Hausdorff dimension of hyperbolic attractors in $\mathbb{R}^3$

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Abstract. One of the most exciting open problems in the field of dimension theory of dynamical systems is to find the Hausdorff dimension of hyperbolic attractors in space. After an introduction we will give an account of the known results.

Contents

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1. Introduction

In this paper we are always working in $\mathbb{R}^d$ where d is usually not bigger than 3. The simplest dynamically defined sets are the self-similar ones. It has been known since the papers Hutchinson [13] or even Moran [15] that for a self-similar iterated function system (IFS) $\mathcal{F} = \{f_1, \dots, f_m\}$, where

$$|f_i(x) - f_i(y)| = \lambda_i \, |x - y| \,, \quad 0 < \lambda_i < 1, \quad x, y \in \mathbb{R}^d,$$

we an compute the Hausdorff dimension of the attractor Λ, which is the unique nonempty compact set satisfying

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda), \tag{1.1}$$

as the solution s of the equation

$$\lambda_1^s + \dots + \lambda_m^s = 1. \tag{1.2}$$

More precisely,

$$\dim_H(\Lambda) = \min\{s, d\} \tag{1.3}$$

holds if the cylinders $\Lambda_i := f_i(\Lambda),\ i = 1, \dots, m$, are pairwise disjoint or, more generally, the so-called open set condition holds. However, if we assume only that the maps f_i above are affine maps (in this case the set defined in (1.1) is called self-affine), then the problem is much more difficult. On the one hand, we can expect only 'almost all' type results, on the other hand, even 'almost all' type results hold only in the case when $\|f_i\| < \frac{1}{2}$, see [11] and [38]. In this note we are interested in the following problem:

Problem: How can we express the dimension of uniformly hyperbolic (axiom A) invariant sets?

On the line and on a surface this problem has similar flavor to the computation of dimension of self-similar sets with disjoint cylinders, and that is why on the line and on a surface the answer has been known for more than 20 years. However, in space (and higher dimension) the problem has similar flavor to the computation of the dimension of self-affine sets, which is much more difficult. That is why in space the problem has been open for more than twenty years. In this note we would like to provide a gentle introduction to this field for everybody interested in fractals but not necessarily having any background in the field of dynamical systems. There are at least three other survey papers [28], [2] and [31] of much wider scope dealing with (among some other things) the problem above.

2. The dimension of self-affine sets

First we study how to compute the dimension of the attractor for a typical self-affine IFS if all contractions are 'stronger' than $\frac{1}{2}$.

2.1. Falconer's Theorem

The most important result in this field is due to Falconer [11]. This is an 'almost all' type result in the following sense:

Let Λ be the attractor of a contractive self-affine IFS $\mathcal{F} = \{f_1, \dots, f_m\}$ on $\mathbb{R}^d$, that is, Λ satisfies (1.1). For every $\mathbf{t} = \{\mathbf{t}_1, \dots, \mathbf{t}_m\} \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ we define $\Lambda^{\mathbf{t}}$ as the attractor of the 'shifted' IFS $\mathcal{F}^{\mathbf{t}} = \{f_1 + \mathbf{t}_1, \dots, f_m + \mathbf{t}_m\}$.

Theorem 2.1 (Falconer [11], Solomyak [38]). *If $\|f_i\| < \frac{1}{2}$ holds*[1] *for all $i = 1, \dots, m$, then for Lebesgue almost all $\mathbf{t} \in \mathbb{R}^{m \cdot d}$ we have*

$$\dim_H \Lambda^{\mathbf{t}} = \dim_B \Lambda^{\mathbf{t}} = \min\{d, \alpha(f_1, \dots, f_m)\}, \tag{2.1}$$

where $\alpha(f_1, \dots, f_m)$ is defined via the singular value function $s \mapsto \varphi^s$ in the following way:

$$\alpha(f_1, \dots, f_m) := \inf\left\{ s : \sum_{k=1}^{\infty} \sum_{i_1 \dots i_k} \varphi^s(f_{i_1} \circ \cdots \circ f_{i_k}) < \infty \right\}. \tag{2.2}$$

We recall that the *singular values* $1 \geq \alpha_1 \geq \cdots \geq \alpha_m > 0$ of a contracting linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ are defined as the square roots of the eigenvalues of the symmetric transformation T^*T. Equivalently, we obtain the singular values of T as the semi-axes of the ellipsoid $T(B)$, where B is the unit ball. Then the singular value function $s \mapsto \varphi^s(T)$ for $0 \leq s \leq d$ is defined as

$$\varphi^s(T) = \alpha_1 \cdots \alpha_{[s]} \alpha_{[s]+1}^{s-[s]}. \tag{2.3}$$

2.2. The importance of the singular value function in computation of dimension

We give a very **heuristic** explanation in the special case when $d = 2$ and $1 < s < 2$. Recall that the s-dimensional Hausdorff measure of Λ is defined as:

$$\mathcal{H}^s(\Lambda) := \lim_{\delta \to 0} \inf\left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s : \text{ where } \Lambda \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \mathrm{diam}(U_i) \leq \delta \right\}. \tag{2.4}$$

For simplicity we assume that the unit ball B contains the attractor Λ . For an arbitrary n the cylinder $B_{i_1 \dots, i_n} := f_{i_1} \circ \cdots \circ f_{i_n}(B)$ is an ellipse of semi-axis α_1, α_2, the singular values of $T := f_{i_1} \circ \cdots \circ f_{i_n}$.

Let us call $\mathcal{C}_{i_1 \dots i_n}$ a cover of $B_{i_1 \dots, i_n}$ with the minimal number of squares of side $2\alpha_2$. Then $\mathcal{C}_{i_1 \dots i_n}$ consists of α_1/α_2 squares of side $2\alpha_2$. Further, $\bigcup_{i_1, \dots, i_n} \mathcal{C}_{i_1 \dots i_n}$ is a cover of the attractor Λ. In (2.4) we take the sum of the diameter of the sets in a cover, raised to the power s and take infimum for all covers with sets of small diameters. Compute this sum for the cover: $\bigcup_{i_1, \dots, i_n} \mathcal{C}_{i_1 \dots i_n}$. The sum in (2.4) is $2^s \sum\limits_{i_1 \dots i_k} \varphi^s(f_{i_1} \circ \cdots \circ f_{i_k})$ since $\varphi^s(T) = \frac{\alpha_1}{\alpha_2} \alpha_2^s$ for $T = f_{i_1} \circ \cdots \circ f_{i_k}$. So, if $s > \alpha(f_1, \dots, f_m)$ then the sum $\sum\limits_{i_1 \dots i_k} \varphi^s(f_{i_1} \circ \cdots \circ f_{i_k})$ tends to zero as $n \to \infty$. Therefore $\alpha(f_1, \dots, f_m)$ is an upper bound of the Hausdorff dimension.

[1] In Falconer's paper this bound was 1/3. Solomyak improved it to 1/2.

However, there are very natural examples (see e.g. [16] and [25]) showing that formula (2.1) does not always hold. Edgar proved that for $d \geq 2$ the upper bound $\frac{1}{2}$ on the norm of the contraction in Falconer Theorem is sharp. In [34] we proved with B. Solomyak that one cannot improve this upper bound even if all the maps in Falconer's Theorem are similarities (assuming again that $d \geq 2$).

3. Hyperbolic attractors on the line

Let $\{f\}_{i=1}^{m}$ be a contracting IFS on the line. We assume that $f_i \in \mathcal{C}^2$ (for simplicity) and

$$0 < \inf_{i,x} |f_i'(x)| < \sup_{i,x} |f_i'(x)| < 1. \tag{3.1}$$

The Hausdorff dimension of the attractor Λ (defined by (1.1)) can be computed in a very similar way to the self similar case, if the cylinders are disjoint:

$$\forall i \neq j, \;\; f_i(\Lambda) \cap f_j(\Lambda) = \emptyset. \tag{3.2}$$

Namely, let $I := \text{hull}(\Lambda)$. Then the most natural cover of Λ is the cover by all n-cylinder intervals $\{I_{i_1\dots i_n}\}$ for some $n \in \mathbb{N}$, where $I_{i_1\dots i_n} := f_{i_1\dots i_n}(I)$. (Throughout the paper we always use the notation: $f_{i_1\dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$.) Let d_n be the exponent that corresponds to the cover $\{I_{i_1\dots i_n}\}$. That is, we define d_n as the solution of the equation:

$$\sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^{d_n} = 1. \tag{3.3}$$

It is well known that

$$\dim_H \Lambda = \lim_{n\to\infty} d_n. \tag{3.4}$$

Obviously, in the special case when all the functions f_i are similarities, for every n we have $d_n = s$, the solution of (1.2). In the general case, unfortunately, the convergence in (3.4) is slow. Namely,

$$|\dim_H \Lambda - d_n| < \frac{const}{n}, \tag{3.5}$$

and in general there is no way to improve this. It is useful to express the Hausdorff dimension using topological pressure. Using that the Hausdorff dimension can be obtained as the root of some pressure equation, Pollicott and Jenkinson find a way [22] to approximate the Hausdorff dimension numerically with much higher speed than in (3.5).

3.1. Computation of the dimension by the pressure formula

Here we give yet again a very heuristic argument, explaining how the topological pressure appears in this field. In this argument we assume that the cover by the cylinder intervals is the cover at which the infimum is attained in (2.4). One can find detailed proofs in the books [9] and [21].
Assume that on the symbolic space $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$ we are given a continuous function $\varphi : \Sigma \to \mathbb{R}$.

Definition 3.1. The pressure of φ is defined by:

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \dots i_n} \exp\left\{ \sup_{\mathbf{j}\in[i_1\dots i_n]} \sum_{k=0}^{n-1} \varphi\left(\sigma^k \mathbf{j}\right) \right\}. \tag{3.6}$$

We use this for the function

$$\varphi_s(\mathbf{j}) := s \log \left| f'_{j_1}\left(\Pi(\sigma \mathbf{j})\right)\right|, \tag{3.7}$$

where $0 \leq s$ and $\Pi : \Sigma \to \Lambda$ is called the natural projection and is defined by:

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} f_{i_1\dots i_n}(0). \tag{3.8}$$

By the chain rule we have

$$\sum_{k=0}^{n-1} \varphi_s\left(\sigma^k \mathbf{j}\right) = \log \left| f'_{i_1\dots i_n}(\Pi(\sigma^n \mathbf{j}))\right|, \tag{3.9}$$

holds for every $\mathbf{j} \in (i_1, \dots, i_n)$. Using the so-called bounded distortion lemma and Lagrange Theorem we obtain that for all $\mathbf{j} \in (i_1, \dots, i_n)$,

$$|I_{i_1\dots i_n}| = |f_{i_1\dots i_n}(I)| \approx \left| f'_{i_1\dots i_n}(\Pi(\sigma^n \mathbf{j}))\right|, \tag{3.10}$$

where $\approx$ means equality up to a multiplicative constant. So,

$$\exp\left[\sup_{\mathbf{j}\in(i_1,\dots,i_n)} \sum_{k=0}^{n-1} \varphi_s\left(\sigma^k \mathbf{j}\right) \right] \approx |I_{i_1\dots i_n}|^s .$$

We see that the topological pressure is the exponential growth rate of the sum of the s-powers of the diameters of the elements of the cover $I_{i_1\dots i_n}$. Assuming again that this cover is the most economical cover in the sense of (2.4), we can see why the notion of topological pressure is so important in the computation of dimension. Namely, one can easily see that the function $s \mapsto P(\varphi_s)$ is a strictly decreasing smooth function, which is positive at 0 and negative at 1. So there is a unique t at which $P(\varphi_t) = 0$. When $s > t$ then $P(\varphi_s) < 0$. This means that the exponential growth rate of the sum $\sum\limits_{i_1\dots i_n} |I_{i_1\dots i_n}|^s$ (in n) is negative, that is, from (2.4) we obtain that $\mathcal{H}^s(\Lambda) = 0$. Similarly, if $s < t$ then the exponential growth rate of $\sum\limits_{i_1\dots i_n} |I_{i_1\dots i_n}|^s$ is positive. So, assuming that this is the most economical cover in the sense of (2.4), we get that the s-dimensional Hausdorff measure is infinity. Therefore we have

Theorem 3.2.

$$\dim_H(\Lambda) = \textit{the zero of the function: } s \mapsto P(\varphi_s), \tag{3.11}$$

where φ_s was defined in (3.7).

4. Uniformly hyperbolic invariant sets

The precise definition of an axiom A diffeomorphism can be found in the books [6, p.68] and [21, p.228]. An important consequence of the definition is that if Λ is an invariant set for a $\mathcal{C}^r$ axiom A diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a smooth finite dimensional Riemannian manifold, then for every $x \in \Lambda$ and small $\varepsilon > 0$, there exist $\mathcal{C}^r$ disks $W_\varepsilon^s(x), W_\varepsilon^u(x) \subset \mathcal{M}$ such that for any $y \in W_\varepsilon^s(x)$ the forward orbit of y is getting exponentially close to the forward orbit of x after sufficiently many steps. The same is true for $y \in W_\varepsilon^u(x)$ if we take the backward orbit. They are called the local stable and unstable manifolds. They are invariant under the dynamics: $f\left(W_\varepsilon^s(x)\right) \subset W_\varepsilon^s(x)$ (the same is true for $W_\varepsilon^u(x)$ with f^{-1}). The intersection of local stable and unstable manifolds at x is transversal and equal to $\{x\}$. Furthermore, the direct sum of their tangent subspaces is equal to the tangent space at x: $T_x\mathcal{M} = E_x^s \bigoplus E_x^u$, where $E_x^s = T_x W_\varepsilon^s(x)$ and $E_x^u = T_x W_\varepsilon^u(x)$. Since locally Λ is the product of the sets $\Lambda \cap W_\varepsilon^s(x)$ and $\Lambda \cap W_\varepsilon^u(x)$ the Hausdorff dimension of Λ is

$$\dim_H(\Lambda \cap B(x, \varepsilon)) = \dim_H(\Lambda \cap W_\varepsilon^s(x)) + \dim_H(\Lambda \cap W_\varepsilon^u(x)), \tag{4.1}$$

where $B(x, \varepsilon)$ is the ball centered at x with radius ε. This is so, because these Cantor sets on the local stable and unstable have the same box and Hausdorff dimension.

5. Hausdorff dimension of Axiom A invariant sets on a surface

Assume that Λ is an invariant set of a $\mathcal{C}^2$ diffeomorphism on a surface. Then for all $x \in \Lambda$ the local stable and unstable manifolds $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ are smooth curves. Let $\Lambda^s(x) := \Lambda \cap W_\varepsilon^s(x)$ and $\Lambda^u(x) := \Lambda \cap W_\varepsilon^u(x)$ and write t_s and t_u for their Hausdorff dimensions respectively. Since the Cantor sets $\Lambda^s(x)$ and $\Lambda^u(x)$ are constructed in a very similar way to the Cantor set in Section 3, their Hausdorff dimension can be computed with essentially the same pressure formulas as in (3.11). Namely, the zeros of the functions

$$s \mapsto P(s \log \left|d_x f|_{E^s(x)}\right|) \quad \text{and} \quad s \mapsto P(-s \log \left|d_x f|_{E^u(x)}\right|) \tag{5.1}$$

are equal to t_s and t_u respectively. Then by (4.1) we obtain the Hausdorff dimension of Λ:

$$\dim_H(\Lambda) = t_s + t_u. \tag{5.2}$$

The same result was obtained by McCluskey and Manning [17] for $\mathcal{C}^1$ axiom A diffeomorphisms. Their result is much more difficult than the argument what appears here, since we always assume that our maps are $\mathcal{C}^2$, so we have the bounded distortion lemma.

6. Hausdorff dimension of Axiom A invariant sets in space

We do not know in general how to compute the dimension of an axiom A invariant set on a three- (or higher) dimensional Riemannian manifold. Let f be a $\mathcal{C}^2$ diffeomorphism on a three-dimensional Riemannian manifold. We may assume that the local stable manifolds are two-dimensional and the local unstable manifolds are one-dimensional submanifolds. Using (4.1) we can compute the Hausdorff dimension as the sum of the dimensions of the parts of Λ falling into the local stable and unstable manifolds. Since by assumption the local unstable manifold is one-dimensional, for any x the dimension $\dim_H (\Lambda \cap W^u_\varepsilon(x))$ can be computed as the zero of the second function in (5.1). The problem is that, in general, we have a difficulty computing $\dim_H (\Lambda \cap W^s_\varepsilon(x))$. Since E^s_x is a plane, it can happen that there are two directions in E^s_x with different contraction coefficients. This is the non-conformal case, so here a difficulty appears similar to that in Section 2.

Since we cannot solve the problem in full generality, it is useful to try to give an answer in some special cases. The most natural axiom A attractor in space is the solenoid.

6.1. Solenoids with two different constant contracting coefficients

If in the definition of a solenoid the two contracting coefficients are the same constant, then the problem is easy (see e.g. [8]). Therefore the first non-trivial example is the solenoid with two different constant contracting coefficients. Y. Pesin in 1982 (in the same year when Manning and McCluskey solved the problem on a surface) asked the following question:

Question (Pesin) Let Λ be the solenoid with two different constant contraction coefficients. How can we compute the Hausdorff dimension of Λ?

The answer was given independently by Bothe [4] and the author [32]. First we give the definition of the solenoid and state the answer for Pesin's question.

Definition 6.1. Suppose $0 < \lambda_2 < \lambda_1 < \frac{1}{2}$. Let $T := S^1 \times D$, where D is the unit disk centered at the origin. We define the map (see Figure **??**) $f : T \to T$ by

$$f(t, x, z) := \left(2t(\mod 1), \lambda_1 x + \frac{1}{2}\cos(2\pi t), \lambda_2 z + \frac{1}{2}\sin(2\pi t) \right). \tag{6.1}$$

We assumed that $\lambda_1 < \frac{1}{2}$ to make the map one-to-one on T. We call the attractor

$$\Lambda = \bigcap_{n=0} f^n(T)$$

the solenoid with two different constant contracting coefficients.

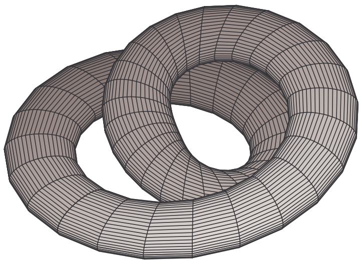

FIGURE 1. $f(T)$, the first image of solenoid map on the torus T

Obviously, on S^1 the map f is expanding with a constant expanding coefficient 2. On D, both in x and in z direction f is contractive with constant contracting coefficients λ_1 and λ_2 respectively.

Theorem 6.2 (Bothe [4], Simon [30, 32])**.** *We have* $\dim_H(\Lambda) = 1 + \frac{\log 2}{-\log \lambda_1}$.

Here we give a heuristic argument for the proof sketch. For an arbitrary $I_{i_n,\dots,i_1}$, a dyadic interval of S^1, we define

$$S_{i_1\dots i_n} := f^n(I_{i_n\dots i_1}).$$

Then $\Lambda = \bigcap_n \bigcup_{i_1\dots i_n} S_{i_1\dots i_n}$. In oder to understand the structure of the solenoid, it is enough to understand the structure of its angular sections. Let us call the set $A_\theta := \{(t,x,z) \in T : t = \theta\}$ the θ-angular section of T. We want to study $A_\theta \cap \Lambda$ for an arbitrary θ. To do this, choose a $\theta \in S^1$. Let $t_1, t_2 \in S^1$ be the inverse images of θ for the function $t \mapsto 2t(\mod 1)$. Then $f(A_{t_i}) \subset A_\theta$ and $f(A_{t_i})$, $i = 1, 2$, are ellipses with semi-axes λ_1, λ_2. In the same way for an n and $i_1, \dots, i_n$ let $\{t_{i_1\dots i_n}\}_{i_1\dots i_n}$ be the inverse images of the function $t \mapsto 2^n t(\mod 1)$. Then $f(A_{t_{i_1\dots i_n}}) \subset A_\theta$ are 2^n congruent ellipses with semi-axes λ_1^n, λ_2^n. Let us call these 2^n ellipses the n-cylinders on A_θ. Obviously, $\Lambda \cap A_\theta \subset \cup_{i_1,\dots,i_n} f(A_{t_{i_1\dots i_n}})$. Using this, we obtain a natural cover of $\Lambda \cap A_\theta$ by the n-cylinders on A_θ. The most natural approach to the computation of the dimension of Λ is to compute first the dimension of $\Lambda \cap A_\theta$ by trying to prove that the cover above is the most economical one in the sense of minimizing the sum in the definition of Hausdorff measure in (2.4). Surprisingly, this most natural approach did not work. Namely, every n-cylinder ellipse contains two 'children' $(n+1)$-cylinder ellipses. Unfortunately, the relative positions of these $(n+1)$-cylinder ellipses are completely different in different n-cylinder ellipses. Therefore the structure of $\Lambda \cap A_\theta$ is different from a self-affine set and difficult to handle. Still we can see from this angular section approach that

$$\dim_H \Lambda \le 1 + \frac{\log 2}{-\log \lambda_1}. \tag{6.2}$$

To get the lower estimate, we need a completely different approach. Observe that the first two component functions of f in (6.1) depend only on the first two variables t and x. (The roles of x and y are different since the contraction in the direction of y is stronger than in the direction of x.) We can consider the projection $\tilde{\Lambda}$ of the solenoid Λ to the (t,x) plane. Then $\tilde{\Lambda}$ is the attractor of

$$\tilde{f}(t,x) := \left(2t(\mod 1), \lambda_1 x + \frac{1}{2}\cos(2\pi t)\right). \tag{6.3}$$

That is

$$\tilde{\Lambda} = \bigcap_{n\geq 0} \tilde{f}^n(\tilde{T}), \tag{6.4}$$

where we denote the projection of T to the (t,x) plane by $\tilde{T}$ (see Figure 2).

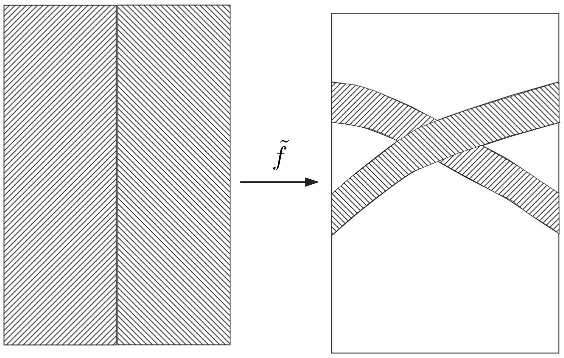

FIGURE 2. $\tilde{f}(\tilde{T})$ the first image of the plane solenoid map on the torus $\tilde{T}$

That is, $\tilde{T} = S^1 \times [-1,1]$. Clearly $\dim_H \tilde{\Lambda} \leq \dim_H \Lambda$. So it is enough to prove that $\dim_H \tilde{\Lambda} \geq 1 + \frac{\log 2}{-\log \lambda_1}$. Let $I_{i_n \ldots i_1}$ be a dyadic interval on S^1. We write $\tilde{S}_{i_1\ldots i_n} := \tilde{f}^n(I_{i_n\ldots i_1})$. Then $\left\{\tilde{S}_{i_1\ldots i_n}\right\}_{i_1\ldots i_n}$ are 2^n strips; each of them intersects every vertical line on the (t,x) plane with $0 \leq t \leq 1$, in an interval of length $2\lambda_1$. **IF** these strips were disjoint (they are not), then we could immediately see that the dimension of $\tilde{\Lambda}$ must be $1 + \frac{\log 2}{-\log \lambda_1}$. Namely, in that case, every vertical line would intersect $\cup_{i_1\ldots i_n} \tilde{S}_{i_1\ldots i_n}$ in a set which is a union of 2^n *disjoint* intervals of length λ_1^n. This would imply that the Hausdorff dimension of the intersection of $\tilde{\Lambda}$ with any vertical line would be $\frac{\log 2}{-\log \lambda_1}$. However, the strips $\tilde{S}_{i_1\ldots i_n}$ intersect each other. Therefore, the intersection of $\tilde{\Lambda}$ with a vertical line consists of 2^n *overlapping* intervals. This causes the difficulty of the problem.

The solution (see [4], [30]) comes from an idea that is now called the transversality condition. Some modifications of this idea for iterated function systems were used later in several other papers, e.g. [23, 37, 38, 35, 36], or see [20] for further applications. To apply this idea we have to check two assumptions:

(i): the contraction is uniformly stronger than the expansion,

(ii): for every $(i_1, i_2, \dots) \in \{1,2\}^{\mathbb{N}}$ we consider the curve: $\Lambda_{i_1,i_2,\dots} := \cap_n S_{i_1\dots i_n}$. If any two of these uncountably many curves intersect each other, then this intersection must be transversal.

It is proved in [32] that if these two conditions hold, then the overlapping between $S_{i_1\dots i_n}$ does not affect the calculation of the dimension of the attractor. Condition (i) follows immediately from the assumption $\lambda_1 < \frac{1}{2}$. Condition (ii) is checked in [32]. In this way we can see that $\dim_H \tilde{\Lambda} \geq 1 + \frac{\log 2}{-\log \lambda_1}$, which completes the proof of Theorem 6.2.

Bothe [4] considered more general solenoids and proved a similar result assuming that the transversality condition (ii) above holds.

6.2. Baker's transformations in space

The result of this section was motivated by Falconer's Theorem [12] (see p. 3.). Observe that if we add one expanding direction to a self-affine set, then we obtain a very special axiom A invariant set. Namely, let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a self-affine IFS on the plane, as in Section 2.1. Suppose that we are given a partition of $[0,1]$ into closed intervals $\{I_k\}_{k=1}^m$ with disjoint interiors and a strictly monotonic, expanding $\mathcal{C}^2$ map ψ whose derivative is separated from zero, such $\psi(I_k) = [0,1]$ for every $k = 1, \dots, m$. We define Λ, the associated axiom A invariant set, as the attractor of the hyperbolic map which is defined for $z \in I_k$ as

$$F(x,y,z) := (f_k(x,y), \psi(z)). \tag{6.5}$$

Then the attractor Λ is the product of the attractor of the self-affine IFS $\mathcal{F} = \{f_1, \dots, f_m\}$ and the unit interval. Let us call $A_\theta := \{(x,y,z) \in \Lambda : z = \theta\}$ the θ section of Λ. In our paper with B. Solomyak [33] we considered more general baker's transformations, where the A_θ sections are different for different θ and none of the component functions are linear. In the rest of this section let $\{I_k\}_{k=1}^m$ and the map ψ be as above. We define $F : [0,1]^3 \to [0,1]^3$ by

$$F(x,y,z) := (\gamma(x,z), \tau(y,z), \psi(z)), \tag{6.6}$$

where the maps are chosen so that: $\gamma, \tau : [0,1] \times [0,1] \to [0,1]$ with

- $0 < \lambda_1 < |\gamma'_x|, |\tau'_y| < \lambda_2 < \frac{1}{2}$;
- $F|_{\Delta_k}$ a smooth 1-1 map, where $\Delta_k := [0,1]^2 \times I_k$.

We denote the non-constant contraction in x and y directions by

$$f_1 := \log |\gamma'_x|, \quad f_2 := \log |\gamma'_y|.$$

To avoid the usage of singular value function we need four pressure formulas.

$$P_{F^{-1}}(s_1 f_1) = 0, \quad P_{F^{-1}}(s_2 f_2) = 0, \tag{6.7}$$

$$P_{F^{-1}}(f_1 + r_1 f_2) = 0, \quad P_{F^{-1}}(r_2 f_1 + f_2) = 0. \tag{6.8}$$

Let $s := \max\{s_1, s_2\}$ and $r := \max\{r_1, r_2\}$. Then

Theorem 6.3 (Simon, Solomyak)**.** *For a typical (in the sense of Theorem 2.1) baker's map F of the form (6.6), the following holds:*

- **(i):** *if $s \le 1$ then $dim_H(\Lambda) = \dim_B(\Lambda) = 1+s$,*
- **(ii):** *if $s > 1$ then $\dim_H(\Lambda) = \dim_B(\Lambda) = 2 + \min\{1, r\}$,*
- **(iii):** *if $r > 1$ then $\mathcal{L}_3(\Lambda) > 0$.*

The restriction in this theorem is that the directions of contraction and expansion are the same at every point. If we consider a more general situation then the problem becomes much more difficult, not only for technical reasons.

One way to try to get more general results is to study the dimension theory of non-conformal repellers. This was done by Falconer [12] and Barreira [1]. Let M be an open subset of $\mathbb{R}^d$ and $f : M \to M$ be a $\mathcal{C}^2$ map. (In Barreira's work $\mathcal{C}^{1+\alpha}$ for $0 < \alpha \le 1$ but here we use Falconer's set-up.) Let J be a mixing repeller for f. That is, J is a compact subset of M such that there exists a bounded neighborhood V of J such that

$$J = \{x \in V : f^n(x) \in V \text{ for all } n \ge 0\}.$$

Furthermore, f is topologically mixing and expanding on J. There is, however, one more very important assumption, which Barreira calls the 1-*bunched property.* Namely,

$$\left\|(D_x f)^{-1}\right\|^2 \cdot \|D_x f\| < 1 \text{ for } x \in J. \tag{6.9}$$

Then Falconer introduces the so-called subadditive pressure. Using that, he gives an upper bound for the Hausdorff dimension of J by an appropriate pressure formulas. Here two problems have been left open: Although in some very special case Falconer proves that the upper bound is sharp, in general, this is still open. On the other hand, it is also a very interesting open problem to give an upper bound without assuming the 1-bunched property (if it is possible).

6.3. Geometric measure-theoretical properties of the SBR measure for solenoids

The SBR measure is the most natural invariant measure (sometimes it is called the natural measure.) If the SBR measure μ exists, then the SBR measure of a set V is the portion of the orbit falling into V for a **LEBESGUE** typical initial point. That is,

$$\mu(V) = \lim_{n\to\infty} \frac{1}{n} \# \left\{0 \le k < n : \; f^k(x) \in V\right\},$$

holds for Lebesgue typical x.

6.3.1. SOLENOID WITH TWO DIFFERENT CONSTANT CONTRACTIVE COEFFICIENTS

First we consider the solenoid map (6.1). We recall that here we assumed that $0 < \lambda_2 < \lambda_1 < \frac{1}{2}$. Above we saw that the Hausdorff dimension of the attractor Λ is $t = 1 + \frac{\log 2}{-\log \lambda_1}$. The SBR measure is supported by Λ. Then it is natural to ask if the SBR measure is equivalent to the appropriate (t-dimensional) Hausdorff measure? This question was answered negatively by M. Rams and the author in [26].

Theorem 6.4 (M. Rams, K. Simon)**.** *Let Λ be the solenoid above. Then*

- **(i):** $\mathcal{H}^t(\Lambda) = 0$,
- **(ii):** $0 < \mathcal{P}^t(\Lambda) < \infty$,
- **(iii):** *The SBR measure is equivalent to the t-dimensional packing measure $\mathcal{P}^t$.*

There have been quite a few examples in the literature of dynamically defined sets having zero Hausdorff but positive and finite packing measure. Our theorem above shows that such a phenomenon can occur even in the case of the most natural axiom A attractor.

6.3.2. Tsuji's result on SBR measures for 'fat' solenoids in the plane. M. Tsuji in [39] considers attractors for maps which are a little bit more general than the ones defined in (6.3). The very important difference is that Tsuji assumes that the solenoid is 'fat', that is, rather expanding than contracting, while in (6.3) the contraction was $\lambda_1 < \frac{1}{2}$ and the expansion was 2. Under this assumption, Tsuji proves that typically the SBR measure is absolutely continuous. More precisely, let $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$,

$$T(x, y) := (\ell x, \lambda y + f(x)), \tag{6.10}$$

where $\ell \geq 2$; and $\frac{1}{\ell} < \lambda < 1$ and f is a $\mathcal{C}^2$ function on S^1. Let

$$\mathcal{D} := \left\{(\lambda, f) : \text{ the SBR is absolutely continuous w.r.t. to } \mathcal{L}eb_2 \text{ with } L^2 \text{ density } \right\}.$$

Theorem 6.5 (Tsuji)**.** *$\mathcal{D}$ contains an open and dense subset of $(\frac{1}{\ell}, 1) \times \mathcal{C}^2(S^1, \mathbb{R})$.*

The method of the proof: Tsuji uses a clever, further developed version of the transversality condition. Originally the transversality condition would require that the contraction ratio be smaller than $\frac{1}{2}$. To avoid such an assumption, Tsuji considers iterates of some perturbation of the map T. Using the special form of the perturbed map, he proves that the iterates of the perturbed map have essentially the same form but have relatively small contraction ratio.

6.3.3. Rams' result on SBR measures for fat solenoids in space. M. Rams used the method of Tsuji and a method of Y. Peres to obtain the result below. For $k \geq 3$ let

$$\Phi : (t, x, z) \to (kt(\mod 1), \lambda_1 x + f(t), \lambda_2 z + g(t)),$$

where $0 < \lambda_2 < \lambda_1 < \frac{1}{2}$; $k\lambda_1 > 1$; $k\lambda_1\lambda_2 < 1$.

The map Φ is a self-map of the torus $T = S^1 \times D$, where D is the unit disk. The functions $f(t)$ and $g(t)$ are $\mathcal{C}^2$ and chosen so that the mapping Φ is an injection. Let Λ be the attractor of Φ; we write μ for the SBR measure of ϕ and ν for the SBR measure of the map defined by the projection of Φ to the (t, x) plane. This solenoid is 'fat' because Φ is more expanding than contracting.

Put

$$t := 2 + \frac{\log k + \log \lambda_1}{-\log \lambda_2}.$$

Then

Theorem 6.6 (M. Rams[27]). *We have* $\dim_P(\Lambda) = t$. *Furthermore, for generic (all but a set of first Baire category)* f, g *the following holds:*

- $\dim_H(\Lambda) = t$
- $\mathcal{P}^s(\Lambda) = \infty$,
- **(i):** *if* $\frac{d\nu}{d\mathcal{L}eb_2}$ *is bounded then* $0 < \mathcal{H}^t(\Lambda) < \infty$ *and* $\mu \sim \mathcal{H}^t|_\Lambda$,
  (ii): *if* $\frac{d\nu}{d\mathcal{L}eb_2}$ *is unbounded then* $\mathcal{H}^t(\Lambda) = 0$.

The case when the contracting coefficients are not constant looks more difficult.

References

[1] L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems* Ergodic Theory and Dyn. Syst. **16** (1996), 871–927.

[2] L. Barreira, *Hyperbolicity and reccurence in dynamical systems: a survey of recent results*, Resenhas IME-USP **5** (2002), 171–230.

[3] T. Bedford and M. Urbański, *The box and Hausdorff dimension of self-affine sets.* Ergodic Theory & Dynam. Sys. **10** (1990), 627–644.

[4] H.G.Bothe, *The Hausdorff-dimension of certain solenoids,* Ergodic Theory and Dyn. Syst. **15** (1995) 449–474.

[5] R. Bowen, *Hausdorff dimension of quasi-circles* Inst Hautes Études Sci. Publ. Math. **50** (1979), 259–273.

[6] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Springer Lecture Notes in Mathematics, **470 (**1979), Springer.

[7] K. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press. (1985).

[8] K. Falconer, *Fractal geometry: mathematical foundations and applications,* John Wiley and Sons, (1990)

[9] K. Falconer, *Techniques in Fractal Geometry*, John Wiley and Sons, (1997).

[10] K. Falconer, *Hausdorff-dimension of some fractals* J. Stat. Phys. **47** (1987), 123–132.

[11] K. Falconer. *The Hausdorff-dimension of self affine fractals.* Math. Proc. Camb. Phil. Soc. **103** (1988), 339-350.

[12] K. Falconer. *Bounded distorsion and dimension for non-conformal repellers*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 315–334.

[13] J.E. Hutchinson, *Fractals and self-similarity.* Indiana University Mathematics Journal. **30** 713–747.

[14] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces,* Cambridge Univ. Press (1995).

[15] P. A. P. Moran, *On plane sets of fractional dimensions.* Proc. London Math. Soc. **51**, (1949). 415–423.

[16] *The Hausdorff dimension of general Sierpiński gasket* Nagoya Math.. J. **96** (1984), 1–9.

[17] H. McCluskey, M. Manning, *Hausdorff-dimension for horseshoes*, Ergodic Theory and Dyn. Syst. **3** (1983), 251–260.

[18] J. Palis, F. Takens, Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations, *Cambridge Univ. Press*, (1993).

[19] Y. Peres, W. Schlag, *Smoothenes of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. **102** (200) 193–251.

[20] Y. Peres, W. Schlag, B. Solomyak, *Sixty years of Bernoulli convolutions,* Fractal Geometry and Stochastics II, Birkhauser (1999) 39–65.

[21] Y. Pesin, *Dimension Theory in Dynamical Systems,* Chicago Lectures in Mathematics (1997).

[22] M. Polliott & O. Jenkinson, *Calculating Hausdorff dimension of Julia sets and Kleinian limit sets.* Amer. J. Math. **124** (2002) 495–545.

[23] M. Pollicott, K. Simon, *The Hausdorff-dimension of λ-expansions with deleted digits*, Trans. Amer. Math. Soc. **347** (1995), 967-983.

[24] M. Pollicott and H. Weiss, *The dimensions of some self affine limit sets in the plane*, J. Stat. Phys. **77** (1994), 841–866.

[25] Przyticki and Urbański, *On the Hausdorf dimension of some fractal sets.* Studia Math. **93** (1989) 155–186.

[26] M. Rams, K. Simon, *Hausdorff and packing measure for solenoids.* Ergodic Theory and Dyn. Syst. **23** (2003) 273–291.

[27] M. Rams, *Absolute continuity for the SBR measure for non-linear fat baker maps.* Nonlinearity (2003) **16** 1649–1655.

[28] J. Schmeling, H. Weiss, *An overview of the dimension theory and dynamical systems.* AMS Symposia in Pure Math. # 69.

[29] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, (1987).

[30] K. Simon, *Hausdorff-dimension for non-invertible maps,* Ergodic Theory and Dyn. Syst. **13** (1993) 199–212.

[31] K. Simon, *Overlapping cylinders: the size of a dynamically defined set.* London Mathematical Society Lecture Notes Series 228, (1996) 259–272.

[32] K. Simon, *The Hausdorff-dimension of the Smale-Williams solenoid with different contraction coefficients,* Proc. Amer. Math. Soc. **125** Number 4, (1997) 1221–1228.

[33] K. Simon and B. Solomyak, *Hausdorff-dimension for Horseshoes in R^3*. Ergodic Theory and Dyn. Syst. **19** (1999) 1343–1363.

[34] K. Simon and B. Solomyak, *On the dimension of self-similar sets.* Fractals, **10** (2002) 59–65.

[35] K. Simon, B. Solomyak, M. Urbanski, *Hausdorff-dimension for limit sets for parabolic IFS with overlaps. Pacific J. of Math.* **201** (2001) 441–477.

[36] K. Simon, B. Solomyak, M. Urbanski, *Invariant meausures for parabolic IFS with overlaps and random continued fractions.* Trans. Amer. Math. Soc. **353** (2001) 5145–5164.

[37] B. Solomyak, *On the random series $\Sigma \pm \lambda^n$ (an Erdős problem),* Annals of Math. **142** (1995), 611–625.

[38] B. Solomyak, *Measures and dimension for some fractal families.* Math. Proc. Cambridge Phil. Soc. **124** (1998), 531–546.

[39] M. Tsuji, *Fat solenoidal attractors*, Nonlinearity **14** (2001) 1011–1027.

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