}(C_2^{1,2} - C_1^{1,2}) = \emptyset\right) \\ &= 1 - \mathbb{P}\left(\mathrm{Int}(C_2^{1,1} - C_1^{1,1}) = \emptyset\right)\mathbb{P}\left(\mathrm{Int}(C_2^{1,2} - C_1^{1,2}) = \emptyset\right) \\ &= 1 - (1-p)^2. \end{aligned}$$

This implies $p \leq p^2$, and hence $p = 0$ or 1. □

3. Notation and the Main Lemma

In the rest of the paper we fix a pair (a, b) satisfying condition (3) and we always deal with Larsson's Cantor sets so we will suppress the labels a, b.

3.1. The geometry of the algebraic difference $C_2 - C_1$.

The $45°$ projection of a point $(x_1, x_2) \in \mathbb{R}^2$ to the x_2-axis, is denoted by $Proj_{45°}$. That is

$$Proj_{45°}(x_1, x_2) := x_2 - x_1.$$

The following trivial fact is the motivation for constructing our branching process of labelled squares:

$$x \in Proj_{45°}\left(C_1 \times C_2\right) \text{ if and only if } x \in C_2 - C_1.$$

So,

$$C_2 - C_1 = \bigcap_{n=0}^{\infty} Proj_{45°}\left(C_1^n \times C_2^n\right)$$

We can naturally label the squares in $C_1^n \times C_2^n$ as follows: We call the top left first level square Q_1 and we continue labeling the first level squares Q_2, Q_3, Q_4 in the clockwise direction. Then within each of these squares we continue in this way. See Figure 3.

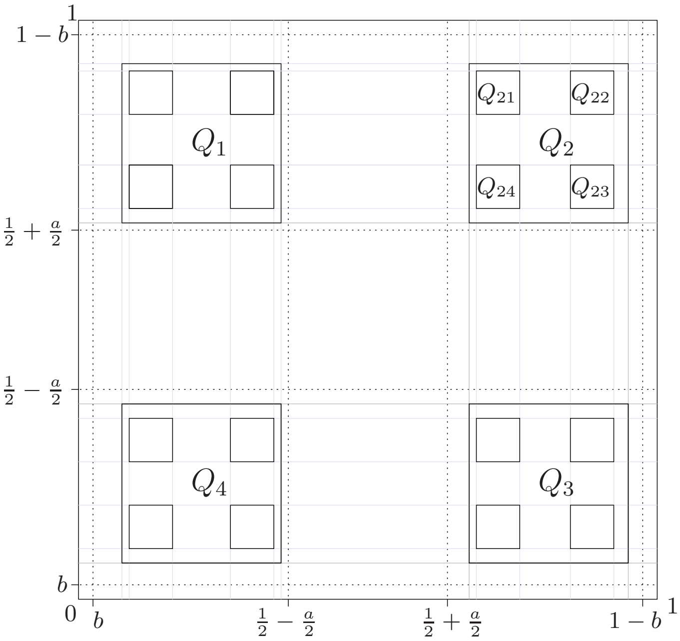

FIGURE 3. The first level squares $Q_1, \ldots, Q_4$ and four second level squares Q_{21}, Q_{22}, Q_{23}, Q_{24}.

For an $x \in [-1, 1]$ we write $e(x)$ for that line with slope 1 which intersects the vertical axis at x. As we observed above

$$x \in C_2 - C_1 \text{ if and only if } e(x) \cap (C_1 \times C_2) \neq \emptyset. \tag{4}$$

Fix x and an arbitrary n. Let $\mathcal{S}_n$ be the set of all $a^n \times a^n$ squares contained in $[0, 1]^2$. Note that for every $Q \in \mathcal{S}_n$ by the statistical self-similarity of the construction the probability of the event $e(x) \cap (Q \cap (C_1 \times C_2)) \neq \emptyset$ is equal to the probability of the event $e(\Phi) \cap (C_1 \times C_2) \neq \emptyset$, where we construct $\Phi = \Phi(Q, x)$ as follows: We rescale the square Q (which is an $a^n \times a^n$ square) by the factor $1/a^n$. Then we choose Φ such that the line segment $e(\Phi) \cap [0, 1]^2$ is the rescaled copy of $e(x) \cap Q$. See Figure 4. More precisely, if (u, v) is the left bottom corner of Q, that is $Q = [u, u + a^n] \times [v, v + a^n]$ then we define

$$\Phi(Q, x) := \begin{cases} \frac{u-v+x}{a^n} & \text{if } e(x) \text{ intersects } Q \\ \Theta & \text{otherwise.} \end{cases} \tag{5}$$

Observe that $\Phi(Q, x) > 0$ if and only if the center of Q is located below the line $e(x)$. Further, $\Phi(Q, x) = 1$, if $e(x)$ intersects Q at the left upper corner and $\Phi(Q, x) = -1$ if $e(x)$ intersects Q at the right bottom corner.

3.2. **The probability space.** Let us define the dyadic tree $\mathcal{T} := \bigcup_{n=0}^{\infty} \{1, 2\}^n$. We will write $\underline{i}_n = i_1 i_2 \ldots i_n$ with i_k is 1 or 2 for the nodes of the tree, and Λ for the root of $\mathcal{T}$. For the construction of Larsson's Cantor set the probability space is $\Omega_1 = \left[0, \frac{1-3a-2b}{2}\right]^{\mathcal{T}}$. An

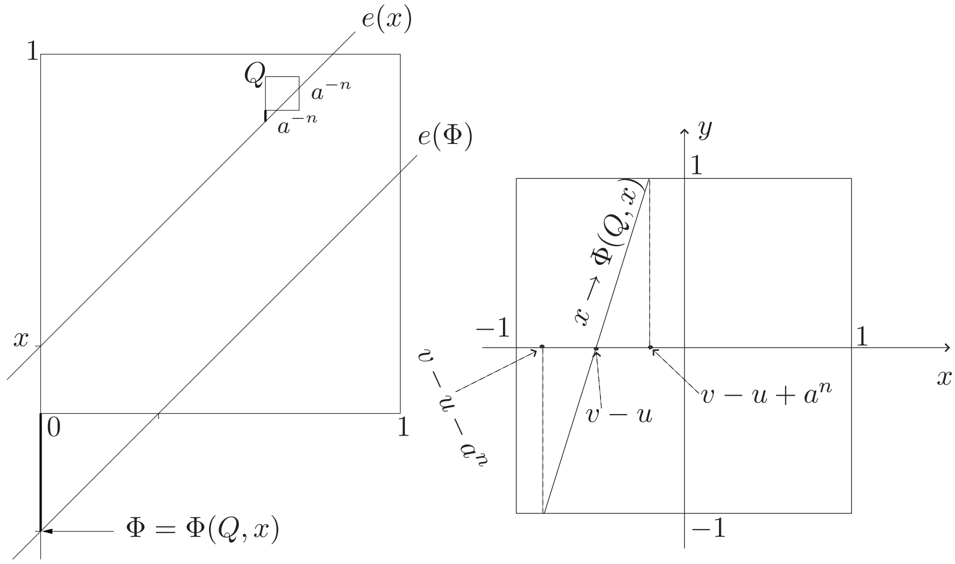

FIGURE 4. A level n square Q, and the function $\Phi(Q,x)$

element of Ω_1 is denoted U, i.e., the value at node $i_1i_2\ldots i_n$ is $U_{i_1i_2\ldots i_n}$. The corresponding σ-algebra is $\mathcal{B}_1 := \prod_{\mathcal{T}} \mathcal{B}\left[0, \frac{1-3a-2b}{2}\right]$. Finally, the probability measure for Larsson's Cantor set is

$$\mathbb{P}_1 := \delta_0 \times \prod_{\mathcal{T}\setminus\{\Lambda\}} \text{Uniform}\left[0, \frac{1-3a-2b}{2}\right].$$

So the probability space for $C_1 \times C_2$ is as follows:

$$\Omega := \Omega_1 \times \Omega_1, \qquad \mathcal{B} := \mathcal{B}_1 \times \mathcal{B}_1, \qquad \mathbb{P} := \mathbb{P}_1 \times \mathbb{P}_1. \tag{6}$$

An element of Ω is a pair of labelled binary trees. The 4^n level n pairs $(i_1i_2\ldots i_n, j_1j_2\ldots j_n)$ are naturally associated with level n squares $Q'_{(i_1i_2\ldots i_n, j_1j_2\ldots j_n)}$ of size $a^n \times a^n$ whose relative positions are given by by $U_{i_1i_2\ldots i_n}$ and $U_{j_1j_2\ldots j_n}$. Note however that (to simplify the notations) we have given new indices to these squares and positions: $Q'_{1,1} =: Q_4, Q'_{1,2} =: Q_1, Q'_{2,1} =: Q_3, Q'_{2,2} =: Q_2$, and similarly for higher order squares, and their positions.

3.3. **The branching process.** On the probability space Ω we define a multi type branching process $\mathcal{Z} = (\mathcal{Z}_n)_{n=0}^{\infty}$. For a Borel set A the natural number $\mathcal{Z}_n(A)$ represents the number of objects in generation n whose type falls into the set A. The type space T is a subset of $[-1,1]$, for the moment think of $T = [-1,1]$. The objects of the n^{th} generation are squares $Q \in \mathcal{S}_n$, and given a fixed $x \in [-1,1]$, their type is $\Phi(Q,x)$, as defined in (5). Note that although we speak of Θ as a type, it is *not* an element of T.

The process $(\mathcal{Z}_n)$ is a Markov chain whose states are collections of squares labelled with their types. The transition mechanism is as described in Section 3.1. The initial condition of the chain is the square

$[0,1] \times [0,1]$, with type x (also called the *ancestor* of the branching process). As usual we then write for $n \geq 1$

$$\mathbb{P}_x\left(\mathcal{Z}_n(A_1) = r_1, \ldots, \mathcal{Z}_n(A_k) = r_k\right) = \\ \mathbb{P}\left(\mathcal{Z}_n(A_1) = r_1, \ldots, \mathcal{Z}_n(A_k) = r_k | \mathcal{Z}_0(\{x\}) = 1\right),$$

for all $k \geq 1, A_1, \ldots, A_k \subset T$ and non-negative integers $r_1, \ldots, r_k$. A collection of squares all with type Θ is an absorbing state: it only generates squares with type Θ. This is obvious from the definition of $\Phi(Q, x)$, but we will extend this property to the case of smaller type spaces T, where by definition a square has type Θ if its type is not in T (this will be further explained in Section 6.1).
A major role in our analysis is played by the expectations $\mathbb{E}_x[\mathcal{Z}_n(A)]$, for $A \subset T$, $n \geq 1$. Let us define for $i = 1, 2, 3, 4$

$$\mathcal{Z}_1^i(A) = \begin{cases} 1 & \text{if } \Phi(Q_i, x) \in A \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Then $\mathcal{Z}_1(A) = \mathcal{Z}_1^1(A) + \cdots + \mathcal{Z}_1^4(A)$, and so

$$\begin{aligned} \mathbb{E}_x[\mathcal{Z}_1(A)] &= \int_\Omega \mathcal{Z}_1(A)\,\mathrm{d}\mathbb{P}_x = \int_\Omega \sum_{i=1}^4 \mathcal{Z}_1^i(A)\,\mathrm{d}\mathbb{P}_x \\ &= \sum_{i=1}^4 \mathbb{P}_x(\Phi(Q_i, x) \in A) = \sum_{i=1}^4 \int_A f_{x,i}(y)\,\mathrm{d}y, \end{aligned}$$

where the $f_{x,i}$ are the densities of the random variables $\Phi(Q_i, x)$ (apart from an atom in Θ). In Section 5.2 these densities will be determined explicitly. It follows that for $n = 1$

$$M_n(x, A) := \mathbb{E}_x[\mathcal{Z}_n(A)]$$

has a density $m_1(x, y)$, called the kernel of the branching process, given by

$$m(x, y) := m_1(x, y) = \sum_{i=1}^4 f_{x,i}. \tag{8}$$

We remark that if M_1 has a density then M_n also has a density. Let us write $m_n(x, \cdot)$ for the density of $M_n(x, \cdot)$. The branching structure of $\mathcal{Z}$ yields (see [4, p.67])

$$m_{n+1}(x, y) = \int_T m_n(x, z) m_1(z, y)\,\mathrm{d}z. \tag{9}$$

The main problem to be solved is that the natural choice of $T = [-1, 1]$ as type space does not work because of condition **C** below, and because we need the uniformity alluded to in Equation (1).
Since the definition of T is complicated we postpone it to Section 6. However, here we collect the most important properties of T:

A: T is the disjoint union of finitely many closed intervals.

B: There exists a $K > 0$ such that $[-K, K] \subset T$.
C: The kernel $m_n(x, y)$ defined in (9) is uniformly positive on $T \times T$ (see Condition (**C1**) below) and it has Perron-Frobenius eigenvalue greater than 1 (see Condition (**C2**)below).

3.4. **The asymptotic behavior of the branching process $\mathcal{Z}$.** We will prove later that there exists an integer n_0 such that m_{n_0} is a uniformly bounded function, that is, there exist $0 < a < b$ such that for all $x, y \in T$ we have

$$0 < a \leq m_{n_0}(x, y) \leq b < \infty. \tag{C1}$$

In the next step we consider the following two operators:

$$g(x) \mapsto \int_{\mathbb{R}} m_1(x, y) \cdot g(y)\, \mathrm{d}y, \qquad h(y) \mapsto \int_{\mathbb{R}} h(x) \cdot m_1(x, y)\, \mathrm{d}x. \tag{10}$$

We use the following theorem proved in [4, Theorem 10.1]:

Theorem 2 (Harris)**.** *It follows from* (**C1**) *that the operators in (10) have a common dominant eigenvalue ρ. Let $\mu(x)$ and $\nu(y)$ be the corresponding eigenfunctions of the first and second operator in (10) respectively. Then the functions $\mu(x)$ and $\nu(y)$ are bounded and uniformly positive. Moreover, apart from a scaling, μ and ν are the only non-negative eigenfunctions of these operators. Furthermore, if we normalize μ and ν so that $\int \mu(x)\nu(x)\,\mathrm{d}x = 1$, which will be henceforth assumed, then for al $x, y \in T$ as $n \to \infty$*

$$m_n(x, y) = \rho^n \mu(x)\nu(y)[1 + O(\Delta^n)], \quad 0 < \Delta < 1,$$

where the bound Δ can be taken independently of x and y.

Later in this paper we will prove that this Perron-Frobenius eigenvalue is greater than one:

$$\rho > 1. \tag{C2}$$

Using this, Harris proves that $\mathcal{Z}_n(A)$ grows in fact exponentially with rate ρ. Introducing

$$W_n(A) := \frac{\mathcal{Z}_n(A)}{\rho^n}.$$

he obtains (see [4, Theorem 14.1])

Theorem 3 (Harris)**.** *If*

$$\sup_{x \in T} \mathbb{E}_x[\mathcal{Z}_1(T)^2] < \infty, \tag{C3}$$

then it follows from (**C1**), (**C2**) *that for all $x \in T$*

$$\mathbb{P}_x\left(\lim_{n\to\infty} W_n(A) =: W(A)\right) = 1. \tag{11}$$

Further, for every Borel measurable $A \subset T$ with $\mathcal{L}\mathbf{eb}_1(A) > 0$ we have

$$\mathbb{P}_x\left(W(A) > 0\right) > 0. \tag{12}$$

Moreover, let A and B be subsets of T such that their Lebesgue measures are positive. Then the relation

$$W(B) = \frac{\int_B \nu(y)\,\mathrm{d}y}{\int_A \nu(y)\,\mathrm{d}y} W(A)$$

holds $\mathbb{P}_x$ almost surely for any $x \in T$.

We are going to use this theorem to prove our Main Lemma which summarizes all that we need about our branching process. Roughly speaking, the Main Lemma says that for the branching process associated to Larsson's Cantor set the statement in Theorem 3 holds uniformly both in n and x for an appropriately chosen small interval of x's.

Main Lemma. *There exist positive numbers δ and q and there exists an $N \in \mathbb{N}$ and a small interval $[-K, K] \subset T$ centered at the origin such that the following inequality holds*

$$\inf_{n \geq N} \inf_{x \in [-K,K]} \mathbb{P}_x \left(\mathcal{Z}_n([-K, 0]) > \delta\rho^n, \mathcal{Z}_n([0, K]) > \delta\rho^n \right) \geq q. \tag{13}$$

4. The proof of Theorem 1

In Section 3.1 we defined the type of a square Q by means of its intersection with a line $e(x)$. Here we will elaborate on this intersection.

4.1. **Nice intersection of a square with a line $e(x)$.** We say that a square Q has a *nice intersection* with $e(x)$ if

$$\Phi(Q, x) \in [-K, K],$$

where K comes from the Main Lemma. For small K this means that the center of Q is close to the line $e(x)$.
Let $\mathcal{A}_x^n$ be the set of squares from $C_1^n \times C_2^n$ having nice intersection with $e(x)$. That is for $x \in T$ and $n \geq 1$ we define

$$\mathcal{A}_x^n := \{Q \in \mathcal{S}_n : |\Phi(Q, x)| \leq K\}.$$

Moreover, for $m \geq 0$ for a square $Q \in \mathcal{S}_m$ we write $l_n^+(Q, x)$, ($l_n^-(Q, x)$) for the number of level $m + n$ squares contained in Q which have nice intersection with $e(x)$ with center below (above) the line $e(x)$ respectively. That is for a $Q = Q_{i_1 \dots i_m}$ let

$$l_n^+(Q, x) = \#\{Q_{i_1 \dots i_m j_1 \dots j_n} \in \mathcal{S}_{m+n} : 0 \leq \Phi(Q_{j_1 \dots j_n}, x) \leq K\}.$$

similarly, let

$$l_n^-(Q, x) = \#\{Q_{i_1 \dots i_m j_1 \dots j_n} \in \mathcal{S}_{m+n} : -K \leq \Phi(Q_{j_1 \dots j_n}, x) \leq 0\}.$$

Finally, for every $n \geq N$, $x \in T$ and $Q \in \mathcal{S}_m$ we define the event

$$A_n(Q, x) := \left\{ l_n^-(Q, x) > \delta\rho^n, l_n^+(Q, x) > \delta\rho^n \right\},$$

where δ and N come from the Main Lemma. Note that the self-similarity of the construction of the squares and the Main Lemma for the underlying branching process imply the following: for $n \geq N$ and a square $Q \in \mathcal{S}_m$ having nice intersection with $e(x)$ we have

$$\mathbb{P}\left(A_n(Q,x)\right) = \mathbb{P}_{\Phi(Q,x)}(\mathcal{Z}_n([-K,0]) > \delta\rho^n,\ \mathcal{Z}_n([0,K]) > \delta\rho^n) \geq q.$$

4.2. **The difference set $C_2 - C_1$ contains an interval with positive $\mathbb{P}$ probability.** We introduce the interval

$$I := \left[-Ka^N, Ka^N\right]$$

with N and K from the Main Lemma. Note that $|I| := \mathcal{L}\mathbf{eb}_1(I) = 2Ka^N$.

Our goal is to prove that

$$\mathbb{P}(C_2 - C_1 \supset I) > 0.$$

First we divide the interval I into 4^{2N} intervals I_{i_1} of equal length. Then we divide all of these intervals into 4^{3N} intervals $I_{i_1 i_2}$ of equal length. If we have already defined the $(k-1)$-th level intervals then we define the k-th level intervals $I_{i_1\dots i_k}$ by subdividing each $(k-1)$-th level interval $I_{i_1\dots i_{k-1}}$ into $4^{(k+1)N}$ intervals of equal length. We denote the center of $I_{i_1\dots i_k}$ by $z_{i_1\dots i_k}$. That is

$$I_{i_1\dots i_k} = \left[z_{i_1\dots i_k} - Ka^N 4^{-[2+\dots+(k+1)]N}, z_{i_1\dots i_k} + Ka^N 4^{-[2+\dots+(k+1)]N}\right]$$

where the $z_{i_1\dots i_k}$ are equally spaced in $I_{i_1\dots i_{k-1}}$.

Note that the interval $I_{i_1\dots i_k}$ has length

$$|I_{i_1\dots i_k}| = 2Ka^N 4^{-[2+\dots+(k+1)]N} < 2Ka^{g_k}$$

where we put

$$g_k := (1 + \ldots + (k+1))N = \frac{1}{2}(k+1)(k+2)N.$$

In the following we will go from generation g_{k-1} to generation g_k.

Definition 1. *We say that the event $B_k(z_{i_1\dots i_k})$ occurs, if there exists some square $Q \in \mathcal{S}_{g_{k-1}}$, $Q \subset C_1^{g_{k-1}} \times C_2^{g_{k-1}}$ having itself nice intersection with $e(z_{i_1\dots i_k})$, such that $A_{(k+1)N}\left(Q, z_{i_1\dots i_k}\right)$ holds. In formulae:*

$$B_k(z_{i_1\dots i_k}) = \bigcup_{Q \in \mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_k}}} A_{(k+1)N}\left(Q, z_{i_1\dots i_k}\right). \tag{14}$$

The following lemma is one of the key statements of the argument.

Lemma 1. *Assume that $B_k(z_{i_1\dots i_k})$ occurs with the square Q. Let $\mathcal{Q}^+$ and $\mathcal{Q}^-$ be the collections of level g_k squares within Q having nice intersection with $e(z_{i_1\dots i_k})$ with center below and above the line $e(z_{i_1\dots i_k})$ respectively. Then*

(1)

$$Proj_{45^\circ}\left(\bigcup_{\widetilde{Q} \in \mathcal{Q}^+} \widetilde{Q}\right) \supset I_{i_1\dots i_k}, \qquad Proj_{45^\circ}\left(\bigcup_{\widetilde{Q} \in \mathcal{Q}^-} \widetilde{Q}\right) \supset I_{i_1\dots i_k}.$$

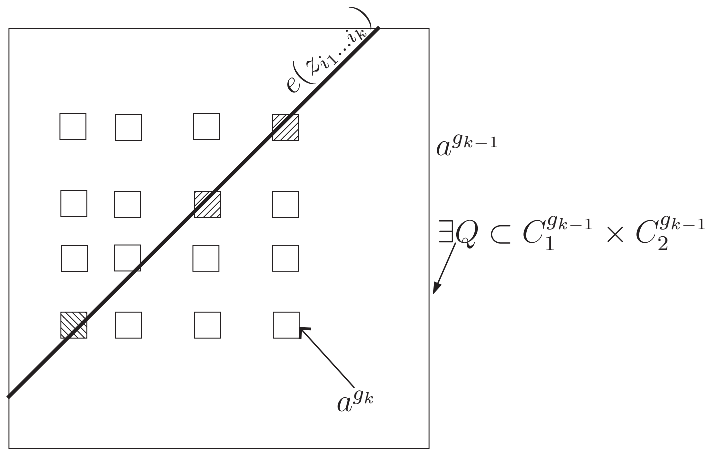

FIGURE 5. Event $B_k(z_{i_1\ldots i_k})$: there is a level g_{k-1} square Q in which the number of striped level g_k squares (the nicely intersecting ones) is at least $\delta\rho^{N(k+1)}$, both for the squares with center above and the squares with center below the line $e(z_{i_1\ldots i_k})$.

(2) For every $i_{k+1} = \pm 1, \ldots, \pm\frac{1}{2}4^{(k+2)N}$ the line $e(z_{i_1\ldots i_k i_{k+1}})$ has nice intersection with all squares from either $\mathcal{Q}^+$ or $\mathcal{Q}^-$. Thus the line $e(z_{i_1\ldots i_k i_{k+1}})$ has nice intersection with at least $\delta\rho^{(k+1)N}$ squares contained in Q such that either all have center below the line $e(z_{i_1\ldots i_k})$ or all have center above the line $e(z_{i_1\ldots i_k})$.

Proof. Pick an arbitrary $y \in I_{i_1\ldots i_k}$. Without loss of generality we may assume that $y \leq z_{i_1\ldots i_k}$. Then it is enough to prove that $e(y)$ has nice intersection with all squares from $\mathcal{Q}^+$. Fix an arbitrary $Q \in \mathcal{Q}^+$. By the definition of $\mathcal{Q}^+$ the square Q is a level g_k square such that its left bottom corner is in between the parallel lines $e(z_{i_1\ldots i_k})$ and $e(z_{i_1\ldots i_k} - Ka^{g_k})$. So for every point $y^* \in [z_{i_1\ldots i_k} - Ka^{g_k}, z_{i_1\ldots i_k}]$ the line $e(y^*)$ has nice intersection with Q—see Figure 6.

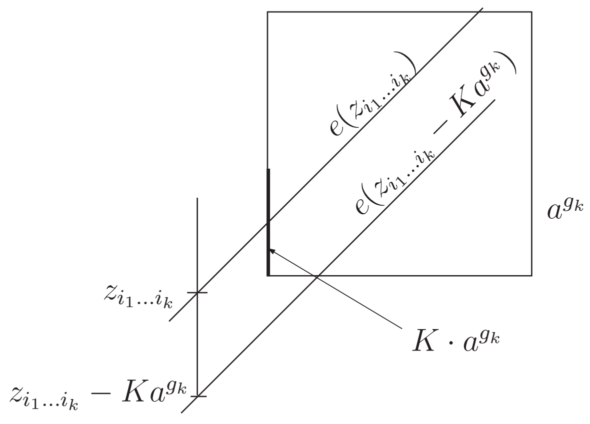

FIGURE 6.

The only thing that remains to be proved is that $y \in [z_{i_1\dots i_k} - Ka^{g_k}, z_{i_1\dots i_k}]$. This is so because we have assumed that $y \le z_{i_1\dots i_k}$ and

$$\frac{1}{2}|I_{i_1\dots i_k}| = Ka^N 4^{-[2+\dots+(k+1)]N} < Ka^N a^{[2+\dots+(k+1)]N} = Ka^{g_k}.$$

□

Definition 2. *Let* $E_0 := A_N([0,1]^2, 0)$*, and let* $E_k := \bigcap_{i_1\dots i_k} B_k(z_{i_1\dots i_k})$*.*

Lemma 2. *The following inequality holds*

$$\mathbb{P}(C_2 - C_1 \supset I) \ge q \prod_{k \ge 1} \mathbb{P}\left(E_k | E_{k-1}\right). \tag{15}$$

Proof. Using that $I = \left[-Ka^N, Ka^N\right] = \bigcup_{i_1\dots i_k} I_{i_1\dots i_k}$ it follows immediately from Lemma 1 that if the event E_k holds then the event

$$S_k := \{Proj_{45^\circ}\left(C_1^{g_k} \times C_2^{g_k}\right) \supset I\}$$

will hold. Therefore $E_k \subset S_k$. Since the sets $C_1^{g_k} \times C_2^{g_k}$ are decreasing we obtain that $S_k \supset S_{k+1}$. Thus

$$\begin{aligned} \mathbb{P}(C_2 - C_1 \supset I) &= \mathbb{P}\left(\cap_{k\ge1} S_k\right) = \lim_{k\to\infty} \mathbb{P}(S_k) \ge \inf_{k\ge1} \mathbb{P}(E_k) \\ &\ge \mathbb{P}(E_0) \prod_{k\ge1} \mathbb{P}\left(E_k | E_{k-1}\right). \end{aligned}$$

The last inequality holds since

$$\begin{aligned} \mathbb{P}(E_0) \prod_{i\ge1} \mathbb{P}\left(E_i | E_{i-1}\right) &\le \mathbb{P}(E_0)\mathbb{P}(E_1|E_0)\cdots\mathbb{P}(E_k|E_{k-1}) \\ &= p\mathbb{P}(E_k E_{k-1}) \le \mathbb{P}(E_k) \end{aligned}$$

where

$$p = \frac{\mathbb{P}(E_0)}{\mathbb{P}(E_0)} \frac{\mathbb{P}(E_1 E_0)}{\mathbb{P}(E_1)} \cdots \frac{\mathbb{P}(E_{k-1}E_{k-2})}{\mathbb{P}(E_{k-1})} \le 1.$$

Since the Main Lemma yields $\mathbb{P}(E_0) \ge q$ one obtains the statement of the lemma. □

In the next Lemma 3 we give a lower bound for $\mathbb{P}\left(E_k | E_{k-1}\right)$ for every k.

Lemma 3. *For any* $k \ge 1$ *we have*

$$\mathbb{P}\left(E_k | E_{k-1}\right) \ge 1 - 4^{2N+\dots+(k+1)N}(1-q)^{\delta\rho^{kN}}.$$

Proof. We recall that E_k was defined as

$$E_k := \bigcap_{i_1\dots i_k} B_k(z_{i_1\dots i_k}).$$

So we have to prove that

$$\mathbb{P}\Big(\bigcup_{i_1\dots i_k} B_k^c(z_{i_1\dots i_k})\Big| E_{k-1}\Big) \le 4^{2N+\dots+(k+1)N}(1-q)^{\delta\rho^{kN}}.$$

Note that the number of indices $i_1 \dots i_k$ on the left hand side is equal to $4^{2N+\dots+(k+1)N}$. Therefore it is enough to show that for each index $i_1 \dots i_k$ we have

$$\mathbb{P}\left(B_k^c(z_{i_1\dots i_k})|E_{k-1}\right) \leq (1-q)^{\delta\rho^{kN}}.$$

By Definition 1 to see this we have to prove that

$$\mathbb{P}\Big(\bigcap_{Q\in\mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_k}}} A^c_{(k+1)N}\left(Q, z_{i_1\dots i_k}\right)\Big| E_{k-1}\Big) \leq (1-q)^{\delta\rho^{kN}}. \tag{16}$$

We assume E_{k-1} so in particular we know that $B_{k-1}(z_{i_1\dots i_{k-1}})$ holds. That is, there exists a level g_{k-2} square Q_{big} such that the event $A_{kN}\left(Q_{\text{big}}, z_{i_1\dots i_{k-1}}\right)$ holds. By definition this means that we can find level g_{k-1} squares $Q_1^+,\dots,Q^+_{\delta\rho^{kN}} \subset Q_{\text{big}}$ having center below the line $e(z_{i_1\dots i_{k-1}})$ and similarly level g_{k-1} squares $Q_1^-,\dots,Q^-_{\delta\rho^{kN}} \subset Q_{\text{big}}$ having center above the line $e(z_{i_1\dots i_{k-1}})$ such that all of these squares have nice intersection with the line $e(z_{i_1\dots i_{k-1}})$. Using the second part of Lemma 1 (for k instead of $k+1$) we obtain that for all i_k the line $z_{i_1\dots i_k}$ has nice intersection either with all the squares $Q_1^+,\dots,Q^+_{\delta\rho^{kN}}$ or with all the squares $Q_1^-,\dots,Q^-_{\delta\rho^{kN}}$. Without loss of generality we may assume the first. That is

$$Q_1^+,\dots,Q^+_{\delta\rho^{kN}} \in \mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_k}}.$$

So, to verify (16) it is enough to show that

$$\mathbb{P}\left(\bigcap_{j=1}^{\delta\rho^{kN}} A^c_{(k+1)N}\left(Q_j^+, z_{i_1\dots i_k}\right)\middle| E_{k-1}\right) \leq (1-q)^{\delta\rho^{kN}}. \tag{17}$$

Observe that all the events in the intersection are conditionally independent. Thus it is enough to verify that

$$\prod_{j=1}^{\delta\rho^{kN}} \mathbb{P}\left(A^c_{(k+1)N}\left(Q_j^+, z_{i_1\dots i_k}\right)|E_{k-1}\right) \leq (1-q)^{\delta\rho^{kN}}.$$

This holds because $A^c_{(k+1)N}(Q_j^+, z_{i_1\dots i_k})$ is independent of E_k and therefore the left hand side is equal to

$$\prod_{j=1}^{\delta\rho^{kN}} \mathbb{P}\left(A^c_{(k+1)N}\left([0,1]^2, \Phi(Q_j^+, z_{i_1\dots i_k})\right)\right) \leq (1-q)^{\delta\rho^{kN}},$$

where the last inequality follows from the Main Lemma. □

Lemma 4. *For all $n \geq 1$:*

$$\prod_{j=1}^{\infty}\left(1 - 4^{[2+\dots+(j+1)]n}(1-q)^{\delta\rho^{jn}}\right) > 0. \tag{18}$$

Proof. We have to show that $\sum_{j=1}^{\infty} a_j$ converges, where

$$a_j = 4^{\frac{1}{2}j(j+1)n}(1-q)^{\delta\rho^{jn}}.$$

It is therefore sufficient that $a_j \le e^{-j}$ for all large j. This is true since

$$\frac{1}{j}\log a_j = \frac{1}{2}(j+1)n\log 4 + \frac{1}{j}\delta\,(\rho^n)^j\log(1-q) \le -1,$$

which holds for j large enough, since $\rho^n > 1$ and $\log(1-q) < 0$. $\square$

Therefore, using Lemma 2 and 3, 4 we obtain that

$$\mathbb{P}(C_2 - C_1 \supset I) \ge q\prod_{k=1}^{\infty}\left(1 - 4^{[2+\cdots+(k+1)]N}(1-q)^{\delta\rho^{kN}}\right) > 0.$$

Combining this with Proposition 1 from Section 2 this completes the proof of Theorem 1.
In the next six sections we prove our Main Lemma.

5. Distribution of types

In this section the density function of $\Phi(Q,x)$ will be determined if $Q \in \mathcal{S}_1$.

5.1. **The distribution of $\Phi(Q,x)$.** Let U_1, U_2, U_3, U_4 be independent $Uniform\left(\frac{1-3a-2b}{2}\right)$ distributed random variables. The left corners of the two level one intervals of the random Cantor set C_i are determined by U_{2i-1}, U_{2i} for $i = 1, 2$. Let (u_i, v_i) be the left bottom corner of the squares Q_i, $i = 1, \ldots, 4$ (see Figure 7). Then

$$\begin{aligned}
(u_1, v_1) &= \left(b + U_1, \frac{1}{2} + \frac{a}{2} + U_4\right)\\
(u_2, v_2) &= \left(\frac{1}{2} + \frac{a}{2} + U_2, \frac{1}{2} + \frac{a}{2} + U_4\right)\\
(u_3, v_3) &= \left(\frac{1}{2} + \frac{a}{2} + U_2, b + U_3\right)\\
(u_4, v_4) &= (b + U_1, b + U_3)
\end{aligned}$$

For an $x \in [-1,1]$ we define $\Phi_i(x) := \Phi(Q_i, x)$. From (5) simple computations yield

$$\Phi_1(x) = \begin{cases} \frac{1}{a}\left(-\frac{1}{2} - \frac{a}{2} + b + U_1 - U_4 + x\right) \\ \qquad\qquad \text{if } \frac{1}{a}\left(-\frac{1}{2} - \frac{a}{2} + b + U_1 - U_4 + x\right) \in [-1,1] \\ \Theta \qquad\qquad \text{otherwise} \end{cases}$$

$$\Phi_2(x) = \begin{cases} \frac{1}{a}(U_2 - U_4 + x) & \text{if } \frac{1}{a}(U_2 - U_4 + x) \in [-1,1] \\ \Theta & \text{otherwise} \end{cases}$$

$$\Phi_3(x) = \begin{cases} \frac{1}{a}\left(\frac{1}{2} + \frac{a}{2} - b + U_2 - U_3 + x\right) \\ \qquad\qquad \text{if } \frac{1}{a}\left(\frac{1}{2} + \frac{a}{2} - b + U_2 - U_3 + x\right) \in [-1,1] \\ \Theta \qquad\qquad \text{otherwise} \end{cases}$$

$$\Phi_4(x) = \begin{cases} \frac{1}{a}(U_1 - U_3 + x) & \text{if } \frac{1}{a}(U_1 - U_3 + x) \in [-1,1] \\ \Theta & \text{otherwise} \end{cases} \tag{19}$$

To get a better geometric understanding of the distribution of the random variables $\Phi_i(x)$ we define the three slanted stripes S_k, $k = 1,2,3$ (see Figure 8) in such a way that $S_k \subset [-1,1]^2$ is bounded by the lines ℓ_{2k-1}, ℓ_{2k}, where

$$\begin{aligned} \ell_1(x) &= \frac{1}{a}x + \frac{1}{a}(1-a-2b), & \ell_2(x) &= \frac{1}{a}x + 2, \\ \ell_3(x) &= \frac{1}{a}x + \frac{1}{2a}(1-3a-2b), & \ell_4(x) &= \frac{1}{a}x - \frac{1}{2a}(1-3a-2b), \\ \ell_5(x) &= \frac{1}{a}x - 2, & \ell_6(x) &= \frac{1}{a}x - \frac{1}{a}(1-a-2b). \end{aligned} \tag{20}$$

An immediate calculation shows that

Lemma 5. *For every $x \in [-1,1]$ and for every $i = 1,\dots,4$ if $\Phi_i(x) \neq \Theta$ then*

$$(x, \Phi_i(x)) \in S_1 \cup S_2 \cup S_3.$$

Let us call ℓ_j the graph of the function $\ell_j(x)$. Observe that the reflection in the origin of ℓ_j is ℓ_{7-j} for $j = 1,\dots,6$. For a point $(x_1,x_2) \in \mathbb{R}^2$ we write $\pi_m(x_1,x_2) := x_m$, $m = 1,2$. Then we define $c > 0$ by

$$-1 + c := \pi_1\left(\ell_1 \cap \{y = x\}\right),$$

and obtain $c = \frac{2b}{1-a}$. By symmetry it follows that

$$1 - c = \pi_1\left(\ell_6 \cap \{y = x\}\right).$$

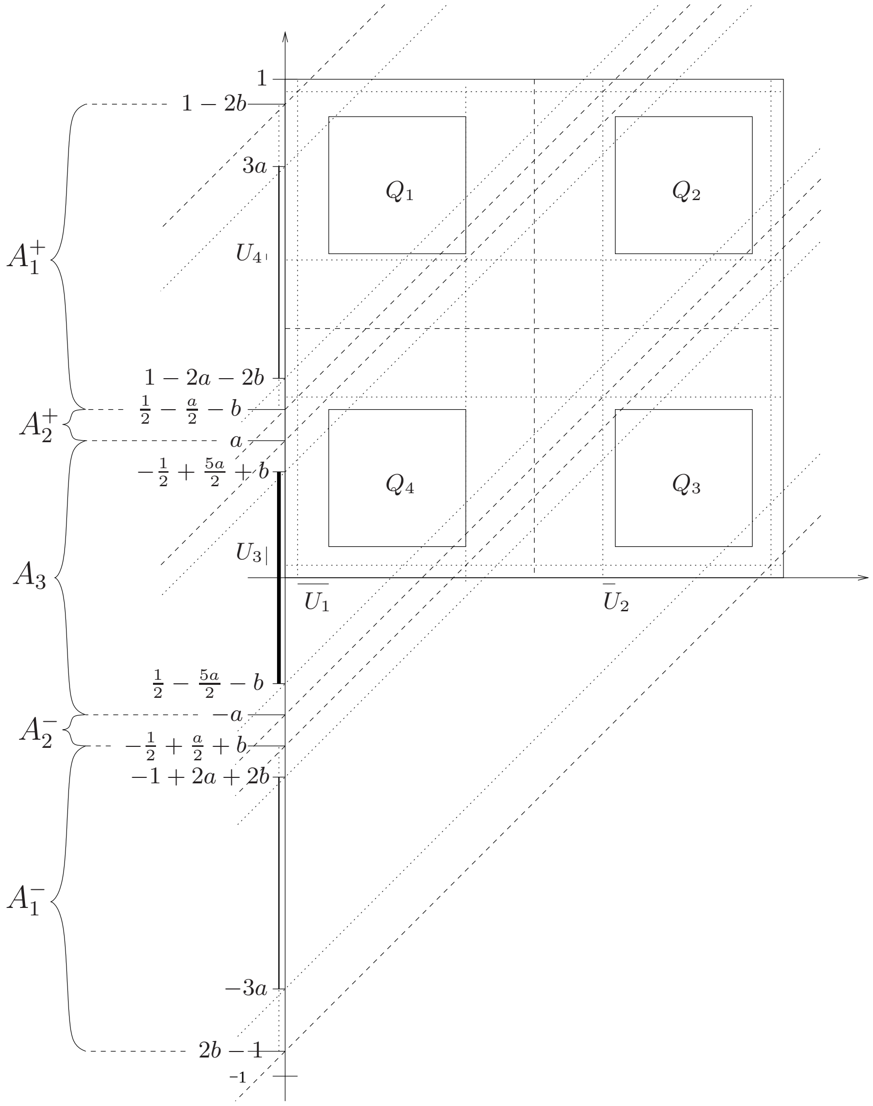

FIGURE 7. If x is an element of the plain bold vertical line then the line $e(x)$ intersects exactly two squares. If x is an element of one of the two plain vertical lines then $e(x)$ intersects one square. If x is an element of one of the four dotted vertical lines then $e(x)$ intersects at most 1 square. If x is such that $a \leq x \leq 1-2a-2b$ or $-1+2a+2b \leq x \leq -a$ then $e(x)$ intersects at most two squares with probability one. If x is such that $-\frac{1}{2}+\frac{5a}{2}+b \leq x \leq a$ or $-a \leq x \leq \frac{1}{2}-\frac{5a}{2}-b$ then $e(x)$ intersects exactly two squares.

Using that $-1+2b = \pi_1\left(\ell_1 \bigcap \{y=-1\}\right)$ it follows from the symmetry mentioned above that
(21)
$x \notin (-1+2b, 1-2b, 1) \Longrightarrow e(x)$ does not intersect any level 1 square.

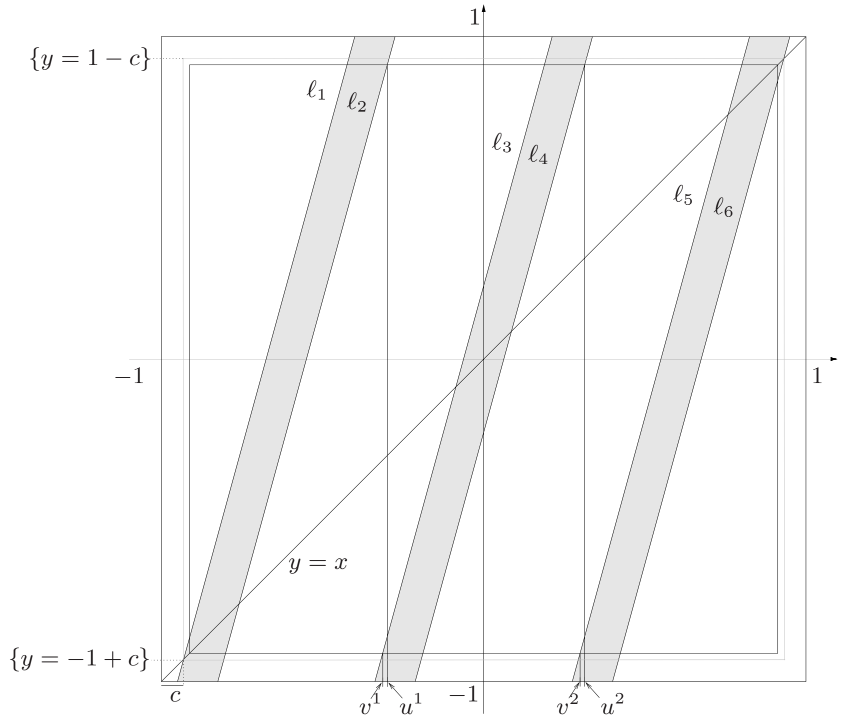

FIGURE 8. The support of the density functions in the simple case.

The functions $\ell_1(x), \ell_6(x)$ have repelling fixed point $-1+c, 1-c$ respectively. Therefore

(22) $x \in [-1,-1+c) \cup (1-c,1] \Longrightarrow \exists n;\ e(x) \cap Q = \emptyset$ for all $Q \in \mathcal{S}_n$.

With probability 1 no line $e(x)$ can intersect more than two descendants, in fact, T can be partitioned into five sets according to which descendants can be produced, where (see also Figure 7)

$$A_1^- = \Big[-1+2b, -\frac{1}{2}+\frac{a}{2}+b\Big) \cap T, \quad A_1^+ = \Big(\frac{1}{2}-\frac{a}{2}-b, 1-2b\Big] \cap T,$$
$$A_2^- = \Big[-\frac{1}{2}+\frac{a}{2}+b, -a\Big) \cap T, \quad A_2^+ = \Big(a, \frac{1}{2}-\frac{a}{2}-b\Big] \cap T,$$
$$A_3 = \Big[-a, a\Big] \cap T.$$

Lemma 6. *If $x \in A_3$ then x can only produce descendants with type $X_2(x)$ and/or $X_4(x)$. If $x \in A_1^+$ ($x \in A_1^-$) then x can produce at most one descendant with type $X_1(x)$ ($X_3(x)$). If $x \in A_2^+$ then there are two possibilities. First, if x produces $X_1(x)$ then $X_2(x)$ and $X_4(x)$ can not be born. Second, if x produces any of $X_2(x)$ and $X_4(x)$ then $X_1(x)$ can not be born. If $x \in A_2^-$ then there are two similar possibilities.*

Proof. Using Figure 7 observe that $Proj_{45^\circ}(Q_1) \cap Proj_{45^\circ}(Q_4) \neq \emptyset$ can happen only in the extreme situation if the bottom of the square Q_1 is the same as the bottom of the dotted square which contains Q_1 on

Figure 3. This means that $U_4 = 0$ which happens with probability zero. Similarly $Proj_{45^\circ}(Q_3) \cap Proj_{45^\circ}(Q_4) \neq \emptyset$ happens only if $U_2 = 0$ which also has probability zero. $Proj_{45^\circ}(Q_1) \cap Proj_{45^\circ}(Q_3) = \emptyset$ always holds which completes the proof of our Lemma. $\square$

5.2. **The density functions.** We will determine the density functions $f_{\Phi_i(x)}(y)$ of the random variables $\Phi_i(x)$, $i = 1, 2, 3, 4$ given explicitly by (19). We do not call them *probability* density functions since the $\Phi_i(x)$ may be equal to Θ with positive probability for some x. The probability density function f of the difference of two independent $Uniform\,(t)$ distributed random variables is the triangular distribution given by $f(z) = 0$ if $|z| > t$ and for $0 \leq |z| \leq t$ by

$$f(z) = \frac{1}{t^2}(t - |z|). \tag{23}$$

To get $f_{\Phi_i(x)}(y)$ we apply simple transformations to $f(z)$ with $t = (1 - 3a - 2b)/2$ and find:

$$\begin{aligned} f_{\Phi_1(x)}(y) &= af\big(ay + \frac{1}{2} + \frac{a}{2} - b - x\big)\, \mathbf{1}_{[-1,1]}(y) \\ f_{\Phi_2(x)}(y) &= f_{\Phi_4(x)}(y) = af(ay - x)\, \mathbf{1}_{[-1,1]}(y) \\ f_{\Phi_3(x)}(y) &= af\big(ay - \frac{1}{2} - \frac{a}{2} + b - x\big)\, \mathbf{1}_{[-1,1]}(y) \end{aligned} \tag{24}$$

From the definition

$$\mathbb{P}(\Phi_i(x) = \Theta) = 1 - \int_{[-1,1]} f_{\Phi_i(x)}(y)\, \mathrm{d}y.$$

6. A uniformly positive kernel

Here and also in the next two sections we are going to define the type space T of the branching process introduced in Section 3.3. In order to ensure that conditions **C1, C2, C3** of Section 3.4 hold we introduce a type space T which also satisfies properties **A, B, C** of Section 3.3. It follows from (22) that we must choose our type space $T \subset [-1+c, 1-c]$.

Unfortunately, the construction of the typespace T satisfying the above conditions is quite involved and technical for those values of the parameters a, b which do not satisfy (2). Therefore, we split the presentation into two parts. In this section we present the construction of T throughout three lemmas: Lemmas 7A, 8A and 9A. In the next section we present the general case with the corresponding Lemmas 7, 8 and 9. The main difference between these lemma lies in the proof of Lemma 7 and Lemma 7A. Lemma 8 is almost literally the same as Lemma 8A. Finally, the proof of Lemma 9 follows the same trace as the proof of Lemma 9A but it is more technical.

6.1. **Descendant distributions and the kernel of the branching process.** We introduce the random variables $X_1(x), X_2(x), X_3(x), X_4(x)$ for $1 \leq i \leq 4$ by

$$X_i(x) = \begin{cases} \Phi_i(x) & \text{if } \Phi_i(x) \in T \\ \Theta & \text{otherwise.} \end{cases}$$

So the density of $X_i(x)$ is

$$f_{x,i}(y) := f_{\Phi_i(x)}(y)\mathbf{1}_T(y)$$

for $i = 1, \ldots, 4$. In general $X_i(x)$ also has an atom: $\mathbb{P}(X_i(x) = \Theta) = 1 - \int_T f_{x,i}(y)\,\mathrm{d}y$.
Recall (see Equation (8)) that the kernel of the branching process can be expressed as the sum of the density functions of the random variables $X_i(x), i = 1, \ldots, 4$:

$$m(x,y) = f_{x,1}(y) + f_{x,2}(y) + f_{x,3}(y) + f_{x,4}(y).$$

The structure of the support of this kernel is very important for the sequel. Since the functions $f_{x,i}(y)$ $(i = 1, 2, 3, 4)$ are piecewise continuous on $[-1, 1]$, $m(\cdot, \cdot)$ is piecewise continuous on $[-1, 1] \times [-1, 1]$. The support of $m(\cdot, \cdot)$ is a subset of the three slanting stripes $S_k, k = 1, 2, 3$ introduced earlier, see also Figure 8.

6.2. **The possible holes in the support of the kernel of $\mathcal{Z}$.** We have seen in (22) that the branching process with ancestor type in the set $[-1, -1 + c]$ or $[1 - c, 1]$ dies out in a finite number of generations almost surely. Therefore, it is reasonable to restrict the type space to $[-1+c+\varepsilon, 1-c-\varepsilon]$ for some small positive ε. However, in some cases we have to make further restrictions. Namely, for $i = 1, 2$ we define

$$u^i := \pi_1\left(\ell_{2i} \cap \{y = 1 - c\}\right), v^i := \pi_1\left(\ell_{2i+1} \cap \{y = -1 + c\}\right), \tag{25}$$

see Figure 8. Clearly $u^1 - v^1 = u^2 - v^2$ and an easy calculation shows that

$$v^1 < u^1 \iff c < \frac{1 - 3a - 2b}{4a}. \tag{26}$$

We remark that this condition is equivalent to the condition in Equation (2) (see also Figure 1). On the other hand, if $u^i < v^i$, $i = 1, 2$ holds then for $x \in [u^i, v^i]$ the set

$$E_1(x) := \{y : m(x, y) > 0\} \tag{27}$$

is contained in $[-1, -1 + c] \cup [1 - c, c]$. This implies that the process dies out in finitely many steps for $x \in [u^i, v^i]$ (see Figure 9). Therefore if the condition stated in (26) does not hold, then we have to make more restrictions on our type space $[-1+c+\varepsilon, 1-c-\varepsilon]$. This is what we are going to do in Section 8. For the convenience of the reader we treat in Section 7 the simpler case when (26) holds.

7. A uniformly positive kernel in the simple case

In the rest of this section we will prove that if (26) holds, i.e., $v^1 < u^1$ then we can choose a sufficiently small $\varepsilon_0 > 0$ such that

$$T = [-1 + c + \varepsilon_0, 1 - c - \varepsilon_0].$$

satisfies conditions (**C1**), (**C2**) and (**C3**) (and also properties **A, B, C**).

Lemma 7A. *Assume that $v^1 < u^1$. Fix an $\varepsilon > 0$ satisfying*

$$\varepsilon < \frac{1 - 3a - 2b}{4a} - c. \tag{28}$$

Further, in this simpler case let

$$T = T(\varepsilon) = [-1 + c + \varepsilon, 1 - c - \varepsilon]. \tag{29}$$

Then the kernel $m(x, y)$ of the branching process $\mathcal{Z}$ has the following property:
(30)
$\exists \kappa > 0$ *such that* $\forall x \in T$ *the set* $E_1(x)$ *contains an interval of length* κ.

Proof. There are two possibilities for the shape of $E_1(x)$ (defined in (27)):

1) $E_1(x)$ *consists of two intervals:*
   $[-1 + c + \varepsilon, \ell_{2k+1}(x)) \cup (\ell_{2k}(x), 1 - c - \varepsilon]$ (for $k = 1$ or $k = 2$). The length of one of these intervals is at least half of $\ell_3(u_1) - (-1 + c + \varepsilon)$, that is, $\kappa_1 = \frac{1}{2} \cdot \left(\frac{1 - 3a - 2b}{2a} - 2c \right)$.
2) $E_1(x) = (\ell_{2k-1}(x), \ell_{2k}(x))$ *(for some $1 \leq k \leq 3$) is an open interval with length $\kappa_2 = \frac{2}{a}(1 - 3a - 2b)$.*

Summarizing these cases, define $\kappa = \min\{\kappa_1, \kappa_2\}$.

□

Lemma 8A. *Let m^ε be the kernel in Lemma 7A with type space $T = T(\varepsilon)$ (29). One can choose $\varepsilon > 0$ which satisfies (28) such that the largest eigenvalue of m^ε is larger than 1. From now on we fix such an ε and we call it ε_0.*

Proof. Let $T(0) := [-1+c, 1-c]$, with corresponding kernel m^0. Define the operator $\mathcal{T}_\varepsilon$ for all $\varepsilon \geq 0$ by

$$\mathcal{T}_\varepsilon f(x) = \int_{\mathbb{R}} m^\varepsilon(x, y) f(y) \, \mathrm{d}y$$

for functions with $supp(f) \subset T(\varepsilon)$.

Note that $4a$ is an eigenvalue of $\mathcal{T}_0$ with eigenfunction $\nu(x) = \mathbf{1}_{T(0)}(x)$:

$$\begin{aligned}\mathcal{T}_0\nu(y) &= \int_{\mathbb{R}} \nu(x) m^0(x,y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \nu(x)\left(f_{x,1}(y) + f_{x,2}(y) + f_{x,3}(y) + f_{x,4}(y)\right)\mathbf{1}_{T(0)}(x)\,\mathrm{d}x \\ &= 4\nu(y)\int_{\mathbb{R}} af(ay-x)\,\mathrm{d}x = 4a\nu(y).\end{aligned}$$

since f is probability density.
The conclusion of the lemma follows from a simple fact noted by Larsson [6]: If the two kernels m^0 and m^ε are close to each other in L^2 sense, then the eigenvalues of the operators $\mathcal{T}_0$ and $\mathcal{T}_\varepsilon$ are close to each other. □

Lemma 9A. *Let T be as in Lemma 8A. There exists an index n such that for all $x \in T$, $\{y : m_n(x,y) > 0\} = T$.*

Since the function $m_n(\cdot,\cdot)$ is piecewise continuous on the compact set T, Lemma 9 implies that $m(x,y) \geq a > 0$ for any $x, y \in T$. Further, using that $m(x,\cdot)$ is bounded we immediately obtain that $\sup_{x\in T}\mathbb{E}_x\mathcal{Z}_1^2(T)$ is finite. So we have

Corollary 1. *Let T be as in Lemma 8. The branching process $\mathcal{Z}$ with type space T satisfies the conditions **C1** and **C3**.*

Proof of Lemma 9A. Basically, we will prove that if (30) holds then Lemma 9A also holds since the slope of the lines ℓ_i is equal to $\frac{1}{a}$ which is bigger than one. Let $E_n(x) = \{y : m_n(x,y) > 0\}$. We will prove that in both cases of the proof of Lemma 7A the sequence $(E_n(x))$ reaches the whole type space in finite number of steps uniformly in n and $x \in T$.
We can derive $E_{n+1}(x)$ from $E_n(x)$ by means of the equation

$$m_{n+1}(x,y) = \int_T m_n(x,z)m_1(z,y)\,\mathrm{d}z$$

which implies the following:

$$E_{n+1}(x) = \bigcup_{y\in E_n(x)} E_1(y) \tag{31}$$

In the proof of Lemma 7A we separated two cases. We continue the proof according to these two cases:

1) *$E_1(x)$ consists of two intervals.* Take the longer one, so its length is at least $\kappa_1 = \frac{1}{2}\cdot\left(\frac{1-3a-2b}{2a} - 2c\right)$. The following two facts hold. This interval contains either $-1+c+\varepsilon$ or $1-c-\varepsilon$ and if $E_n(x)$ contains one of these point then $E_{n+1}(x)$ also contains the same point because of (31). Therefore, if $E_n(x) \neq T$ and is of the

form e.g. $[-1+c+\varepsilon, -1+c+\varepsilon+s)$ for some positive s then $E_{n+1}(x) \supset \left[-1+c+\varepsilon, -1+c+\varepsilon+\frac{1}{a}s\right)$ or $E_{n+1}(x)=T$. Hence, if $E_1(x)=[-1+c+\varepsilon, -1+c+\varepsilon+s)$ then in

$$n_1(x) = \left\lceil \log_{\frac{1}{a}} \left(\frac{2(1-c-\varepsilon)}{s} \right) \right\rceil$$

number of steps $E_n(x)$ reaches T, that is, $E_{n_1(x)}(x)=T$. $s \geq \kappa_1$ implies $n_1(x) \leq \left\lceil \log_{\frac{1}{a}} \left(\frac{2(1-c-\varepsilon)}{\kappa_1} \right) \right\rceil = n_1^*$.

2) $E_1(x) = (\ell_{2k-1}(x), \ell_{2k}(x))$ *(for some* $1 \leq k \leq 3$*) is an open interval with length* $\kappa_2 = \frac{2}{a}(1-3a-2b)$. If for some n $E_n(x)$ does not contain neither $-1+c+\varepsilon$ nor $1-c-\varepsilon$ then we have three possibilities for $E_{n+1}(x)$: it does not contain any of these two points; or it contains one of them; or it equals T. In the first case the length of $E_{n+1}(x)$ equals $\frac{1}{a}|E_n(x)| + \frac{1-3a-2b}{a}$; in the second case we have $E_{n+n_1^*}(x)=T$ by case 1). So we estimate the number of necessary iterations from below if we suppose the first case happens in each step then the second in n_1^* number of steps. As in case 1) we have a uniform bound for the first number in 2): $n_2^* = \left\lceil \log_{\frac{1}{a}} \left(\frac{2(1-c-\varepsilon)}{\kappa_2} \right) \right\rceil$. Therefore, in this case we have $E_{n_1^*+n_2^*}(x)=T$ for any x.

Summarizing these considerations one obtains that for $n \geq n_1^*+n_2^*$ one has $E_n(x)=T$.

□

8. A uniformly positive kernel in the general case

The construction of T consists of two steps. We will call any open subset of $[-1,1]$ a pre-type space. First we inductively construct a sequence of pre-type spaces $T^0 \supset T^1 \supset \cdots \supset T^l$ and we prove that T^r, $r=0,\dots,l$ consists of 3^r disjoint open intervals of equal length. Those elements of T^l which are "far" from the endpoints of the components of T^l satisfy (32). Unfortunately, the same does not hold for the points close the the boundary of the components of T^l. So, as a second step of the construction of T we remove a small neighborhood of the boundary of T^l from T^l.

Lemma 7. *There exists a restriction of the pre-type space* $(-1+c, 1-c)$ *to a closed set* T *such that the kernel* m *of the branching process* $\mathcal{Z}$ *with type space* T *satisfies*

(32)

$\exists \kappa > 0$ *such that* $\forall x \in T$ *the set* $E_1(x)$ *contains an interval of length* κ.

Further, T *consists of* 3^l *disjoint closed intervals of equal length for some* $l \in \mathbb{N}$. *Moreover,* 0 *is contained in the interior of* T.

Proof of Lemma 7. We recall that u^1, v^1 were defined in (25) and we take the pre-type space $T^0 := (-1+c, 1-c)$. If $v^k < u^k$ then we

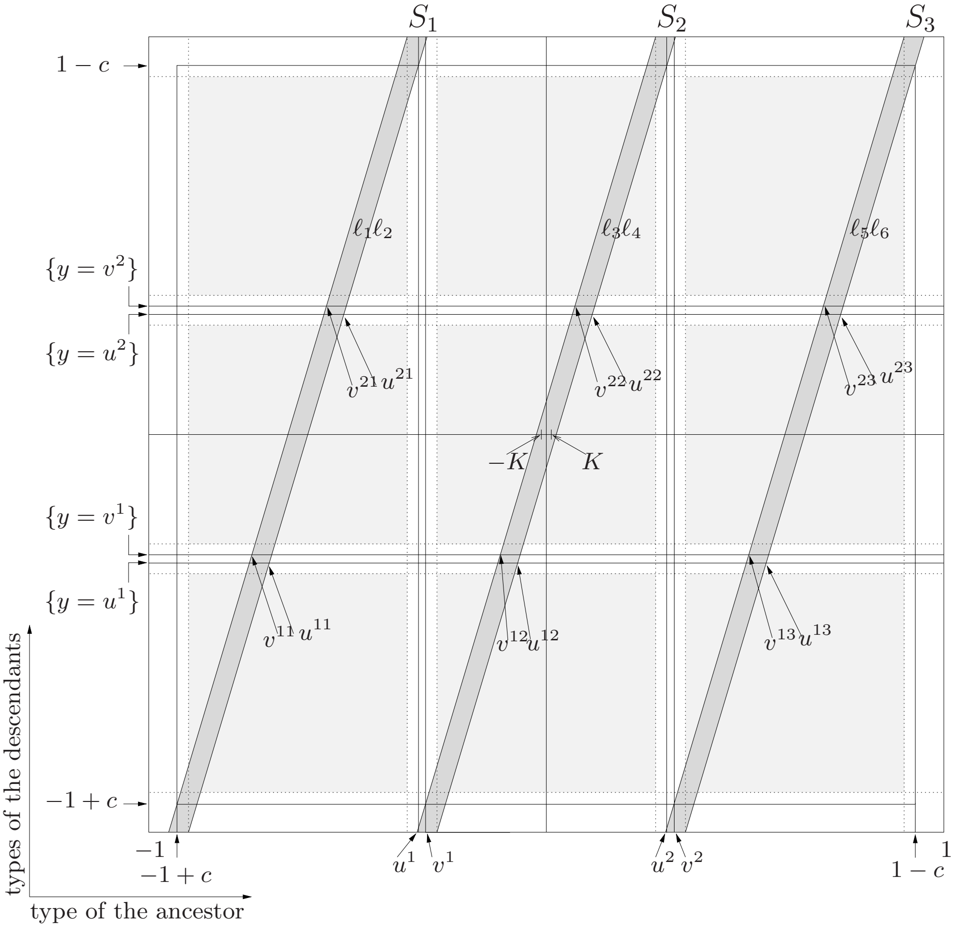

FIGURE 9. Some points and lines related to the kernel $m(x,y)$ if $l=1$

define $l := 0$ and the proof of (32) was settled in Lemma 7A. So, we can assume that $u^k \le v^k$, $k=1,2$. To insure that (32) holds we need to remove the intervals $[u^1,v^1]$ and $[u^2,v^2]$ from the pre-type space T^0 (cf. Figure 9). So, we restrict ourselves to the next level pre-type space: $T^1 = T^0 \backslash \{[u^1,v^1] \cup [u^2,v^2]\}$. The size of each of the intervals removed is $\varrho_1 := v^1 - u^1 = v^2 - u^2$. We define the second generation endpoints u^{i_1k} and v^{i_1k} as follows

$$u^{i_1k} = \pi_1(\{y = u^{i_1}\} \cap \ell_{2k}) \text{ and } v^{i_1k} = \pi_1(\{y = v^{i_1}\} \cap \ell_{2k-1}),$$

where $i_1 = 1,2$ and $k = 1,2,3$, see Figure 9. If $v^{i_1k} < u^{i_1k}$ then we define $l := 1$. Otherwise, we continue defining the sets T^r and the endpoints of the subtracted intervals $v^{i_1\dots i_r}$ and $u^{i_1\dots i_r}$ ($i_1 = 1,2$, $i_2,\dots,i_r = 1,2,3$) as follows: Assuming that $u^{i_1\dots i_{r-1}} \le v^{i_1\dots i_{r-1}}$ then define the level r endpoints as

$$\begin{aligned} u^{i_1\dots i_{r-1}k} &= \pi_1(\{y = u^{i_1\dots i_{r-1}}\} \cap \ell_{2k}), \quad \text{and} \\ v^{i_1\dots i_{r-1}k} &= \pi_1(\{y = v^{i_1\dots i_{r-1}}\} \cap \ell_{2k-1}). \end{aligned} \tag{33}$$

for $i_1 = 1, 2,\ i_2, \ldots, i_{r-1}, k = 1, 2, 3$. Put

$$T_r = T_{r-1} \setminus \left\{[u^{i_1 i_2 \ldots i_r}, v^{i_1 i_2 \ldots i_r}],\ i_1 = 1, 2,\ i_2, \ldots, i_r = 1, 2, 3\right\}.$$

The size of each of the intervals removed is $\varrho_r := v^{i_1 i_2 \ldots i_r} - u^{i_1 i_2 \ldots i_r}$. One can easily check that

$$\forall r \geq 1, \quad \rho_{r+1} = a\rho_r - (1 - 3a - 2b) \text{ and } \rho_1 = v^1 - u^1. \tag{34}$$

Consider the smallest $r \geq 1$ for which $v^{i_1 \ldots i_{r+1}} < u^{i_1 \ldots i_{r+1}}$ or equivalently $\rho_{r+1} < 0$. Then we set $l = r - 1$ and the recursion ends. The fact that l is finite is immediate from (34) (see Figure 10).

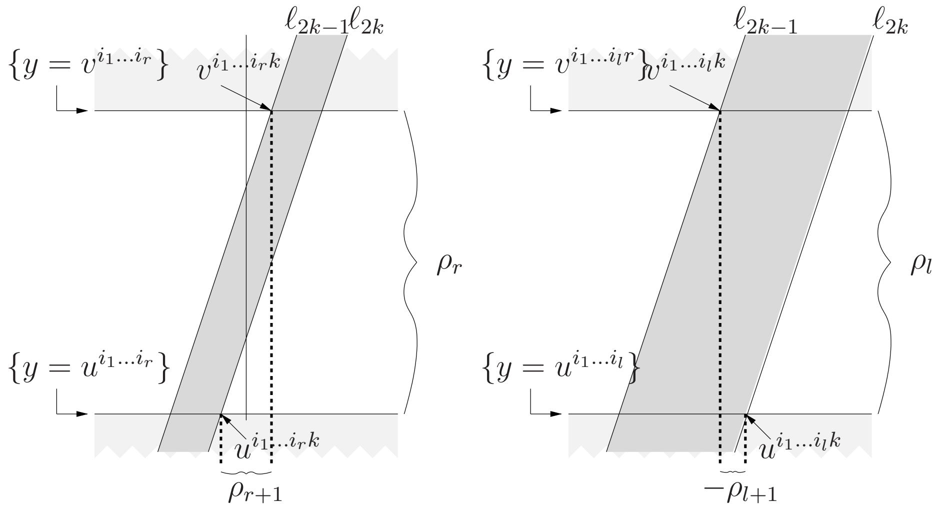

FIGURE 10. The recursion of $\{\rho_r\}_r$. On the left hand side $r \leq l - 1$.

We can represent T^{l-1} and T^l as follows

$$T^{l-1} = \bigcup_{j=1}^{3^{l-1}} (\gamma_j, \delta_j), \qquad T^l = \bigcup_{i=1}^{3^l} (\alpha_i, \beta_i).$$

Using (33) it follows from elementary geometry (see Figure 12) that

$$\forall i, \exists j, \exists k : \qquad \begin{aligned} \alpha_i &= \pi_1\left(\{(x, y) : y = \gamma_j\} \cap \ell_{2k-1}\right), \\ \beta_i &= \pi_1\left(\{(x, y) : y = \delta_j\} \cap \ell_{2k}\right). \end{aligned} \tag{35}$$

We need further restrictions because around the endpoints α_i, β_i condition (32) is not satisfied. Therefore we remove sufficiently small intervals from both ends of each of the 3^l intervals of T^l. Namely, we define the type space of the process by

$$T(\varepsilon) := \bigcup_{i=1}^{3^l} [\alpha_i + \varepsilon, \beta_i - \varepsilon], \tag{36}$$

where

$$0 < \varepsilon < \min\left\{-\frac{\rho_{l+1}}{2a}, \frac{1 - 3a - 2b}{2}\right\}. \tag{37}$$

For any $j \in \{1, \dots, 3^{l-1}\}$ we can find $i' \in \{1, \dots, 3^l\}$ such that

$$[\gamma_j + \varepsilon, \delta_j - \varepsilon] = \bigcup_{m=0}^{2} [\alpha_{i'+m} + \varepsilon, \beta_{i'+m} - \varepsilon] \cup \bigcup_{h=1}^{2} R_h^{(j)}, \tag{38}$$

where $R_h^{(j)}$, $h = 1, 2$ are intervals of length $\rho_l + 2\varepsilon$, see Figure 12. Further, for every $1 \le i \le 3^l$, $1 \le j \le 3^{l-1}$ the set $(\alpha_i + \varepsilon, \beta_i - \varepsilon) \times (\gamma_j + \varepsilon, \delta_j - \varepsilon) \cap T(\varepsilon) \times T(\varepsilon)$ consists of three congruent squares aligned on top of each other of side

$$s := \beta_i - \alpha_i - 2\varepsilon.$$

The distance between two neighboring squares is $\rho_l + 2\varepsilon$.
Now, we prove that (32) holds. That is, we want to estimate the length of the longest interval in $E_1(x)$ from below. The argument uses only elementary geometry.
For any $x \in T(\varepsilon)$ there is a unique $k \in \{1, 2, 3\}$ such that $E_1(x) \subseteq (\ell_{2k}(x), \ell_{2k-1}(x))$ holds. Using (20) one can immediately see that the length of the interval $(\ell_{2k}(x), \ell_{2k-1}(x))$ is $\frac{1}{a}(1 - 3a - 2b)$. Geometrically this means that the vertical line through x intersects the stripe S_k in a (vertical) interval of length $\frac{1}{a}(1 - 3a - 2b)$.
Since there are many holes in $T(\varepsilon)$, for some $x \in T(\varepsilon)$, the set $E_1(x)$ consists of at most three subintervals of $(\ell_{2k}(x), \ell_{2k-1}(x))$, see Figure 12. We prove that the maximum length of these intervals is uniformly bounded away from zero.
Fix a component $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset T(\varepsilon)$ and let $x \in [\alpha_i + \varepsilon, \beta_i - \varepsilon]$. For this i we choose j and k according to the formula (35). Now we distinguish three possibilities:

- **(a):** First we assume that the intersection of the vertical line through x with the stripe S_k is not contained in the rectangle $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times [\gamma_j + \varepsilon, \delta_j - \varepsilon]$ (see Figure (12)). Then using that the slope of the lines ℓ_m, $m = 1, \dots, 6$ is $1/a > 3$ by elementary geometry we obtain that the set $E_1(x)$ contains an interval of length $\kappa := \frac{1}{a}\varepsilon - \varepsilon > 2\varepsilon > 0$ (see Figure 12 B).
- **(b):** Next we assume that there exists $m \in \{0, 1, 2\}$ such that the intersection of the vertical line through x with the stripe S_k is contained in the square
$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times [\alpha_{i'+m} + \varepsilon, \beta_{i'+m} - \varepsilon]$, where i' is defined as in (38). In this case the set $E_1(x) = (\ell_{2k}(x), \ell_{2k-1}(x))$ and then the assertion holds with the choice of $\kappa := \frac{1}{a}(1 - 3a - 2b) > 0$ (see (37)).
- **(c):** Finally, we assume that the intersection of the vertical line through x with the stripe S_k has a non-empty intersection with one of the rectangles $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times R_h^{(j)}$, $h = 1, 2$. In this case, by elementary geometry (see Figure 12) $E_1(x)$ contains

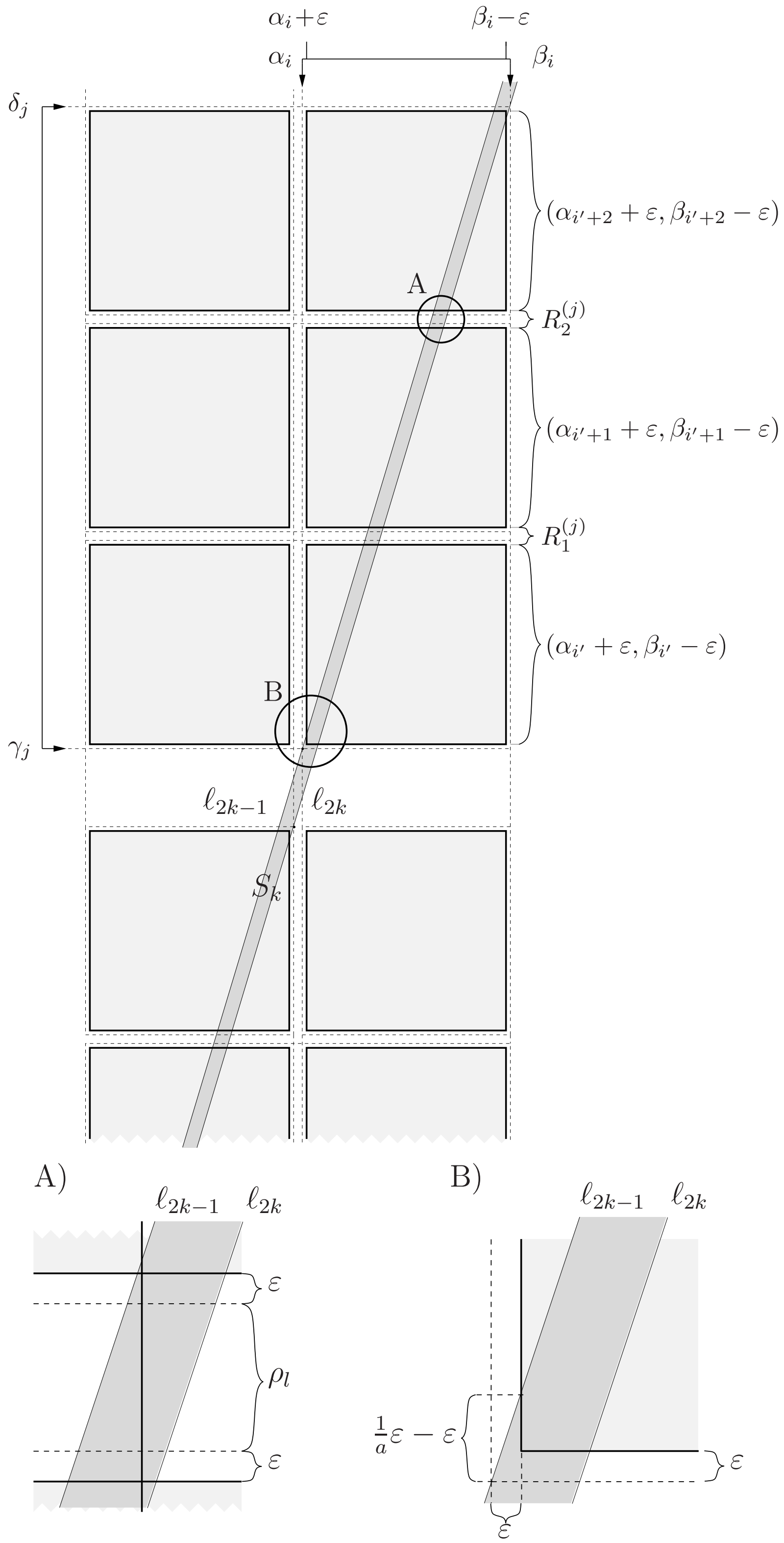

FIGURE 11. Stripe S_k and level l squares.

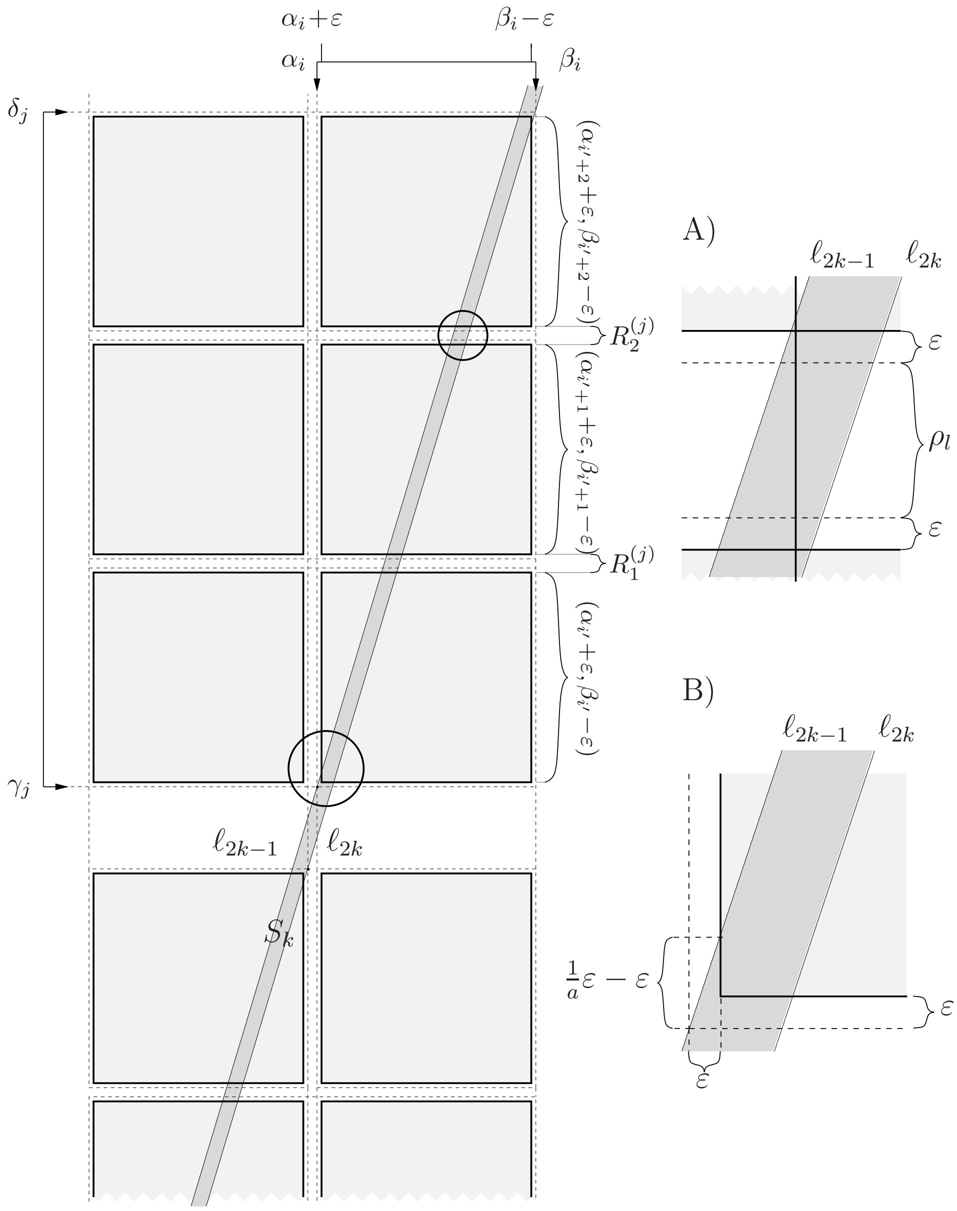

FIGURE 12. Stripe S_k and level l squares.

an interval of length at least

$$\begin{aligned}\kappa &:= \min\left\{s, \frac{1}{2}\cdot(\ell_{2k-1}(x)-\ell_{2k}(x))-(\rho_l+2\varepsilon)\right\}\\ &= \min\left\{s, \frac{1}{2}\left(\frac{1}{a}(1-3a-2b)-(\rho_l+2\varepsilon)\right)\right\}.\end{aligned}$$

It follows from (37) that $\kappa > 0$.

□

We will now deal with the problem of still having a kernel with largest eigenvalue larger than 1.

Lemma 8. *Let m^ε be the kernel in Lemma 7 with type space $T = T(\varepsilon)$. One can choose ε so small that the largest eigenvalue of m^ε is larger than 1.*

Proof. Changing T^0 to T^l in the proof of Lemma 8A we obtain the proof of Lemma 8. □

Lemma 9. *Let T be as in Lemma 8. Then there exists an n such that for all $x \in T$, $\{y : m_n(x, y) > 0\} = T$.*

Since the function $m_n(\cdot, \cdot)$ is piecewise continuous on the compact set T, Lemma 9 implies that $m(x, y) \geq a > 0$ for any $x, y \in T$. Further, using that $m(x, \cdot)$ is bounded we immediately obtain that $\sup_{x \in T} \mathbb{E}_x[\mathcal{Z}_1^2(T)]$ is finite. So we have

Corollary 2. *Let T be as in Lemma 8. The branching process $\mathcal{Z}$ with type space T satisfies the conditions **C1** and **C3**.*

Proof of Lemma 9. We recall the definition of $E_n(x)$: $E_n(x) = \{y : m_n(x, y) > 0\}$. We will prove the lemma in two steps.

Step 1 $\forall x \in T, \exists i, n$ such that $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_n(x)$ implies that $E_{n+l}(x) = T$.

Step 2 There exists an N such that for every $x \in T$ we can find a positive integer $n(x) \leq N$ such that the following holds

$$\exists i, \ [\alpha_i + \varepsilon, \beta_i + \varepsilon] \subset E_{n(x)}(x).$$

As a corollary of these two statements we obtain the assertion of the lemma holds with the choice of $n = N + l$. Namely, for any $x \in T$ we have $E_{N+l}(x) = T$.

To verify *Step 1* first we observe that by (31) we have

$$\begin{aligned} E_{n+1}(x) &= \bigcup_{y \in E_n(x)} E_1(y) \\ &= \bigcup_{y \in E_n(x)} ((\ell_2(y), \ell_1(y)) \cup (\ell_4(y), \ell_3(y)) \cup (\ell_6(y), \ell_5(y))) \cap T. \end{aligned} \tag{39}$$

Fix an $i \in \{1, \ldots, 3^l\}$. First we define $\alpha_{i,l-r}$ and $\beta_{i,l-r}$ for $r = 0, \ldots, l$ inductively. For $r = 0$ let $(\alpha_{i,l}, \beta_{i,l}) := (\alpha_i, \beta_i)$. Assume that we have already defined $(\alpha_{i,l-r}, \beta_{i,l-r})$. Using (33) we define $\alpha_{i,l-(r+1)}$ and $\beta_{i,l-(r+1)}$ as the unique numbers satisfying:

$$\begin{aligned} \alpha_{i,l-r} &= \pi_1\big(\big\{(x, y) \colon y = \alpha_{i,l-(r+1)}\big\} \cap \ell_{2k(r)-1}\big), \\ \beta_{i,l-r} &= \pi_1\big(\big\{(x, y) \colon y = \beta_{i,l-(r+1)}\big\} \cap \ell_{2k(r)}\big), \end{aligned} \tag{40}$$

where $k(r) = 1, 2, 3$. Then by the construction we have $(\alpha_{i,0}, \beta_{i,0}) = (-1 + c, 1 - c)$. Let $x \in T$. According to the assumption of *Step 1* we can find i, n such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] = (\alpha_i, \beta_i) \cap T \subset E_n(x) \tag{41}$$

holds. Using induction we prove that

$$E_{n+r}(x) \supset (\alpha_{i,l-r}, \beta_{i,l-r}) \cap T \text{ for } 0 \leq r \leq l. \tag{42}$$

Namely, for $r = 0$ the assertion in the induction is identical to (41). Now we suppose that (42) holds for $r < l$. By (39) and (40) we have

$$\begin{aligned} E_{n+r+1}(x) &= \bigcup_{y \in E_{n+r}(x)} E_1(y) \supset \bigcup_{y \in (\alpha_{i,l-r}, \beta_{i,l-r}) \cap T} (\ell_{2k(r)}(y), \ell_{2k(r)-1}(y)) \cap T \\ &= (\alpha_{i,l-(r+1)}, \beta_{i,l-(r+1)}) \cap T, \end{aligned}$$

which completes the proof of (42). We apply (42) for $r = l$. This yields that $E_{n+l} = (-1 + c, 1 - c) \cap T = T$ holds.

Next, we prove *Step 2*. First, observe that the largest interval in $E_1(x)$ either has an endpoint that is an endpoint of a connected component of T (this happens in case (a) and (c) in the end of the proof of Lemma 7) or $E_1(x) = (\ell_{2k_1}(x), \ell_{2k_1-1}(x))$ (which is case (b) in the same proof). However, in the last case using (39), after N_1 steps, where N_1 is the smallest solution of the inequality $\left(\frac{2}{a}\right)^{N_1} \cdot \frac{1}{a}(1 - 3a - 2b) > s$, we obtain that the largest interval contained in $E_{N_1}(x)$ has and endpoint of a connected component of T (see Figure 12) and its length is longer than κ. In this way because of the symmetry between the endpoints of the connected components of T from now on we may assume that $[\alpha_i + \varepsilon, \alpha_i + \varepsilon + z_1) \subset E_1(x)$ where $z_1 \geq \kappa$. Using (39) we can write

$$\begin{aligned} E_2(x) \supset &\bigcup_{y \in [\alpha_i + \varepsilon, \alpha_i + \varepsilon + z_1)} (\ell_{2k_1}(y), \ell_{2k_1-1}(y)) \cap T = \\ &(\ell_{2k_1}(\alpha_i + \varepsilon), \ell_{2k_1-1}(\alpha_i + \varepsilon + z_1)) \cap T = [\alpha^{(2)} + \varepsilon, \alpha^{(2)} + \varepsilon + z_2) \cap T \end{aligned} \tag{43}$$

for some $k_1 \in \{1, 2, 3\}$, endpoint $\alpha^{(2)} \in T$ and $z_2 > \frac{1}{a} z_1 \geq \frac{1}{a}\kappa$. If $z_2 < s$ then the largest connected component of $E_2(x)$ has an endpoint of one of the connected components of T, let us say $\alpha^{(2)}$, but the other endpoint is in the interior of the same connected component of T. If $z_2 \geq s$ then $E_2(x)$ clearly contains a connected component of T. For $E_n(x)$, $n \geq 3$ we can define inductively k_n, $\alpha^{(n)}$, z_n in the same way as above. Observe that $z_n > \left(\frac{1}{a}\right)^n \kappa$. Let N_2 the smallest solution of the inequality $\left(\frac{1}{a}\right)^{N_2} \kappa > s$. Then $E_{N_2}(x)$ contains a connected component of T.

Let $N = N_1 + N_2$. Then $E_N(x)$ contains a connected component of T. $\square$

9. Uniform exponential growth

In this section we want to prove an extension of Theorem 3 stating that the population can grow uniformly exponentially starting from any element of a special interval. For the precise statement see Lemma 12.

First, we will determine the density of the measure $\mathbb{P}_x(\mathcal{Z}_1(A) \in \cdot)$. We use the notations of Lemma 6, and define for $x_1, x_2 \in T$

$$\mathbb{P}_{x_1,x_2} := \mathbb{P}_{x_1} \otimes \mathbb{P}_{x_2},$$

the convolution of the measures $\mathbb{P}_{x_1}$ and $\mathbb{P}_{x_1}$.

Lemma 10. *For $x \in T, A \subset T$ and a natural number L we have the following equation for any $n \geq 1$*

$$\begin{aligned}(44)\quad \mathbb{P}_x(\mathcal{Z}_{n+1}(A) = L) = &\\ &\int_T \mathbb{P}_z\left(\mathcal{Z}_n(A) = L\right) h_1(x,z)\,\mathrm{d}z + \\ &\qquad \int_T\int_T \mathbb{P}_{z_1,z_2}\left(\mathcal{Z}_n(A) = L\right) h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2.\end{aligned}$$

where $h_1(x,z) : T \times T \to \mathbb{R}_+$ and $h_2(x,z_1,z_2) : T \times T \times T \to \mathbb{R}_+$ are defined as follows

$$h_1(x,z) = \begin{cases} f_{x,1}(z) & \text{if } x \in A_1^+ \\ f_{x,1}(z) + 2f_{x,2}(z)\left(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\right) & \text{if } x \in A_2^+ \\ 2f_{x,2}(z)\left(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\right) & \text{if } x \in A_3 \\ f_{x,3}(z) + 2f_{x,2}(z)\left(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\right) & \text{if } x \in A_2^- \\ f_{x,3}(z) & \text{if } x \in A_1^- \end{cases},$$

$$h_2(x,z_1,z_2) = \begin{cases} 2f_{x,2}(z_1)f_{x,4}(z_2) & \text{if } x \in A_3 \cup A_2^+ \cup A_2^- \\ 0 & \text{otherwise} \end{cases}.$$

Both are bounded and piecewise uniformly continuous functions in x on T for any fixed $z, z_1, z_2 \in T$.

Proof. The decomposition (44) is obtained from the Chapman-Kolmogorov equation, i.e., by conditioning on the first generation. In the following formula we use one of the conclusions of Lemma 6, i.e., that exactly two squares in generation 1 can only be generated by Q_2 and Q_4.

$$\begin{aligned}(45)\quad \mathbb{P}_x(\mathcal{Z}_{n+1}(A) = L) = &\int_T \mathbb{P}_z(\mathcal{Z}_n(A) = L)\mathbb{P}_x(\mathcal{Z}_1(\,\mathrm{d}z) = 1) + \\ &\int_T\int_T \mathbb{P}_{z_1,z_2}(\mathcal{Z}_n(A) = L)\mathbb{P}_x(\mathcal{Z}_1^2(\,\mathrm{d}z_1) = 1, \mathcal{Z}_1^4(\,\mathrm{d}z_2) = 1).\end{aligned}$$

We have to determine the density function $h_1(x,z)$ of exactly one descendant with type $\mathrm{d}z$ and $h_2(x,z_1,z_2)$ the density function of exactly two descendants with type $\mathrm{d}z_1\,\mathrm{d}z_2$. One can decompose the probability of having exactly one descendant such that the type of this descendant falls into the set $(-\infty, z]$ (for any real z) as follows

$$\mathbb{P}_x\left(\mathcal{Z}_1((-\infty, z]) = 1\right) = \sum_{i=1}^{4} \mathbb{P}(X_i(x) \in (-\infty, z], X_j(x) = \Theta, j \neq i).$$

The decomposition in Lemma 6 implies $\{X_2(x) \neq \Theta\} \cup \{X_4(x) \neq \Theta\}$, $\{X_1(x) \neq \Theta\}$, and $\{X_3(x) \neq \Theta\}$ are disjoint events for any $x \in T$. Therefore, one obtains

$$\mathbb{P}_x(\mathcal{Z}_1((-\infty, z]) = 1) = \mathbb{P}(X_1(x) \in (-\infty, z])+ \\ 2\mathbb{P}(X_2(x) \in (-\infty, z])\mathbb{P}(X_4(x) = \Theta) + \mathbb{P}(X_3(x) \in (-\infty, z])$$

using that $X_2(x)$ and $X_4(x)$ are independent and identically distributed. Since $X_i(x)$ has density $f_{x,i}$ one gets that this equals

$$\int_{(-\infty,z]} f_{x,1}(y)\,\mathrm{d}y \cdot \mathbf{1}_{A_1^+ \cup A_2^+}(x) + \\ 2\int_{(-\infty,z]} f_{x,2}(y)\,\mathrm{d}y \left(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\right) \cdot \mathbf{1}_{A_3 \cup A_2^+ \cup A_2^-}(x) + \\ \int_{(-\infty,z]} f_{x,3}(y)\,\mathrm{d}y \cdot \mathbf{1}_{A_1^- \cup A_2^-}(x) = \int_{(-\infty,z]} h_1(x,y)\,\mathrm{d}y.$$

Let us next deal with exactly two descendants with types falling into $(-\infty, z_1]$, respectively $(-\infty, z_2]$. This probability equals

$$2\mathbb{P}(X_2(x) \in (-\infty, z_1], X_4(x) \in (-\infty, z_2]).$$

Since $X_2(x)$ and $X_4(x)$ are independent and identically distributed one obtains that this equals

$$2\int_{(-\infty,z_1]} f_{x,2}(y)\,\mathrm{d}y \int_{(-\infty,z_2]} f_{x,4}(y)\,\mathrm{d}y \cdot \mathbf{1}_{A_3 \cup A_2^+ \cup A_2^-}(x) = \\ \int_{(-\infty,z_1]} \int_{(-\infty,z_2]} h_2(x, y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Summarizing these considerations one obtains (44).

Now, we will prove that $h_1(x, z)$ and $h_2(x, z_1, z_2)$ are piecewise uniformly continuous in x on T. By the definition of f (23) and $f_{x,i}$ (24), for i $(1 \leq i \leq 4)$, if $x_1, x_2 \in [\alpha_j, \beta_j]$ for some $1 \leq j \leq 3^l$ and $|x_1 - x_2| \leq \delta$ we have

$$\sup_{z \in T} |f_{x_1,i}(z) - f_{x_2,i}(z)| \leq \delta\left(\tfrac{1-3a-2b}{2}\right)^{-2},$$

$$\left|\left(1 - \int_T f_{x_1,3}(y)\,\mathrm{d}y\right) - \left(1 - \int_T f_{x_2,4}(y)\,\mathrm{d}y\right)\right| \leq |T|\frac{1}{\delta}\left(\tfrac{1-3a-2b}{2}\right)^{-2}.$$

Therefore,

$$\sup_{z\in T}|h_1(x_1,z)-h_1(x_2,z)| \leq (|T|+1)\,\delta\left(\tfrac{1-3a-2b}{2}\right)^{-4}, \tag{46}$$

$$\sup_{z_1,z_2\in T}|h_2(x_1,z_1,z_2)-h_2(x_2,z_1,z_2)| \leq 4\delta\left(\tfrac{1-3a-2b}{2}\right)^{-3}$$

for $x_1,x_2\in[\alpha_j,\beta_j]$ for some $1\leq j\leq 3^l$ and $|x_1-x_2|\leq\delta$. □

Let $A\subset T$ such that the Lebesgue measure of A is positive. Let $W_n(A)=\mathcal{Z}_n(A)\rho^{-n}$ and $W(A)=\lim_{n\to\infty}W_n(A)$ which almost surely exists by Theorem 3. We need a stronger result: the random variable $W(A)$ is strictly separated from 0 with uniformly positive probability for some neighborhood of the initial type 0. This is shown in the next lemma.

Lemma 11. *For some neighborhood $J\subset T$ of 0 and positive numbers y and r we have*

$$\inf_{x\in J}\mathbb{P}_x(W(A)>y)\geq r. \tag{47}$$

Proof. Lemma 10 implies

$$\begin{aligned}\mathbb{P}_x(W_{n+1}(A)\leq y)=\mathbb{P}_x(\mathcal{Z}_{n+1}(A)\leq\rho^{n+1}y)=&\\ \int_T\mathbb{P}_z\left(W_n(A)\leq\rho y\right)h_1(x,z)\,\mathrm{d}z+&\\ \int_T\int_T\mathbb{P}_{z_1,z_2}\left(W_n(A)\leq\rho y\right)h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2.&\end{aligned} \tag{48}$$

We will investigate the convergence of the last two terms in (48). Theorem 3 implies that we have for all $z\in T$

$$\lim_{n\to\infty}\mathbb{P}_z(W_n(A)\leq y)=\mathbb{P}_z(W(A)\leq y) \tag{49}$$

if $y\in Cont(\mathbb{P}_{z,A})$ where $Cont(\mathbb{P}_{z,A})$ denotes the set of continuity points of the distribution function on the right side of (49).
Next, we seek the weak convergence of the measure $\mathbb{P}_{z_1,z_2}(W_n(A)\in\cdot)$. This measure is the convolution of the measures $\mathbb{P}_{z_1}(W_n(A)\in\cdot)$ and $\mathbb{P}_{z_2}(W_n(A)\in\cdot)$. Since they are weakly convergent the convolution is also weakly convergent. So

$$\lim_{n\to\infty}\mathbb{P}_{z_1,z_2}(W_n(A)\leq y)=\mathbb{P}_{z_1,z_2}(W(A)\leq y). \tag{50}$$

if $y\in Cont(\mathbb{P}_{z_1,z_2,A})$.
Let $t(z,z_1,z_2;y,\varepsilon)$ for $z,z_1,z_2\in T$, $y>0$, $\varepsilon>0$ be a real number such that $y\leq t(z,z_1,z_2;y,\varepsilon)<y+\varepsilon$ and $\rho\,t(z,z_1,z_2;y,\varepsilon)\in Cont(\mathbb{P}_{z,A})\cap Cont(\mathbb{P}_{z_1,z_2,A})$. In the sequel, we use the simplified notation $y_t=t(z,z_1,z_2;y,\varepsilon)$.

Let us define the following two functions

$$\begin{aligned}
\beta_{n+1}(x,y,A) &= \int_T \mathbb{P}_z\left(W_n(A) \le \rho y_t\right) h_1(x,z)\,\mathrm{d}z + \\
&\quad \int_T\int_T \mathbb{P}_{z_1,z_2}\left(W_n(A) \le \rho y_t\right) h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
\beta(x,y,A) &= \int_T \mathbb{P}_z\left(W(A) \le \rho y_t\right) h_1(x,z)\,\mathrm{d}z + \\
&\quad \int_T\int_T \mathbb{P}_{z_1,z_2}\left(W(A) \le \rho y_t\right) h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2. \qquad (51)
\end{aligned}$$

Using the decomposition (48) and the right continuity of distribution functions we can derive the following bounds

$$\mathbb{P}_x(W_{n+1}(A) \le y) \le \beta_{n+1}(x,y,A) \le \mathbb{P}_x(W_{n+1}(A) \le y+\varepsilon).$$

By using (49), (50) and the bounded convergence theorem we get that $\beta_n(x,y,A)$ converges as $n \to \infty$ so

$$\mathbb{P}_x(W(A) \le y) \le \beta(x,y,A) \le \mathbb{P}_x(W(A) \le y+\varepsilon). \qquad (52)$$

Using (46), piecewise continuity of h_1 and h_2 in x, and bounded convergence one can see that $\beta_n(x,y,A)$ and $\beta(x,y,A)$ are piecewise continuous on T in x.

Using Theorem 12 and the right continuity of distribution functions we can find two positive numbers r,u such that $\mathbb{P}_0(W(A) > u) > 2r$ or equivalently $\mathbb{P}_0(W(A) \le u) \le 1-2r$. Let $y = u-\varepsilon$ for some positive $\varepsilon < u$. Using the second inequality of (52) one gets $\beta(0,y,A) \le \mathbb{P}_0(W(A) \le y+\varepsilon) \le 1-2r$. Since $\beta(x,y,A)$ is piecewise continuous on T there exist an interval $J \subset T$ neighborhood of 0 such that the bound $\beta(x,y,A)$ is uniformly smaller that 1 on this interval, that is, $\sup_{x\in J}\beta(x,y,A) \le 1-r$. The first inequality of (52) implies $\sup_{x\in J}\mathbb{P}_x(W(A) \le y) \le \sup_{x\in J}\beta(x,y,A) \le 1-r$ which yields the required bound in (47).

$\square$

Lemma 12. *There exist two positive numbers η, r, an integer N and a number K with $0 < K < \frac{1}{8}$ such that the following inequality holds*

$$\inf_{n\ge N}\inf_{x\in[-K,K]}\mathbb{P}_x\left(\mathcal{Z}_n([-K,K]) > \eta\rho^n\right) > \frac{r}{2}.$$

Proof. We apply Lemma 11 with $A = T$ and obtain the numbers y, r, and the set J. Let K be a positive number such that $K < \frac{1}{8}$ and $[-K,K] \subset J$. So we have

$$\inf_{x\in[-K,K]}\mathbb{P}_x(W(T) > y) \ge r.$$

Using Theorem 3 one gets that

$$W([-K,K]) = \gamma W(T)$$

holds $\mathbb{P}_x$ almost surely for any $x \in T$, where

$$\gamma = \frac{\int_{[-K,K]} \nu(z)\,\mathrm{d}z}{\int_T \nu(z)\,\mathrm{d}z}.$$

Hence, we have the following bound

$$\inf_{x\in[-K,K]} \mathbb{P}_x(W([-K,K]) > \eta + \varepsilon) > r,$$

where $\eta + \varepsilon = \gamma y$ for some positive η and ε. This and the second inequality of (52) implies that $\beta(x, \eta, [-K,K])$ is uniformly smaller than 1:

(53)

$$\sup_{x\in[-K,K]} \beta(x, \eta, [-K,K]) \le \sup_{x\in[-K,K]} \mathbb{P}_x(W([-K,K]) \le \eta + \varepsilon) \le 1 - r.$$

We will show that $\beta_n(x, \eta, [-K,K])$ defined in (51) converges uniformly to $\beta(x, \eta, [-K,K])$ on $[-K,K]$ as n tends to infinity. Using trivial estimations one gets the following chain of inequalities:

$$\begin{aligned}
&\sup_{x\in[-K,K]} |\beta_{n+1}(x,\eta,[-K,K]) - \beta(x,\eta,[-K,K])| \le \\
&\sup_{x\in[-K,K]} \int_T |\mathbb{P}_z\left(W_n([-K,K]) \le \rho\eta_t\right) - \\
&\qquad\qquad \mathbb{P}_z\left(W([-K,K]) \le \rho\eta_t\right)| \, h_1(x,z)\,\mathrm{d}z + \\
&\sup_{x\in[-K,K]} \int_T \int_T |\mathbb{P}_{z_1,z_2}\left(W_n([-K,K]) \le \rho\eta_t\right) - \\
&\qquad\qquad \mathbb{P}_{z_1,z_2}\left(W([-K,K]) \le \rho\eta_t\right)| \, h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \le \\
&\sup_{x,z\in T} h_1(x,z) \int_T |\mathbb{P}_z\left(W_n([-K,K]) \le \rho\eta_t\right) - \\
&\qquad\qquad \mathbb{P}_z\left(W([-K,K]) \le \rho\eta_t\right)| \,\mathrm{d}z + \\
&\sup_{x,z_1,z_2\in T} h_2(x,z_1,z_2) \cdot \int_T \int_T |\mathbb{P}_{z_1,z_2}\left(W_n([-K,K]) \le \rho\eta_t\right) - \\
&\qquad\qquad \mathbb{P}_{z_1,z_2}\left(W([-K,K]) \le \rho\eta_t\right)| \,\mathrm{d}z_1\,\mathrm{d}z_2.
\end{aligned}$$

By bounded convergence both integrals in the last expression converge to 0. The suprema are finite since h_1 and h_2 are bounded (see Lemma 11). So $\beta_n(x, \eta, [-K,K])$ uniformly converges to $\beta(x, \eta, [-K,K])$ on $[-K,K]$. Therefore, there exist an index N such that for $n \ge N$

$$\sup_{x\in[-K,K]} |\beta_n(x,\eta,[-K,K]) - \beta(x,\eta,[-K,K])| \le \frac{r}{2}.$$

Using first inequality of (52), the triangular inequality, (53), and Lemma 11 one can write

$$\sup_{x\in[-K,K]} \mathbb{P}_x(W_n([-K,K]) \le \eta) \le \sup_{x\in[-K,K]} \beta_n(x,\eta,[-K,K]) \le$$
$$\sup_{x\in[-K,K]} \beta(x,\eta,[-K,K]) + \sup_{x\in[-K,K]} |\beta_n(x,\eta,[-K,K]) - \beta(x,\eta,[-K,K])| \le$$
$$1 - r + \frac{r}{2} = 1 - \frac{r}{2}$$

for $n \ge N$. This gives the conclusion of the lemma. □

10. The proof of the Main Lemma

We repeat the Main Lemma:

Main Lemma. *There exist three positive numbers δ, q, K and an index N such that the following inequality holds*

$$\inf_{n>N} \inf_{x\in[-K,K]} \mathbb{P}_x\left(\mathcal{Z}_n([0,K]) > \delta\rho^n \;\&\; \mathcal{Z}_n([-K,0]) > \delta\rho^n\right) > q.$$

Proof. Take K as defined in Lemma 12. Since $[-K,K] = [-K,0] \cup [0,K]$, and type 0 has probability 0 to occur, it follows directly from Lemma 12 that one of $\mathbb{P}_x\left(\mathcal{Z}_n([0,K]) > \delta\rho^n\right)$ and $\mathbb{P}_x\left(Z_n([-K,0]) > \delta\rho^n\right)$ is larger than $r/4$ for all $x \in [-K,K]$, and $n > N$. But then, by symmetry, *both* of these probabilities are larger than $r/4$.
Now take any $x \in [-K,K]$. Since $K < \frac{1}{8}$, it follows that with a positive probability, denoted $p_{2,4}$, in the first generation the squares Q_2 and Q_4—with types x_2 and x_4 from a subinterval of $[-K,K]$—will be present. But by the above, these two squares will, independently of each other and with probability at least $r/4$, generate more than $\delta\rho^n$ squares with type in $[0,K]$, respectively $[-K,0]$ in generation $n+1$. Thus for all $x \in [-K,K]$, and $n > N$

$$\mathbb{P}_x\left(\mathcal{Z}_{n+1}([0,K]) > \delta\rho^n \;\&\; \mathcal{Z}_{n+1}([-K,0]) > \delta\rho^n\right) > p_{2,4} \cdot \frac{r}{4} \cdot \frac{r}{4}.$$

So replacing δ by δ/ρ, N by $N+1$, and defining $q = p_{2,4} r^2/16$ this proves the Main Lemma. □

References

[1] Carlos Gustavo T. de A. Moreira and Jean-Christophe Yoccoz. Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. of Math. (2)*, 154(1):45–96, 2001.
[2] F.M. Dekking and K. Simon. On the size of the algebraic difference of two random Cantor sets. *Random Structures and Algorithms*, 32: 205–222, 2008.
[3] Chao Du. The difference set of two Cantor sets revisited. Master thesis, Delft University of Technology, 2003.
[4] Theodore E. Harris. *The theory of branching processes.* Dover Phoenix Editions. Springer, Berlin, 1963. Corrected reprint of the 1963 original [Springer, Berlin; MR0163361 (29 #664)].

[5] Per Larsson. *The difference set of two Cantor sets.* PhD thesis, U.U.D.M. Report, 1991:11.

[6] Per Larsson. L'ensemble différence de deux ensembles de Cantor aléatoires. (French. English summary) [The difference set of two random Cantor sets]. *C. R. Acad. Sci. Paris Sér. I Math.*, 310 (10): 735–738, 1990.

[7] Jacob Palis and Floris Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, volume 35 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993. Fractal dimensions and infinitely many attractors.

Michel Dekking, Delft Institute of Applied Mathematics, Technical University of Delft, The Netherlands
F.M.Dekking@ewi.tudelft.nl

Károly Simon, Institute of Mathematics, Technical University of Budapest, H-1529 B.O.box 91, Hungary simonk@math.bme.hu

Balázs Székely, Institute of Mathematics, Technical University of Budapest, H-1529 B.O.box 91, Hungary szbalazs@math.bme.hu