THE DIMENSION OF GRAPH DIRECTED ATTRACTORS WITH OVERLAPS ON THE LINE, WITH AN APPLICATION TO A PROBLEM IN FRACTAL IMAGE RECOGNITION.

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Abstract. Consider a graph directed iterated function system (GIFS) on the line which consists of similarities. Assuming neither any separation conditions, nor any restrictions on the contractions, we compute the almost sure dimension of the attractor. Then we apply our result to give a partial answer to an open problem in the field of fractal image recognition concerning some self-affine graph directed attractors in space.

1. Introduction

Mauldin and Williams [6] computed the Hausdorff dimension of the attractor for a Graph Directed Iterated Function System (GIFS) of similarities in $\mathbb{R}^d$ assuming that the cylinders, in some sense, are well separated. If we drop the separation condition in the Mauldin-Williams Theorem, we can expect only "almost all" type results. In this paper we prove such a result on the line assuming neither any separation conditions nor any restrictions on the contraction ratios. The only thing we have to assume is that the matrix of the directed graph is irreducible. Our research was motivated by an open problem in the field of fractal image compression. This problem is about the box-counting dimension of a certain graph directed non-conformal attractor in space. Using techniques similar to the ones in [4], we can reduce this problem to the computation of the attractor of some GIFS on the line with overlapping cylinders.

2. The results and the motivating example

This section is organized as follows: First we state our theorem about the almost sure dimension of GIFS on the line. Then we introduce the non-conformal graph directed attractor coming from fractal image compression which motivated our research. Then we compute the almost sure box-counting dimension of this attractor.

2.1. **GIFS on the line.** Let Γ be a directed graph with vertex set $\mathcal{V} = \{1, \dots, m\}$ and edge set $\mathcal{E}$. We assume that Γ is strongly connected. For every $\omega \in \mathcal{E}$ there exists a contracting similarity $f_\omega : \mathbb{R} \to \mathbb{R}$,

$$f^{\mathbf{t}}_\omega(x) := \lambda_\omega \cdot x + t_\omega \tag{2.1}$$

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$0 < \lambda_\omega < 1$. The reason for this notation is that we consider the contraction ratios λ_ω as fixed and the translations $\mathbf{t} = \{t_\omega\}_{\omega\in\mathcal{E}}$ as parameter. Let $\ell := \#\mathcal{E}$. By a fixed enumeration of $\mathcal{E}$ we assign to any $\mathbf{t} \in \mathbb{R}^\ell$ a family of maps $\left\{f_\omega^{\mathbf{t}}\right\}_{\omega\in\mathcal{E}}$ of the form (2.1).

Then, as proved in [6] (see also [3]), there exists a family of non-empty compact sets $\Lambda_1^{\mathbf{t}}, \dots, \Lambda_m^{\mathbf{t}}$ such that for every $1 \le i \le m$:

$$\Lambda_i^{\mathbf{t}} = \bigcup_{j=1}^{m} \bigcup_{\omega\in\mathcal{E}_{ij}} f_\omega^{\mathbf{t}}(\Lambda_j), \tag{2.2}$$

where $\mathcal{E}_{ij}$ is the set of directed edges in Γ connecting i to j, with entries 0 if $\mathcal{E}_{ij} = \emptyset$. We call the family $\left\{\Lambda_1^{\mathbf{t}}, \dots, \Lambda_m^{\mathbf{t}}\right\}$ a family of graph directed sets and say that

$$\Lambda^{\mathbf{t}} = \bigcup_{j=1}^{m} \Lambda_j^{\mathbf{t}} \tag{2.3}$$

is the attractor of $\left\{f_\omega^{\mathbf{t}}\right\}_{\omega\in\mathcal{E}}$. Next we introduce a number α which plays the role of similarity dimension for graph directed systems (see [6]).

Definition 1 (The definition of α)**.** For $\beta \ge 0$ consider the following $m \times m$ matrix (recall that Γ has m vertices):

$$A_\beta := \left[\sum_{\omega\in\mathcal{E}_{ij}} \lambda_\omega^\beta\right]_{i,j\le m}. \tag{2.4}$$

Let α be the number for which the spectral radius ρ of the matrix A_α satisfies

$$\rho(A_\alpha) = 1.$$

The symbol $\mathcal{L}_n$ denotes the Lebesgue measure in $\mathbb{R}^n$. Our main result is:

Theorem 1. *For $\mathcal{L}_\ell$ almost every $\mathbf{t} \in \mathbb{R}^\ell$ we have*
- (i) $\dim_H \Lambda^{\mathbf{t}} = \min\{1, \alpha\}$,
- (ii) *if $\alpha > 1$ then $\mathcal{L}_1(\Lambda^{\mathbf{t}}) > 0$.*

2.2. **An affine GIFS in space.** The following attractor arose naturally in the field of fractal image compression:

Fix $K \in \mathbb{N}$. Let $m := 2^K \times 2^K$. By definition, the *domain squares* $Q_1, \dots, Q_m$ are the $2^{-K} \times 2^{-K}$ grid squares of the unit square $[0,1]\times[0,1]$. More precisely, for all $1 \le \ell \le m$ there exist some $0 \le i, j \le K-1$ such that $Q_\ell = [i/2^{-K}, (i+1)/2^{-K}] \times [j/2^{-K}, (j+1)/2^{-K}]$. If we divide all the domain squares Q_ℓ into four identical squares, we obtain the family of *range squares* Q_ℓ^i ($\ell = 1, \dots, m; \quad i = 1, \dots, 4$) by the following convention about the placement of Q_ℓ^i:

$$Q_\ell = \begin{array}{|c|c|}\hline Q_\ell^1 & Q_\ell^2 \\ \hline Q_\ell^3 & Q_\ell^4 \\ \hline\end{array}$$

We associate a domain square to every range square (see Figure 1). That is, we are given a function

$$\varphi : \{1, \dots, m\} \times \{1, \dots, 4\} \to \{1, \dots, m\}.$$

Using φ we define a directed graph $\Gamma := (\mathcal{V}, \mathcal{E})$ in the following way: The set of vertices is $\mathcal{V} := \{1, \dots, m\}$. By definition, there is a directed edge $\omega \in \mathcal{E}$ from vertex ℓ to the vertex k iff there exists $i \in \{1, \dots, 4\}$ such that $k = \varphi(\ell, i)$. In this case we write $\omega = \omega(\ell^i, k)$. So there are exactly 4 edges leading out of every vertex.

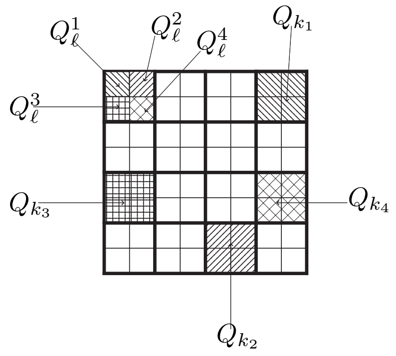

FIGURE 1. We assign a big square to every small square. Here $k_i := \varphi(\ell, i)$.

Principal Assumption: We always assume that the directed graph $(\mathcal{V}, \mathcal{E})$ is strongly connected.

The reason we call Q_k and Q_ℓ^i a domain square and a range square is that for every $1 \leq \ell \leq m$ and $1 \leq i \leq 4$ we are given a one-parameter family of affine maps (see Figure 2)

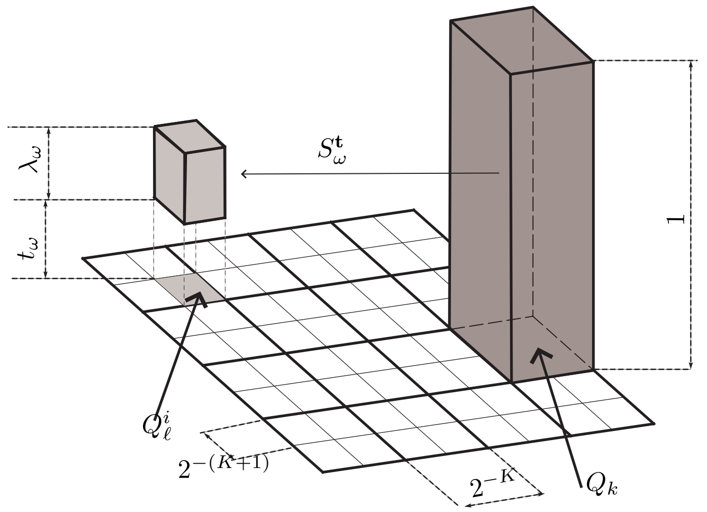

FIGURE 2. $S_\omega^{\mathbf{t}}$, maps the box on the right onto the box on the left, where $k = \varphi(\ell, i)$ and $\omega = \omega(l^i, k)$.

$$S_\omega^{\mathbf{t}} : Q_k \times \mathbb{R} \to Q_\ell^i \times \mathbb{R},$$

where $\omega = \omega(\ell^i, k)$ and the parameter $\mathbf{t}$ is in $\mathbb{R}^{\#\mathcal{E}}$. Moreover,

$$S_\omega^{\mathbf{t}}(\mathbf{x}, y) := \left(T_\omega(\mathbf{x}), f_\omega^{\mathbf{t}}(y)\right),$$

where T_ω is the similarity which maps Q_k onto Q_ℓ^i with $DT_\omega = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $f_\omega^{\mathbf{t}}(y) = \lambda_\omega \cdot y + t_\omega$, $0 < \lambda_\omega < 1$. That is, the affine map $S_\omega^{\mathbf{t}}$ in the horizontal direction is a similarity with contraction ratio 1/2 and in the vertical direction it is of the form (2.1). The vector whose components are the translations t_ω is used as a parameter.

As on the line, we define the graph directed sets $\left\{\hat{\Lambda}_1^{\mathbf{t}}, \dots, \hat{\Lambda}_m^{\mathbf{t}}\right\}$ as the unique non-empty compact sets satisfying

$$\hat{\Lambda}_i^{\mathbf{t}} = \bigcup_{j=1}^{m} \bigcup_{\omega \in \mathcal{E}_{ij}} S_\omega^{\mathbf{t}}(\hat{\Lambda}_j^{\mathbf{t}})$$

for all $i = 1, \dots, m$, and the attractor of $\left\{S_\omega^{\mathbf{t}}\right\}_{\omega \in \mathcal{E}}$ is

$$\hat{\Lambda}^{\mathbf{t}} := \bigcup_{j=1}^{m} \hat{\Lambda}_j^{\mathbf{t}}.$$

Note that the projection of $\hat{\Lambda}^{\mathbf{t}}$ to the z axis is $\Lambda^{\mathbf{t}}$, the attractor of $\left\{f_\omega^{\mathbf{t}}\right\}_{\omega \in \mathcal{E}}$ defined in (2.3).

2.3. **Calculating the almost sure box-counting dimension of $\hat{\Lambda}^{\mathbf{t}}$.** Similarly to [4], we can express the almost sure box-counting dimension of $\hat{\Lambda}^{\mathbf{t}}$ in terms of the almost sure dimension of $\Lambda^{\mathbf{t}}$. Denote

$$d(\mathbf{t}) := \dim_H \Lambda^{\mathbf{t}}.$$

Note that the box-counting dimension $\dim_B \Lambda^{\mathbf{t}}$ exists and equals $d(\mathbf{t})$ as well, by [3, Corollary 3.5].

Proposition 1. *For all $\mathbf{t} \in \mathbb{R}^\ell$, the box-counting dimension of $\hat{\Lambda}^{\mathbf{t}}$ exists and equals*

$$\dim_B \hat{\Lambda}^{\mathbf{t}} = \max\left\{2, d(\mathbf{t}) + \frac{\log \rho(A_{d(\mathbf{t})})}{\log 2}\right\}.$$

This, together with Theorem 1, allows us to compute the almost sure box-counting dimension of $\hat{\Lambda}^{\mathbf{t}}$:

Theorem 2. *For almost all $\mathbf{t} \in \mathbb{R}^\ell$ we have*

$$\dim_B \hat{\Lambda}^{\mathbf{t}} = \max\left\{2, 1 + \frac{\log \rho(A_1)}{\log 2}\right\}. \tag{2.5}$$

Remark 1. Recall our principal assumption that the directed graph $(\mathcal{V}, \mathcal{E})$ is strongly connected. Without this assumption we could not verify the results above.

The rest of the paper is organized as follows: In the next section we prove Theorem 1. Then in the last section we prove Proposition 1 and Theorem 2.

3. Proof of Theorem 1

First we need some notation. If $\omega \in \mathcal{E}_{\ell k}$ then we call ℓ and k the source and the target of edge ω respectively, and we write $s(\omega) = \ell$ and $t(\omega) = k$. When we want to refer to an edge in Γ we usually write ω or τ. On the other hand, when we refer to a finite or infinite path in Γ we usually write $\bar{\omega}$ or $\bar{\tau}$.

3.1. **Symbolic space and invariant measure.** Our symbolic space is

$$\Sigma := \{\bar{\omega} \in \mathcal{E}^\infty : t(\omega_j) = s(\omega_{j+1})\}.$$

That is, the alphabet of our symbolic space is the edge set, and Σ is the set of all possible infinite paths in Γ. Obviously this is a subshift of finite type. In fact, let $\ell := \#\mathcal{E}$ and

$$\mathcal{E} := \{e_1, \dots, e_\ell\}.$$

Further, let $R = [r_{ij}]_{i,j \le \ell}$ be the $\ell \times \ell$ matrix defined by

$$r_{ij} = \begin{cases} 1, & \text{if } t(e_i) = s(e_j); \\ 0, & \text{otherwise.} \end{cases}$$

Then Σ can be identified in a trivial way with

$$\Sigma_R^\ell := \left\{(i_1, i_2, \dots) \in \{1, \dots, \ell\}^{\mathbb{N}} : r_{i_k i_{k+1}} = 1, \forall k \ge 1\right\}.$$

The set of all k-paths of the directed graph Γ is denoted Σ^k. We denote by Σ_p the set of all infinite paths in Γ which originate at the vertex p, for $p = 1, \dots, m$. That is, for $k \in \mathbb{N}$ and $1 \le p \le m$,

$$\Sigma^k := \left\{(\omega_1, \dots, \omega_k) \in \mathcal{E}^k : t(\omega_j) = s(\omega_{j+1})\right\} \text{ and } \Sigma_p := \{\bar{\omega} \in \Sigma : s(\omega_1) = p\}.$$

For a path $(\omega_1, \dots, \omega_k) \in \Sigma^k$ we write

$$\lambda_{\omega_1 \dots \omega_k} := \lambda_{\omega_1} \cdots \lambda_{\omega_k},$$

and let $\lambda_\emptyset := 1$ where $\emptyset$ is the empty word. Further, let $f^{\mathbf{t}}_{\omega_1 \dots \omega_k} := f^{\mathbf{t}}_{\omega_1} \circ \cdots \circ f^{\mathbf{t}}_{\omega_k}$. The natural projection $\Pi^{\mathbf{t}} : \Sigma \to \mathbb{R}$ is defined by

$$\Pi^{\mathbf{t}}(\bar{\omega}) := \lim_{n \to \infty} f^{\mathbf{t}}_{\omega_1 \dots \omega_n}(0) = \sum_{k=1}^{\infty} t_{\omega_k} \cdot \lambda_{\omega_1 \dots \omega_{k-1}}.$$

As usual, we write σ for the left shift on Σ.

There is a natural Markov measure on Σ associated with the GIFS. Recall that α is such that $\rho(A_\alpha) = 1$. By the Perron-Frobenius theorem, since A_α is irreducible, there is a unique strictly positive right eigenvector $(v_i)_{i=1}^\ell$ for A_α, with the eigenvalue 1, normalized by $\sum_{i=1}^\ell v_i = 1$. Then one can set, for any $k \ge 1$ and any $\omega_1, \dots, \omega_k \in \mathcal{E}^k$:

$$\mu([\omega_1, \dots, \omega_k]) = \lambda^\alpha_{\omega_1 \dots \omega_k} v_{\omega_k},$$

where $[\omega_1, \dots, \omega_k] := \{\bar{\tau} \in \Sigma : \tau_1 = \omega_1, \dots, \tau_k = \omega_k\}$. It is easy to check (see [6]) that μ is consistently defined and σ-invariant. Alternatively, we can define μ as the Gibbs measure for the Hölder potential $\bar{\omega} \mapsto \alpha \log(f^{\mathbf{t}}_{\omega_1})'(\Pi^{\mathbf{t}}(\sigma\bar{\omega})) = \alpha \log \lambda_{\omega_1}$. In any case, there exist $c_1, c_2 > 0$ such that for all k and all $(\omega_1, \dots, \omega_k) \in \Sigma^k$,

$$c_1 \lambda^\alpha_{\omega_1 \dots \omega_k} \le \mu([\omega_1, \dots, \omega_k]) \le c_2 \lambda^\alpha_{\omega_1 \dots \omega_k}. \tag{3.1}$$

Let $\mathcal{P}$ be the set of pairs from Σ having the same initial vertex. That is,

$$\mathcal{P} := \{(\bar{\omega}, \bar{\tau}) : s(\omega_1) = s(\tau_1)\},$$

where $\bar{\omega} = \omega_1\omega_2 \dots$ and $\bar{\tau} = \tau_1\tau_2 \dots$ Let $\bar{\eta} = (\eta_1, \dots, \eta_k) \in \Sigma^k$. Then we write

$$\mathcal{P}_{\bar{\eta}} := \{(\bar{\omega}, \bar{\tau}) \in \mathcal{P} : \bar{\omega} \wedge \bar{\tau} = \bar{\eta}\}$$

where $\bar{\omega} \wedge \bar{\tau}$ denotes the common initial segment of the words $\bar{\omega}$ and $\bar{\tau}$. In particular, we consider $\mathcal{P}_\emptyset$ for $\eta = \emptyset$, the empty word. Obviously,

$$\mathcal{P} = \mathcal{P}_\emptyset \bigcup \left(\cup_{k \ge 1} \cup_{\bar{\eta} \in \Sigma^k} \mathcal{P}_{\bar{\eta}}\right). \tag{3.2}$$

For $\bar{\eta} \in \Sigma^k$ we define $\Phi^{\bar{\eta}} : \mathcal{P}_{\bar{\eta}} \to \mathcal{P}_{\emptyset}$ by $\Phi^{\bar{\eta}}(\bar{\omega}, \bar{\tau}) := \left(\sigma^k \bar{\omega}, \sigma^k \bar{\tau}\right)$. Put $\mu_2 := (\mu \times \mu)|_{\mathcal{P}}$ and consider $\mu_2^{\bar{\eta}} := \Phi_*^{\bar{\eta}}(\mu_2|_{\mathcal{P}_{\bar{\eta}}})$. Then we obtain from (3.1) that there exist constants $c_3, c_4 > 0$ such that for every $A \in \mathcal{P}_{\emptyset}$,

$$c_3 \lambda_{\bar{\eta}}^{2\alpha} \leq \frac{\mu_2^{\bar{\eta}}(A)}{\mu_2(A)} \leq c_4 \lambda_{\bar{\eta}}^{2\alpha}. \tag{3.3}$$

3.2. A condition which implies that $\Lambda^{\mathbf{t}}$ contains an interval almost surely.

Definition 2. The weight of a path $\bar{\omega} \in \Sigma^*$ is defined by $w(\bar{\omega}) := \lambda_{\bar{\omega}}$. For an edge $\omega \in \mathcal{E}$ let $K(\omega)$ be a cycle (i.e. a path with identical initial and terminal vertices) starting with ω and having maximal weight. Let

$$r_\omega := w(K(\omega)).$$

Lemma 1. *If there exist distinct $\omega, \tau \in \mathcal{E}$, with $s(\omega) = s(\tau)$, such that*

$$r_\omega + r_\tau \geq 1, \tag{3.4}$$

then for all $\mathbf{t} \in \mathbb{R}^\ell \setminus F$ the attractor $\Lambda^{\mathbf{t}}$ contains an interval, where F is a hyperplane of codimension 1 (hence for Lebesgue almost all $\mathbf{t}$).

Proof. Let $K(\omega) = (\omega_1, \ldots, \omega_k)$ and $K(\tau) = (\tau_1, \ldots, \tau_p)$, so that $\omega_1 = \omega$ and $\tau_1 = \tau$. Then

$$f^{\mathbf{t}}_{\omega_1,\ldots,\omega_k}(x) = \lambda_{\omega_1,\ldots,\omega_k} x + t_{\omega_1,\ldots,\omega_k}, \quad f^{\mathbf{t}}_{\tau_1,\ldots,\tau_p}(x) = \lambda_{\tau_1,\ldots,\tau_p} x + t_{\tau_1,\ldots,\tau_p},$$

where

$$t_{\omega_1,\ldots,\omega_k} = t_{\omega_k} \lambda_{\omega_1,\ldots,\omega_{k-1}} + \cdots + t_{\omega_1}, \quad t_{\tau_1,\ldots,\tau_p} = t_{\tau_p} \lambda_{\tau_1,\ldots,\tau_{p-1}} + \cdots + t_{\tau_1}.$$

Consider the iterated function system (IFS) $\{f^{\mathbf{t}}_{\omega_1,\ldots,\omega_k}, f^{\mathbf{t}}_{\tau_1,\ldots,\tau_p}\}$ on the line. Its attractor consists of all points $\Pi^{\mathbf{t}}(\bar{\eta})$ where $\bar{\eta} \in \Sigma_{s(\omega)}$ is an arbitrary infinite concatenation of the cycles $K(\omega)$ and $K(\tau)$. Thus, the attractor of $\{f^{\mathbf{t}}_{\omega_1,\ldots,\omega_k}, f^{\mathbf{t}}_{\tau_1,\ldots,\tau_p}\}$ is contained in $\Lambda^{\mathbf{t}}_{s(\omega)}$. The sum of the contraction ratios of the maps of the IFS is greater than one by assumption. It follows that the attractor of the IFS contains an interval, as long as $f^{\mathbf{t}}_{\omega_1,\ldots,\omega_k}$ and $f^{\mathbf{t}}_{\tau_1,\ldots,\tau_p}$ have different fixed points. (This is an elementary fact, see e.g. [9].)

It remains to note that the set of $\mathbf{t}$, for which the fixed points coincide, is a hyperplane. Indeed, the fixed points coincide if and only if

$$t_{\omega_1,\ldots,\omega_k}(1 - \lambda_{\omega_1,\ldots,\omega_k})^{-1} = t_{\tau_1,\ldots,\tau_p}(1 - \lambda_{\tau_1,\ldots,\tau_p})^{-1},$$

which is a linear equation in $\mathbf{t}$. It is non-trivial, i.e. not identically satisfied, since $t_\omega = t_{\omega_1}$ occurs in $t_{\omega_1,\ldots,\omega_k}$ only once and does not occur in $t_{\tau_1,\ldots,\tau_p}$, by the choice of $K(\omega)$ and $K(\tau)$ as cycles of maximal weight. $\square$

3.3. Establishing a kind of transversality condition.

We fix an arbitrary $\rho > 0$ for the rest of the paper. In all the proofs it is sufficient to restrict ourselves to $\mathbf{t} \in B_{\rho_0} = \left\{\mathbf{x} \in \mathbb{R}^\ell : |\mathbf{x}| < \rho_0\right\}$. Recall that $\ell = \#\mathcal{E}$. We will need the following elementary fact.

Lemma 2. *For any $\rho_0 > 0$ there exists $c_5 = c_5(\rho_0) > 0$ such that for every $\delta > 0$ and for every $\mathbf{v} \in \mathbb{R}^\ell$,*

$$\mathcal{L}_\ell \left\{\mathbf{t} \in B_{\rho_0} : |\mathbf{t} \cdot \mathbf{v}| \leq \delta\right\} \leq c_5 \delta \left|\mathbf{v}\right|^{-1}. \tag{3.5}$$

The following function will play an important role in the rest of the proof.

Definition 3 (The definition of the function $\mathbf{v} : \mathcal{P} \to \mathbb{R}^\ell$)**.** For any $(\bar{\omega}, \bar{\tau}) \in \mathcal{P}$ let $\mathbf{v}(\bar{\omega}, \bar{\tau}) := (v_1, \dots, v_\ell)$ be such that for every $1 \leq n \leq \ell$,

$$v_n = v_n(\bar{\omega}, \bar{\tau}) := \sum_{\omega_k = e_n} \lambda_{\omega_1, \dots, \omega_{k-1}} - \sum_{\tau_p = e_n} \lambda_{\tau_1, \dots, \tau_{p-1}},$$

where $\mathcal{E} = \{e_1, \dots, e_\ell\}$ and $\lambda_\emptyset = 1$ by definition.

The reason we will use the function $\mathbf{v}$ often is that

$$\Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau}) = \mathbf{t} \cdot \mathbf{v}(\bar{\omega}, \bar{\tau}). \tag{3.6}$$

It is immediate from the definitions that if $|\bar{\omega} \wedge \bar{\tau}| = k$ then

$$\mathbf{v}(\bar{\omega}, \bar{\tau}) = \lambda_{\omega_1, \dots, \omega_k} \mathbf{v}\left(\sigma^k \bar{\omega}, \sigma^k \bar{\tau}\right). \tag{3.7}$$

Lemma 3. *Assume that for all $\omega, \tau \in \mathcal{E}$, with $s(\omega) = s(\tau)$ and $\omega \neq \tau$, we have*

$$r_\omega + r_\tau < 1.$$

Then there exists $c^* > 0$ such that for all $(\bar{\omega}, \bar{\tau}) \in \mathcal{P}_\emptyset$ we have:

$$v_n(\bar{\omega}, \bar{\tau}) > c^*, \quad \forall n \leq \ell.$$

Proof. Let

$$c^* := \min \left\{ 1 - \frac{r_\omega}{1 - r_\tau} : \omega, \tau \in \mathcal{E}; s(\omega) = s(\tau); \omega \neq \tau \right\}.$$

By assumption, $c^* > 0$. Let $\bar{\omega}, \bar{\tau} \in \mathcal{P}_\emptyset$. Assume that $\omega_1 = e_n$ and $\tau_1 = e_p$; then we know that $s(e_n) = s(e_p)$. Then, since $\lambda_j > 0$ for all $j \in \mathcal{E}$, we have

$$v_n \geq 1 - r_{e_p} - r_{e_p} r_{e_n} - r_{e_p} r_{e_n}^2 \cdots = 1 - \frac{r_{e_p}}{1 - r_{e_n}} \geq c^* > 0.$$

$\square$

Under the assumption of Lemma 3, it follows from (3.7) that for $(\bar{\omega}, \bar{\tau}) \in \mathcal{P}$, $|\bar{\omega} \wedge \bar{\tau}| = k$, we have

$$v_n(\bar{\omega}, \bar{\tau}) > \lambda_{\omega_1, \dots, \omega_k} c^*, \quad \forall n \leq \ell. \tag{3.8}$$

Lemma 4. *Under the assumption of Lemma 3, for any $\rho_0 > 0$ there exist $c_6 > 0$ such that*

$$A'(r) := \iint_{\mathcal{P}_\emptyset} \mathcal{L}_\ell \left\{ \mathbf{t} \in B_{\rho_0} : \left| \Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau}) \right| \leq r \right\} d\mu(\bar{\tau}) d\mu(\bar{\omega}) \leq c_6 r.$$

Proof. Using (3.5) and (3.6) we get that

$$\mathcal{L}_\ell \left\{ \mathbf{t} \in B_{\rho_0} : \left| \Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau}) \right| \leq r \right\} \leq c_5 \cdot r \cdot (\mathbf{v}(\bar{\omega}, \bar{\tau}))^{-1} \leq r \cdot c_5 \cdot (c^*)^{-1}.$$

So we can chose $c_6 := c_5 \cdot (c^*)^{-1}$. $\square$

For $k \geq 1$ and $\bar{\eta} \in \Sigma^k$ we define

$$A_{\bar{\eta}}(r) := \iint_{\mathcal{P}_{\bar{\eta}}} \mathcal{L}_\ell \left\{ \mathbf{t} \in B_{\rho_0} : \left| \Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau}) \right| \leq r \right\} d\mu(\bar{\omega}) d\mu(\bar{\tau}).$$

Lemma 5. *There exists $c_7 > 0$ such that for all $k \geq 1$ and for all $\bar{\eta} \in \Sigma^k$ we have*

$$A_{\bar{\eta}}(r) \leq c_7 r \left(\lambda_{\max}\right)^{k(\alpha - 1)} \mu([\bar{\eta}]).$$

Proof. For every $r > 0$ and $(\mathbf{i},\mathbf{j}) \in \mathcal{P}$ we introduce the function

$$g_r(\mathbf{i},\mathbf{j}) := \mathcal{L}_\ell \left\{ \mathbf{t} \in B_{\rho_0} : \left|\Pi^{\mathbf{t}}(\bar\omega) - \Pi^{\mathbf{t}}(\bar\tau)\right| \le r \right\}.$$

Then, since $\bar\omega \wedge \bar\tau = \bar\eta$, by (3.7),

$$g_r(\mathbf{i},\mathbf{j}) = g_{r/\lambda_{\bar\eta}}(\sigma^k \mathbf{i}, \sigma^k \mathbf{j}).$$

Thus, first making a change of variables, then using (3.3), definition of $A'(r)$, and finally Lemma 4, we obtain:

$$\begin{aligned} A_{\bar\eta}(r) &= \iint_{\mathcal{P}_{\bar\eta}} g_{r/\lambda_{\bar\eta}}\left(\sigma^k \mathbf{i}, \sigma^k \mathbf{j}\right) d\mu_2(\mathbf{i},\mathbf{j}) = \iint_{\mathcal{P}_\emptyset} g_{r/\lambda_{\bar\eta}}\left(\mathbf{i},\mathbf{j}\right) d\mu_2^{\bar\eta}(\mathbf{i},\mathbf{j}) \\ &\le c_4 \lambda_{\bar\eta}^{2\alpha} \iint_{\mathcal{P}_\emptyset} g_{r/\lambda_{\bar\eta}}\left(\mathbf{i},\mathbf{j}\right) d\mu_2(\mathbf{i},\mathbf{j}) = c_4 \lambda_{\bar\eta}^{2\alpha} A'\left(\frac{r}{\lambda_{\bar\eta}}\right) \\ &\le c_6 c_4 r \lambda_{\bar\eta}^{2\alpha-1}. \end{aligned}$$

Since $\mu\left([\bar\eta]\right) \asymp \lambda_{\bar\eta}^{\alpha}$ by (3.1), there exists $c_7 > 0$ such that

$$A_{\bar\eta}(r) \le c_7 r \lambda_{\bar\eta}^{\alpha-1} \mu\left([\bar\eta]\right) \le c_7 r \left(\lambda_{\max}\right)^{k(\alpha-1)} \mu\left([\bar\eta]\right).$$

$\square$

Now we are in a position to prove Theorem 1. The argument involves integration over the parameters and Fubini Theorem, and uses the "transversality condition." By now such arguments are rather standard, but we provide all the details for the reader's convenience. Some of the early references for this method are [8] and [7].

3.4. **Proof of Theorem 1(ii).** Here we assume that $\alpha > 1$. If the assumption of Lemma 3 does not hold, then by Lemma 1, for almost all $\mathbf{t} \in \mathbb{R}^\ell$, the attractor $\Lambda^{\mathbf{t}}$ contains an interval, which proves the claim of Theorem 1(ii). So, from now on we may also assume that the assumption of Lemma 3 holds.

We recall that we defined $\Sigma_1 = \{\bar\omega \in \Sigma : s(\omega) = 1\}$. Let $\nu_{\mathbf{t}} := \left(\Pi^{\mathbf{t}}|_{\Sigma_1}\right)_* (\mu)$. Let

$$\underline{D}\left(\nu_{\mathbf{t}}, x\right) := \liminf_{r \to 0} \frac{\nu_{\mathbf{t}}\left([x-r, x+r]\right)}{2r}.$$

Fix an arbitrary $\rho_0 > 0$. We will be done if we can prove that

$$I := \int_{B_{\rho_0}} \int_{\mathbb{R}} \underline{D}\left(\nu_{\mathbf{t}}, x\right) d\nu_{\mathbf{t}}\, d\mathbf{t} < \infty.$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}} \nu_{\mathbf{t}}\left([x-r, x+r]\right) d\nu_{\mathbf{t}} &= \iint_{\Sigma_1 \times \Sigma_1} 1_{\{(\bar\omega,\bar\tau):|\Pi^{\mathbf{t}}(\bar\omega) - \Pi^{\mathbf{t}}(\bar\tau)| \le r\}}\, d\mu(\bar\omega) d\mu(\bar\tau) \\ &\le \iint_{\mathcal{P}} 1_{\{(\bar\omega,\bar\tau):|\Pi^{\mathbf{t}}(\bar\omega) - \Pi^{\mathbf{t}}(\bar\tau)| \le r\}}\, d\mu(\bar\omega) d\mu(\bar\tau). \end{aligned}$$

Using Fatou's Lemma, (3.2), and Lemmas 4 and 5 we get

$$
\begin{aligned}
I &\le \liminf_{r\to 0}(2r)^{-1} \iint\limits_{\mathcal{P}} \mathcal{L}_\ell \left\{\mathbf{t}\in B_{\rho_0} : \left|\Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau})\right| \le r\right\} d\mu(\bar{\omega}) d\mu(\bar{\tau}) \\
&= \liminf_{r\to 0}(2r)^{-1} \left(A'(r) + \sum_{k=1}^{\infty} \sum_{\bar{\eta}\in\Sigma^k} A_{\bar{\eta}}(r) \right) \\
&\le \liminf_{r\to 0}(2r)^{-1} \left(c_6 r + \sum_{k=1}^{\infty} \sum_{\bar{\eta}\in\Sigma^k} c_7 r \left(\lambda_{\max}\right)^{k(\alpha-1)} \mu([\bar{\eta}]) \right) \\
&\le \mathrm{const}\cdot \sum_{k=1}^{\infty} \left(\lambda_{\max}\right)^{k(\alpha-1)} < \infty,
\end{aligned}
$$

since we assumed that $\alpha > 1$. □

3.5. **Proof of Theorem 1(i).** Now we may assume that $\alpha \le 1$. Otherwise, from the second part of Theorem 1 proved above, we would immediately get the first part. Further, we may also assume that the assumption of Lemma 3 holds. Indeed, otherwise Lemma 1 yields that $\Lambda^{\mathbf{t}}$ contains an interval for almost all $\mathbf{t}\in\mathbb{R}^\ell$ and hence $\dim_H \Lambda^{\mathbf{t}} = 1$ almost surely. Using the trivial observation that α is always an upper bound for the dimension of $\Lambda^{\mathbf{t}}$, we would get that $\alpha \ge 1$, proving the statement of Theorem 1(i).

As above, we write $\nu_{\mathbf{t}} := \left(\Pi^{\mathbf{t}}|_{\Sigma_1}\right)_* (\mu)$ and recall that $\Pi^{\mathbf{t}} : \Sigma_1 \to \Lambda_1^{\mathbf{t}}$ is onto. Fix an arbitrary $\rho_0 > 0$. It follows from the potential-theoretic characterization of the Hausdorff dimension (see e.g. [5] or [3]) that it is enough to prove that the following integral is finite:

$$
I := \int\limits_{B_{\rho_0}} \iint\limits_{\Lambda_1^{\mathbf{t}}\times\Lambda_1^{\mathbf{t}}} |x-y|^{-s}\, d\nu_{\mathbf{t}}(x) d\nu_{\mathbf{t}}(y) d\mathbf{t} < \infty,
$$

for every $0 < s < \alpha$, since in this case $\dim_H \Lambda^{\mathbf{t}} = \dim_H \Lambda_1^{\mathbf{t}} \ge s$ almost surely. Thus, it is enough to show that

$$
I < I_2 := \int\limits_{B_{\rho_0}} \iint\limits_{\mathcal{P}} \left|\Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau})\right|^{-s} d\mu(\bar{\omega}) d\mu(\bar{\tau}) d\mathbf{t} < \infty.
$$

By (3.6), Lemma 2, and (3.8) we obtain that there exists $c_8 > 0$ such that

$$
\mathcal{L}_\ell \left\{\mathbf{t}\in B_{\rho_0} : \left|\Pi^{\mathbf{t}}(\bar{\omega}) - \Pi^{\mathbf{t}}(\bar{\tau})\right| < r\right\} \le c_8 \min\left\{1, r\cdot \lambda_{\bar{\omega}\wedge\bar{\tau}}^{-1}\right\}.
$$

Thus,

$$\begin{aligned}
\int\limits_{B_{\rho_0}} \left|\Pi^{\mathbf{t}}(\bar\omega)-\Pi^{\mathbf{t}}(\bar\tau)\right|^{-s} d\mathbf{t} &= \int\limits_{r=0}^{\infty} \mathcal{L}_\ell \left\{\mathbf{t}\in B_{\rho_0} : \left|\Pi^{\mathbf{t}}(\bar\omega)-\Pi^{\mathbf{t}}(\bar\tau)\right| \le r^{-1/s}\right\} \\
&\le c_8 \int\limits_{r=0}^{\infty} \min\left\{1, r^{-1/s}\cdot \lambda_{\bar\omega\wedge\bar\tau}^{-1}\right\} dr \\
&\le c_8 \int\limits_{r=0}^{\lambda_{\bar\omega\wedge\bar\tau}^{-s}} 1\, dr + \lambda_{\bar\omega\wedge\bar\tau}^{-1} \int\limits_{r=\lambda_{\bar\omega\wedge\bar\tau}^{-s}}^{\infty} r^{-1/s} dr \\
&= c_9 \lambda_{\bar\omega\wedge\bar\tau}^{-s},
\end{aligned}$$

for some $c_9>0$. Therefore, using (3.2) and (3.1) we have

$$\begin{aligned}
I_2 &\le c_9 \iint\limits_{\mathcal{P}} \lambda_{\bar\omega\wedge\bar\tau}^{-s}\, d\mu(\bar\omega) d\mu(\bar\tau) \\
&\le c_9 + c_9 \sum_{k=1}^{\infty} \sum_{\bar\eta\in\Sigma^k} \mu([\bar\eta])^2\cdot \lambda_{\bar\eta}^{-s} \le c_9 + c_2 c_9 \cdot \sum_{k=1}^{\infty} \sum_{\bar\eta\in\Sigma^k} \mu([\bar\eta]) \lambda_{\bar\eta}^{\alpha-s} \\
&\le c_9 + c_2 c_9 \sum_{k=1}^{\infty} \lambda_{\max}^{k(\alpha-s)} \sum_{\bar\eta\in\Sigma^k} \mu([\bar\eta]) = c_9 + c_2 c_9 \sum_{k=1}^{\infty} \lambda_{\max}^{k(\alpha-s)} < \infty,
\end{aligned}$$

since $s<\alpha$. This completes the proof of Theorem 1(i). □

4. Proof of Proposition 1 and Theorem 2

Proof of Proposition 1. This proof is very similar to that of [4, Theorem 5.2], which is credited to Falconer in [4]. In this proof we always omit the superscript $\mathbf{t}$. When we write $\Lambda, \hat\Lambda, S$ we always think of $\Lambda^{\mathbf{t}}, \hat\Lambda^{\mathbf{t}}, S^{\mathbf{t}}$ for an arbitrary fixed $\mathbf{t}\in\mathbb{R}^\ell$.

First we need some additional notation. For a bounded set $A\subset\mathbb{R}^3$ we denote by $M_{2^{-k}}(A)$ the smallest number of cubes of side 2^{-k} needed to cover A, such that their sides are parallel to the axes and the projections of the cubes to the xy plane are dyadic squares. Clearly, $\dim_B(A)=\lim_{k\to\infty}\frac{1}{k\log 2}\log M_{2^{-k}}(A)$ if the limit exists. For a bounded set $B\subset\mathbb{R}$ and $r>0$, we denote by $N_r(B)$ the smallest number of intervals of length r required to cover the set B. It is immediate that for any $c>0$,

$$N_r(c\cdot B)=N_{r/c}(B).$$

Let $d:=\dim_B(\Lambda)$ (recall that d exists and equals $\dim_H(\Lambda)$ by [3, Corollary 3.5]). Fix $\varepsilon>0$. By the definition of the box-counting dimension, there exist $c_1', c_2'>0$ such that for every $1\le i\le m$:

$$c_1' \max\left\{1, \left(\frac{1}{r}\right)^{d-\varepsilon}\right\} \le N_r(\Lambda), N_r(\Lambda_i) \le c_2' \max\left\{1, \left(\frac{1}{r}\right)^{d+\varepsilon}\right\}. \tag{4.1}$$

Let I and I_i be the hulls of Λ and Λ_i respectively (i.e. the smallest closed intervals containing the respective sets). The union of boxes $R_i := Q_i\times I_i$ provides a natural first level approximation of $\hat{\Lambda}$. The k-th level approximation of $\hat{\Lambda}$ is the union of all boxes $R_{\omega_1,\ldots,\omega_k} := S_{\omega_1,\ldots,\omega_k}\left(R_{t(\omega_k)}\right)$. Since

$$\hat{\Lambda}_i = \bigcup_{\substack{(\omega_1,\ldots,\omega_k)\in\Sigma^k \\ s(\omega_1)=i}} S_{\omega_1,\ldots,\omega_k}\left(\hat{\Lambda}_{t(\omega_k)}\right)$$

we have

$$\hat{\Lambda} \subset \bigcup_{(\omega_1,\ldots,\omega_k)\in\Sigma^k} R_{\omega_1,\ldots,\omega_k}.$$

The dimensions of the box $R_{\omega_1,\ldots,\omega_k}$ are $2^{-(K+k)} \times 2^{-(K+k)} \times \lambda_{\omega_1,\ldots,\omega_k} \cdot \left|I_{t(\omega_k)}\right|$. Therefore,

$$\begin{aligned} M_{2^{-(K+k)}}\left(R_{\omega_1,\ldots,\omega_k} \cap \hat{\Lambda}\right) &= N_{2^{-(K+k)}}\left(\lambda_{\omega_1,\ldots,\omega_k}\Lambda_{t(\omega_k)}\right) \\ &= N_{2^{-(K+k)}(\lambda_{\omega_1,\ldots,\omega_k})^{-1}}\left(\Lambda_{t(\omega_k)}\right). \end{aligned}$$

By the definition of our GIFS (see subsection 2.2), the projections of the boxes $R_{\omega_1,\ldots,\omega_k}$ to the xy plane are distinct dyadic squares whose union is the unit square. Thus we can sum over all possible sequences $\omega_1,\ldots,\omega_k$ and use (4.1) to obtain

$$\begin{aligned} c_1' \max\left\{\#\Sigma^k,\ 2^{(k+K)\cdot(d-\varepsilon)} \cdot \sum_{(\omega_1,\ldots,\omega_k)\in\Sigma^k} \lambda_{\omega_1,\ldots,\omega_k}^{d-\varepsilon}\right\} & \\ \leq M_{2^{-(K+k)}}(\hat{\Lambda}) & \\ \leq c_2'\left(\#\Sigma^k + 2^{(k+K)\cdot(d+\varepsilon)} \cdot \sum_{(\omega_1,\ldots,\omega_k)\in\Sigma^k} \lambda_{\omega_1,\ldots,\omega_k}^{d+\varepsilon}\right). & \end{aligned} \tag{4.2}$$

We claim that

$$\lim_{k\to\infty} \frac{1}{k} \log \#\Sigma^k = \log 4.$$

In fact, consider the $m \times m$ matrix $B = [b_{i,j}]_{i,j\leq m}$ defined by $b_{ij} := \#\mathcal{E}_{ij}$. Let $B^k = \left[b_{ij}^{(k)}\right]_{i,j\leq m}$. It is well known that $\#\Sigma^k = \sum_{i,j\leq m} b_{ij}^{(k)}$. So,

$$\frac{1}{k}\log\#\Sigma^k = \frac{1}{k}\log\sum_{i,j\leq m} b_{ij}^{(k)} \to \log\rho(B) = \log 4,$$

where the last equality follows from the fact that every row sum of B is four (the out-degree of every vertex in Γ is four), hence the spectral radius of B equals 4.

Now we take the logarithm in (4.2), divide by $k\log 2$, and pass to the limit to obtain:

$$\begin{aligned} c_3' \max\left\{2,\ d-\varepsilon+\frac{1}{\log 2}\log\rho\left(A_{d-\varepsilon}\right)\right\} & \\ \leq \frac{1}{\log 2}\lim_{k\to\infty}\frac{1}{k}\log M_{2^{-(K+k)}}(\hat{\Lambda}) & \\ \leq c_4' \max\left\{2,\ d+\varepsilon+\frac{1}{\log 2}\log\rho\left(A_{d+\varepsilon}\right)\right\}, & \end{aligned}$$

where we used that for $\beta > 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \sum_{(\omega_1,\dots,\omega_k)\in\Sigma^k} \lambda^{\beta}_{\omega_1,\dots,\omega_k} = \log\rho(A_\beta).$$

Since $\varepsilon > 0$ is arbitrary,

$$\dim_B(\hat\Lambda) = \max\left\{2, d + \frac{\log(A_d)}{\log 2}\right\},$$

as desired. □

Proof of Theorem 2. By Theorem 1, $d(\mathbf{t}) = \dim_H \Lambda^{\mathbf{t}} = \min\{1,\alpha\}$ for almost all $\mathbf{t} \in \mathbb{R}^\ell$, where α is such that $\rho(A_\alpha) = 1$. If $\alpha \geq 1$, then $d(\mathbf{t}) = 1$ for a.e. $\mathbf{t} \in \mathbb{R}^\ell$, and Proposition 1 implies (2.5). If $\alpha < 1$, then for a.e. $\mathbf{t} \in \mathbb{R}^\ell$ we have

$$d(\mathbf{t}) + \frac{\log\rho(A_{d(\mathbf{t})})}{\log 2} = \alpha + \frac{\log\rho(A_\alpha)}{\log 2} = \alpha < 1 < 2,$$

so $\dim_B(\hat\Lambda^{\mathbf{t}}) = 2$ by Proposition 1. On the other hand, $\rho(A_1) \leq \rho(A_\alpha) = 1$, since $\beta \mapsto \rho(A_\beta)$ is decreasing, hence the right-hand side of (2.5) equals 2 as well. □

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