ON THE DIMENSION OF SELF-SIMILAR SETS

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ABSTRACT. We extend a theorem of Falconer to give a complete result on the dimension and Lebesgue measure of typical self-similar attractors on the line. Further, we present some examples to point out that in higher dimensions one cannot expect the same result in this generality.

1. INTRODUCTION

Beginning from the late 1980's the field of fractal encoding and analysis has emerged. It is based on the work of Hutchinson [10] and Diaconis and Shahshahani [2], and was further developed by Barnsley and co-authors, see [1] and references therein. The scope of the subject in the mid-1990's was reflected in the two volumes edited by Fisher [8, 9]. The idea, roughly, is to approximate a given image by an attractor of an **iterated function system** (IFS). Recall [10] that, given a family of contractions $F = \{S_1, \ldots, S_k\}$ in $\mathbb{R}^d$, there is a unique non-empty compact set $\Lambda(F)$, called the **attractor of the IFS** F, such that

$$\Lambda(F) = \bigcup_{i=1}^{k} S_i(\Lambda(F)).$$

In applications the contractions S_i are usually **affine**, that is,

$$S_i(x) = T_i x + b_i, \quad \text{for} \quad i \leq k, \tag{1.1}$$

where T_i are contractive linear maps on $\mathbb{R}^d$ and $b_i \in \mathbb{R}^d$. The attractor of an IFS of affine contractions is called a **self-affine set**. Let $F(\mathbf{b})$ denote the attractor of the IFS (1.1) where the dependence on translation vectors $\mathbf{b} = (b_1, \ldots, b_k)$ is emphasized.

Both from theoretical and applied viewpoints, it is essential to compute (or estimate) the fractal dimensions of the attractor, such as the Hausdorff dimension $\dim_H$ and the box-counting (Minkowski) dimension $\dim_B$. Falconer obtained the following important result:

Theorem 1.1. [5] *Suppose that* $\|T_i\| < \frac{1}{3}$ *for all* $i \leq k$. *Then* $\dim_H(\Lambda(F(\mathbf{b})) = \dim_B(\Lambda(F(\mathbf{b})) = \min\{\alpha(T_1, \ldots, T_k), d\}$ *for almost all* $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbb{R}^{dk}$ *in the sense of* dk*-dimensional Lebesgue measure.*

2000 *Mathematics Subject Classification.* Primary 28A80.

Key words and phrases. Iterated function systems, attractors, Hausdorff dimension.

Research of Simon was supported in part by the OTKA foundation grant T032022. Research of Solomyak was supported in part by NSF grant #DMS 9800786.

Here $\|T_i\|$ is the operator norm of T_i. See [5] for the general formula for $\alpha(T_1,\ldots,T_k)$, which is given in terms of singular numbers of products of T_i. In the special case when T_i are **similitudes**, that is, $|T_i x| = |\lambda_i|\cdot|x|$ for all $x\in\mathbb{R}^d$, we have $\alpha(T_1,\ldots,T_k) = s(\lambda_1,\ldots,\lambda_k)$ where

$$s = s(\overline{\lambda}) = s(\lambda_1,\ldots,\lambda_k) \quad \text{is such that} \quad \sum_{i=1}^k |\lambda_i|^s = 1. \tag{1.2}$$

In this case the attractor $\Lambda(F(\mathbf{b}))$ is a **self-similar set**, and $s(\overline{\lambda})$ is called its **similarity dimension**.

How sharp is the condition $\|T_i\| < \frac{1}{3}$ in Theorem 1.1? It was noted in [17] that Falconer's proof, with a minor modification, extends to the case $\|T_i\| < \frac{1}{2}$. On the other hand, Edgar [3] showed that the conclusion of Theorem 1.1 may break down if we assume $\|T_i\| < \frac{1}{2}+\varepsilon$ for any $\varepsilon > 0$. In Edgar's example, based on the work of Przytycki and Urbański [15], $d=2$, $k=2$, and $T_1 = T_2$ is not a similitide.

The good news is that no such problems can occur on the line. Let $S_i(x) = \lambda_i x + b_i$ for $x\in\mathbb{R}$. Assume that $0 < |\lambda_1| \le \ldots \le |\lambda_k| < 1$. In the next section we prove:

$$\dim_H(\Lambda(F(\mathbf{b}))) = \min\{s(\overline{\lambda}), 1\} \qquad \text{for a.e. } \ \mathbf{b}\in\mathbb{R}^k. \tag{1.3}$$

Observe that $s(\overline{\lambda}) < 1$ if and only if $\sum_{i=1}^k |\lambda_i| < 1$. In this case (1.3) was proved by Falconer [4, Th.1], who used an elegant projection argument, and it turns out that his argument extends to the case when $|\lambda_k| + |\lambda_{k-1}| < 1$. On the other hand, if $|\lambda_k| + |\lambda_{k-1}| \ge 1$, then the attractor $\Lambda(F(\mathbf{b}))$ contains an interval for all $\mathbf{b}$ such that S_{k-1} and S_k have distinct fixed points, hence its dimension equals 1. We also obtain a topological analog of the result, by showing that the dimension formula holds for a residual set of translation parameters.

Fisher *et al* [9, Ch.5] used (1.3) in the case when $s(\overline{\lambda}) > 1$, although this was not justified at that time. They relied on the discussion of Theorem 1.1 in [7] which does not contain all the details.

Section 3 contains new higher-dimensional examples in which the almost sure dimension formula does not hold. In these examples the contractions are similitudes, and we can take any $d\ge 2$ and $k\ge 3$. The simplest, perhaps, is the "fat Sierpiński gasket," see Example 3.2.

2. Self-similar sets on the line

Theorem 2.1. *Let $F(\mathbf{b}) = \{S_i\}_{i\le k}$, where $S_i(x) = \lambda_i x + b_i$, $\mathbf{b} = (b_1,\ldots,b_k)$, and $0 < |\lambda_1| \le \ldots \le |\lambda_k| < 1$. Let $s(\overline{\lambda}) = s(\lambda_1,\ldots,\lambda_k)$ be the similarity dimension, defined by (1.2).*

(a) *If $|\lambda_k| + |\lambda_{k-1}| \ge 1$ and the fixed points of S_{k-1} and S_k are distinct, then $\Lambda(F(\mathbf{b}))$ contains an interval.*

(b) $\dim_H(\Lambda(F(\mathbf{b}))) = \min\{s(\overline{\lambda}), 1\}$ *for a.e.* $\mathbf{b}\in\mathbb{R}^k$.

(c) *If $|\lambda_k| + |\lambda_{k-1}| < 1$ and $\sum_{i=1}^k |\lambda_i| > 1$ then $\Lambda(F(\mathbf{b}))$ has positive Lebesgue measure for a.e. $\mathbf{b}\in\mathbb{R}^k$.*

The proof of part (a) immediately follows from the following lemma. It is folklore, but we include a proof for completeness.

Lemma 2.2. *Let $f_1(x) := \lambda_1 x + a$ and $f_2(x) := \lambda_2 x + b$, be such that their fixed points are different, that is,*

$$\frac{a}{1-\lambda_1} \neq \frac{b}{1-\lambda_2}. \tag{2.1}$$

Assume that $|\lambda_1| + |\lambda_2| \geq 1$. Then the attractor Λ of the IFS $\{f_1, f_2\}$ is a nonempty interval.

Proof. Introduce a new variable $y = cx + d$. Then a straightforward computation shows that for $c = \left(b - \frac{a(1-\lambda_2)}{1-\lambda_1}\right)^{-1}$ (exists because of (2.1)) and $d = -\frac{ac}{1-\lambda_1}$ we have $f_1(y) = \lambda_1 y$ and $f_2(y) = \lambda_2 y + 1$. Therefore, without loss of generality we may assume that $f_1(x) = \lambda_1 x$ and $f_2(x) = \lambda_2 x + 1$. According to the signs of λ_1 and λ_2 we distinguish four cases. In each case we present a nonempty interval I such that

$$I = f_1(I) \cup f_2(I). \tag{2.2}$$

Case 1. $\lambda_1 > 0, \lambda_2 > 0$. Then $I = [0, \frac{1}{1-\lambda_2}]$.
Case 2. $\lambda_1 > 0, \lambda_2 < 0$. Then $I = [0, 1]$.
Case 3. $\lambda_1 < 0, \lambda_2 > 0$. Then $I = [\frac{\lambda_1}{1-\lambda_2}, \frac{1}{1-\lambda_2}]$.
Case 4. $\lambda_1 < 0, \lambda_2 < 0$. Then $I = [\frac{\lambda_1}{1-\lambda_1\lambda_2}, \frac{1}{1-\lambda_1\lambda_2}]$.

In each of the four cases above it is easy to check that (2.2) holds, which implies that the attractor $\Lambda = I$ and completes the proof of the lemma. □

Now we prove the two remaining parts of Theorem 2.1.

Proof of Theorem 2.1(b),(c). Our proof is in fact a slight modification of Falconer's proof (see [4, Th.1]). Using part (a), without loss of generality we can assume that

$$|\lambda_k| + |\lambda_{k-1}| < 1. \tag{2.3}$$

Let $Q := [-1, 1]^{k-1}$. We claim that there exist vectors $\mathbf{a}_i \in \mathbb{R}^{k-1}$, for $i = 1, \ldots, k$, such that $f_i(\mathbf{y}) = \lambda_i \mathbf{y} + \mathbf{a}_i$ satisfy:

(i) $f_i(Q) \subset Q$ for all $i \leq k$;
(ii) $f_i(Q) \cap f_j(Q) = \emptyset$ for all $i \neq j$;
(iii) The vectors $(\mathbf{a}_i, 1 - \lambda_i) \in \mathbb{R}^{k-1} \times \mathbb{R}$, for $1 \leq i \leq k$, are linearly independent.

Indeed, let $\mathbf{c}_1, \ldots, \mathbf{c}_k$ be k different vertices of the cube $Q = [-1, 1]^{k-1}$; further, let $\mathbf{a}_i := (1 - |\lambda_i|)\mathbf{c}_i$. Geometrically this means that $f_i(Q)$, for $i = 1, \ldots, k$, are in different corners of the cube Q, so that (i) holds. Then (ii) follows from (2.3). If (iii) is violated, we can perturb $\mathbf{a}_i$ slightly, to satisfy (iii) but also keep (i) and (ii). The claim is verified.

Thus the IFS $\{\widetilde{S}_i\}_{i\le k}$ in $\mathbb{R}^k$, defined by $\widetilde{S}_i(x,\mathbf{y}) := (\lambda_i x, \lambda_i \mathbf{y} + \mathbf{a}_i)$, satisfies the open set condition with the open set $(-1,1)^k$ (see [10, 7]). Therefore,

$$\dim_H(\widetilde{\Lambda}) = s,$$

where $\widetilde{\Lambda}$ is the attractor of the IFS $\{\widetilde{S}_i\}_{i\le k}$ and s is the similarity dimension for both IFS $\{\widetilde{S}_i\}_{i\le k}$ and $\{S_i\}_{i\le k}$. By the classical projection theorems [12, 13], the projection of $\widetilde{\Lambda}$ to almost every line in $\mathbb{R}^k$ through the origin, has the Hausdorff dimension equal to $\min\{s,1\}$. Moreover, if $\sum_{i=1}^k |\lambda_i| > 1$, that is $s > 1$, then the projection of $\widetilde{\Lambda}$ to almost every line is a set of positive Lebesgue measure. It was shown in Falconer's proof of [4, Th.1] that there is a linear bijection between the set of projections of $\widetilde{\Lambda}$ to lines in $\mathbb{R}^k$ through the origin and the set of attractors of the IFS $F(\mathbf{b})$. This completes the proof of Theorem 2.1. □

Next we prove the topological version of Theorem 2.1(b). A set is considered large in a complete metric space if it is residual, that is, if its complement is of first Baire category, see [14]. Residual sets are topological analogs of full measure sets. If a set is of full Lebesgue measure in $\mathbb{R}^k$ (large from the measure-theoretic point of view), this does not imply that the set is residual (large from the topological point of view) or vice versa, see [14, Th.1.6].

Denote by $\mathcal{F}$ the set of all IFS $\{S_i\}_{i\le k}$ in $\mathbb{R}$ such that $S_i(x) = \lambda_i x + b_i$. The topology on $\mathcal{F}$ is induced by the natural bijection from $\mathcal{F}$ to $(-1,1)^k \times \mathbb{R}^k$. Denote by $s(F)$ the similarity dimension of $F \in \mathcal{F}$.

First observe that Theorem 2.1(b) implies that $\dim_H(\Lambda(F)) = \min\{s(F),1\}$ holds for a dense subset of $F \in \mathcal{F}$. In fact, much more is true.

Theorem 2.3. *There is a residual subset $\mathcal{F}' \subset \mathcal{F}$ such that for all $F \in \mathcal{F}'$,*

$$\dim_H(\Lambda(F)) = \min\{s(F),1\}. \tag{2.4}$$

Proof. Fix $\varepsilon > 0$. It is enough to prove that (2.4) holds for a residual subset of

$$\mathcal{F}_\varepsilon := \{F \in \mathcal{F} :\ \varepsilon \le |\lambda_1| \le |\lambda_k| \le 1-\varepsilon\}.$$

It follows from a theorem of Jonker and Veerman [11] that the function $\alpha : \mathcal{F}_\varepsilon \to \mathbb{R}$, defined by $\alpha(F) := \dim_H(\Lambda(F))$ is lower semi-continuous. (The lower semi-continuity can also be deduced from [6, Th.4].) Let $\beta(F) := \min\{s(F),1\}$. Obviously, $\beta : \mathcal{F}_\varepsilon \to \mathbb{R}$ is a continuous function, and $\alpha(F) \le \beta(F)$ holds for all F. First we observe that

$$\{F \in \mathcal{F}_\varepsilon :\ \alpha(F) = \beta(F)\} = \{F \in \mathcal{F}_\varepsilon :\ \alpha \text{ is continuous at } F\}. \tag{2.5}$$

Indeed, suppose that $\alpha(F) = \beta(F)$ holds for some $F \in \mathcal{F}_\varepsilon$. We know that β is continuous and $\alpha \le \beta$ is lower semi-continuous, hence α is continuous at F. On the other hand, if α is continuous at F and $\alpha(F) < \beta(F)$, then $\alpha < \beta$ holds on a whole neighborhood of F. However, this is impossible, since by Theorem 2.1(b) $\alpha = \beta$ on a dense subset of $\mathcal{F}_\varepsilon$. This completes the proof of (2.5). Using this and the fact that the set of continuity points of any function is a G_δ set, see [14, Th.7.1], we obtain that the set of $F \in \mathcal{F}$ where (2.4) holds, contains a dense G_δ set; therefore, it is residual in $\mathcal{F}_\varepsilon$. $\square$

3. Examples in $\mathbb{R}^d$, $d \geq 2$.

Let $S_i(x) = \lambda_i x + b_i$ in $\mathbb{R}^d$. As above, we consider the IFS $F(\mathbf{b}) = \{\lambda_i x + b_i\}_{i \leq k}$ and its attractor $\Lambda(F(\mathbf{b}))$. Let $s(\overline{\lambda})$ be the similarity dimension given by (1.2). In this section we assume that $s(\overline{\lambda}) \leq d$. The inequality $\dim_H(\Lambda(F(\mathbf{b})) \leq s(\overline{\lambda})$ is always true; if it is strict we say that there is a "dimension drop." A sufficient condition for dimension drop is that two "cylinders" of the self-similar set coincide. Let us make this precise.

Let $\mathcal{A} = \{1, \dots, k\}$ and consider $\mathcal{A}^n$ as the set of "words" of length n in the "alphabet" $\mathcal{A}$. For $u = u_1 \dots u_n \in \mathcal{A}^n$ let

$$S_u = S_{u_1} \circ \dots \circ S_{u_n} \quad \text{and} \quad \lambda_u = \lambda_{u_1} \cdot \ldots \cdot \lambda_{u_n}.$$

We write $S_u \equiv S_v$ if $S_u(x) = S_v(x)$ for all $x \in \mathbb{R}^d$.

Lemma 3.1. *Suppose that there exists $n \in \mathbb{N}$ and two distinct words $u, v \in \mathcal{A}^n$ such that $S_u \equiv S_v$. If $s = s(\overline{\lambda}) \leq d$, then $\dim_H(\Lambda(F(\mathbf{b})) < s$. In fact,*

$$\dim_H(\Lambda(F(\mathbf{b})) \leq t, \quad \textit{where} \quad \sum_{|w|=n,\, w \neq u} \lambda_w^t = 1.$$

Proof. This is a standard fact in the theory of self-similar IFS. We can choose an efficient covering by cylinder sets restricting ourselves to cylinders whose "address" of length ℓn does not contain the word u. This gives the upper bound of t for the Hausdorff dimension. $\square$

We begin with a simple example which illustrates the idea, and then show how it can be extended.

Example 3.2. Let $S_i(x) = \gamma x + b_i$ for $i = 1, 2, 3$, so that $\lambda_1 = \lambda_2 = \lambda_3 = \gamma$. We assume that $\gamma > \frac{1}{2}$. Then the attractor is a "fat Sierpiński gasket." See Figure 1 for the first stage of the construction.

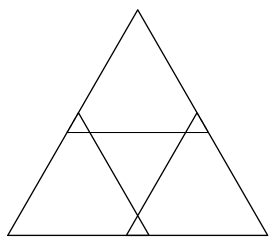

Figure 1. Constructing a "fat Sierpiński gasket."

We took $b_1 = (0,0),\ b_2 = (1,0),\ b_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Observe that the attractor will be the same, up to an affine transformation, for any choice of b_1, b_2, b_3, as long as the fixed points of S_1, S_2, S_3 are distinct. Let $\gamma \approx .543689$ be the positive solution of the equation $1 = x + x^2 + x^3$. Then we have

$$S_{1222} \equiv S_{2111}, \quad S_{1333} \equiv S_{3111}, \quad S_{2333} \equiv S_{3222}.$$

Thus, we can use $(3^4 - 3)^n$ balls of diameter $c\gamma^{4n}$ to cover $\Lambda(F(\mathbf{b}))$, hence

$$\dim_H(\Lambda(F(b)) \le \frac{\log 78}{4\log(1/\gamma)} < \frac{\log 3}{\log(1/\gamma)} = s(\overline{\lambda}).$$

We have $\frac{\log 78}{4\log(1/\gamma)} \approx 1.78736$ and $\frac{\log 3}{\log(1/\gamma)} \approx 1.80284$. The attractor is shown in Figure 2.

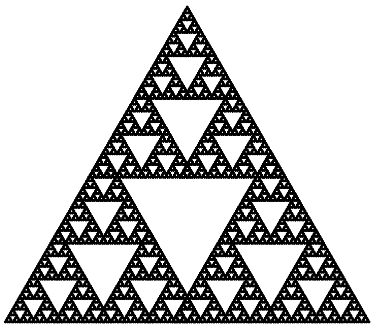

FIGURE 2. A "fat Sierpiński gasket" with dimension drop.

This example can be generalized. We no longer assume that $|\lambda_1| \le \ldots \le |\lambda_k|$. We try to find λ_1, λ_2 and two words $u, v \in \{1,2\}^n$ so that $S_u \equiv S_v$. Then there will be a dimension drop for any choice of $\lambda_i,\ 3 \le i \le k$, and $b_i,\ 1 \le i \le k$, assuming that $s(\overline{\lambda}) \le d$. Observe that for $u \in \mathcal{A}^n$,

$$S_u(x) = b_{u_1} + \lambda_{u_1} b_{u_2} + \lambda_{u_1 u_2} b_{u_3} + \ldots + \lambda_{u_1 \ldots u_{n-1}} b_{u_n} + \lambda_u x. \tag{3.1}$$

We start with the easier case of $\lambda_1 = \lambda_2$ and then consider the general case.

Proposition 3.3. *There is a dense subset $\mathcal{E} \subset (\frac{1}{2}, 1)$ with the following property: For any $\gamma \in \mathcal{E}$, any $k \ge 3$, any $d \ge 2$, any $\lambda_3, \ldots, \lambda_k$, and any $b_1, \ldots, b_k \in \mathbb{R}^d$, if $\lambda_1 = \lambda_2 = \gamma$ and $\sum_{i=1}^k |\lambda_i|^d \le 1$, then*

$$\dim_H(\Lambda(F(\mathbf{b})) < s(\overline{\lambda}). \tag{3.2}$$

Proof. We are going to use Lemma 3.1. Without loss of generality (by making the change of variable $y = x + a$ for appropriate $a \in \mathbb{R}^d$), we can assume that $b_1 = \mathbf{0} \in \mathbb{R}^d$. Suppose that $u, v \in \{1,2\}^n$. Then, since $\lambda_1 = \lambda_2 = \gamma$, it follows from (3.1) that

$$S_u(x) = \sum_{i=1}^{n} b_{u_i}\gamma^{i-1} + \gamma^n x,$$

and there is an analogous formula for S_v. Suppose that $u_1 = 1$ and $v_1 = 2$. Then

$$\begin{aligned} S_v(x) - S_u(x) &= b_2 + \sum_{i=2}^{n}(b_{v_i} - b_{u_i})\gamma^{i-1} \\ &= b_2\left(1 + \sum_{i=2}^{n} c_i\gamma^{i-1}\right), \end{aligned} \tag{3.3}$$

where

$$c_i = \begin{cases} 1, & \text{if } v_i = 2,\ u_i = 1; \\ -1, & \text{if } v_i = 1,\ u_i = 2; \\ 0, & \text{if } v_i = u_i. \end{cases} \tag{3.4}$$

Let

$$\mathcal{E} = \left\{\gamma \in (1/2, 1) :\ \exists\, n \geq 2,\ \exists\, c_2, \ldots, c_n \in \{-1, 0, 1\} \ \text{ with } \ 1 + \sum_{i=1}^{n} c_i\gamma^{i-1} = 0\right\}.$$

If $\gamma \in \mathcal{E}$, then we can easily find u, v so that (3.4) holds, and so by (3.1) and Lemma 3.1, we have (3.2), provided $s(\overline{\lambda}) \leq d$. Note that the latter is equivalent to $\sum_{i=1}^{k} |\lambda_i|^d \leq 1$. Thus, it suffices to check that $\mathcal{E}$ is dense in $(\frac{1}{2}, 1)$. This is well-known, but we provide a short proof for completeness.

Fix $\beta \in (1,2)$. It is enough to show that for any $\varepsilon > 0$, we have $\mathcal{E} \cap (\frac{1}{\beta+\varepsilon}, \frac{1}{\beta-\varepsilon}) \neq \emptyset$. One can always find a representation

$$1 = \sum_{i=1}^{\infty} a_i\beta^{-i} \quad \text{with} \quad a_i \in \{0, 1\},$$

using the "β-expansion" [16], i.e. $a_1 = \lfloor\beta\rfloor = 1$ and $a_\ell = \lfloor\beta^n - \sum_{i=1}^{\ell-1} a_i\beta^{n-i}\rfloor$ for $\ell \geq 2$. Consider $p_N(x) = 1 - \sum_{i=1}^{N} a_i x^i$. It is an easy exercise to check that $p_N(x)$ has a zero x_0 in $(\beta - \varepsilon, \beta + \varepsilon)$ for N sufficiently large. Then clearly, $\gamma = \frac{1}{x_0} \in \mathcal{E}$. □

Next we consider the case of different contraction ratios. Let

$$F(\mathbf{b}) = \{\lambda_i x + b_i\}_{i \leq k}, \quad F'(\mathbf{b}) = \{\lambda'_i x + b_i\}_{i \leq k} \quad \text{in } \ \mathbb{R}^d,$$

where $d \geq 2$ and $k \geq 3$.

Proposition 3.4. *For any $\varepsilon > 0$, for any $\overline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 > 1$ and $\sum_{i=1}^{k} |\lambda_i|^d < 1$, there exists λ'_1, with $|\lambda_1 - \lambda'_1| < \varepsilon$, so that setting $\lambda'_i = \lambda_i,\ i \geq 2$, we have*

$$\dim_H(\Lambda(F'(\mathbf{b})) < s(\overline{\lambda}') \quad \textit{for all} \ \ \mathbf{b} \in \mathbb{R}^{dk}.$$

Proof. As in the proof of Proposition 3.3, we can assume without loss of generality that $b_1 = \mathbf{0}$. We want to find $u, v \in \{1,2\}^n$ such that $S_u \equiv S_v$ for λ_1' close to λ_1. Let

$$u = 12^{n_1}1^{m_1}\ldots 1^{m_{r-1}}2^{n_r},$$

where $r \geq 2$, $n_i \geq 1$ for $i = 1, \ldots, r$ and $m_i \geq 1$ for $i = 1, \ldots, r-1$. Let

$$N = n_1 + \ldots + n_r \quad \text{and} \quad M = m_1 + \ldots + m_{r-1}.$$

By (3.1),

$$S_u(x) = \lambda_u x + b_u, \quad \text{where} \;\; \lambda_u = \lambda_1^{1+M}\lambda_2^N$$

and

$$b_u = \left(\lambda_1 \sum_{i=0}^{n_1-1} \lambda_2^i + \sum_{j=1}^{r-1} \lambda_2^{n_1+\ldots+n_j}\lambda_1^{1+m_1+\ldots+m_j} \sum_{i=0}^{n_{j+1}-1} \lambda_2^i\right) b_2. \tag{3.5}$$

Let

$$v = 21^{M+1}2^{N-1}$$

(note that $N \geq r \geq 2$). Then

$$S_v(x) = \lambda_v x + b_v, \quad \text{where} \;\; \lambda_v = \lambda_u$$

and

$$b_v = \left(1 + \lambda_2\lambda_1^{M+1} \sum_{i=0}^{N-2} \lambda_2^i\right) b_2. \tag{3.6}$$

We have $S_u \equiv S_v$ if and only if $b_u = b_v$. Notice that b_v is very close to b_2 when M is large. Thus, if b_u is also very close to b_2, by a small perturbation of λ_1 we will be able to achieve $b_u = b_v$ as desired. This is the strategy; now let us implement it. Let

$$n_1 = \max\left\{n \geq 1 : \; y_1 := \lambda_1 \sum_{i=0}^{n-1} \lambda_2^i < 1\right\}.$$

Notice that $\frac{\lambda_1}{1-\lambda_2} > 1$ by assumption, so $1 \leq n_1 < \infty$ exists. Fix a small $\delta > 0$ such that $y_1 < 1 - \delta$. Next, let

$$m_1 = \min\{m \geq 0 : \; y_1' := y_1 + \lambda_2^{n_1}\lambda_1^{1+m} < 1\}.$$

Notice that $y_1 + \lambda_1\lambda_2^{n_1} \geq 1$ by the choice of n_1, so $m_1 \geq 1$. We continue by induction. Suppose that for some $\ell \geq 1$ we have found n_i, m_i, y_i, and y_i' for $i \leq \ell$. Let

$$n_{\ell+1} = \max\left\{n \geq 0 : \; y_{\ell+1} := y_\ell + \lambda_2^{n_1+\ldots+n_\ell}\lambda_1^{1+m_1+\ldots+m_\ell} \sum_{i=0}^{n-1} \lambda_2^i < 1\right\}.$$

If $y_{\ell+1} > 1 - \delta$, then we stop and let $r := \ell + 1$. Otherwise, let

$$m_{\ell+1} = \min\left\{m \geq 0 : \; y_{\ell+1}' := y_{\ell+1} + \lambda_2^{n_1+\ldots+n_{\ell+1}}\lambda_1^{1+m_1+\ldots+m_\ell+m} < 1\right\}.$$

Note that $\sum_{i=0}^{\infty}\lambda_2^i = \frac{1}{1-\lambda_2} > \frac{1}{\lambda_1}$ and $y_\ell + \lambda_1\lambda_2^{n_1+\ldots+n_\ell}\lambda_1^{m_1+\ldots+m_\ell-1} \geq 1$ by the choice of m_ℓ, so $1 \leq n_\ell < \infty$ exists. We also have $m_{\ell+1} \geq 1$ by the choice of $n_{\ell+1}$. We continue in this manner until we get

$$y_r := y_{\ell+1} \in (1-\delta, 1).$$

This process has to stop, when $\lambda_2^{n_1+\ldots+n_\ell}\lambda_1^{1+m_1+\ldots+m_\ell}$ gets less than δ, or earlier, and we obtain the integers $n_1, \ldots, n_r$, $m_1, \ldots, m_{r-1}$. Observe that $b_u = y_r b_2$ where b_u is given by (3.5). Notice that as $\delta \to 0$, we have $r \to \infty$, since we used strict inequalities in the definition of n_i and m_i. Thus, $M \geq (r-1) \to \infty$, which implies that b_v approaches b_2 by (3.6). It remains to implement the perturbation argument. It follows from (3.5) that

$$b_u = q_u(\lambda_1)b_2, \quad \text{where} \quad q_u(x) = x\sum_{i=0}^{n_1-1}\lambda_2^i + \sum_{j=1}^{r-1} x^{1+m_1+\ldots+m_j}\lambda_2^{n_1+\ldots+n_j}\sum_{i=0}^{n_{j+1}-1}\lambda_2^i,$$

and (3.6) implies

$$b_v = q_v(\lambda_1)b_2, \quad \text{where} \quad q_v(x) = 1 + x^{M+1}\sum_{i=1}^{N-1}\lambda_2^i.$$

Notice that all coefficients of the polynomials q_u, q_v are non-negative, $q_u(\lambda_1) \in (1-\delta, 1)$, and

$$1 < q_v(\lambda_1) < 1 + \lambda_1^{M+1}\lambda_2(1-\lambda_2)^{-1}.$$

Thus,

$$0 \leq (q_v - q_u)(\lambda_1) \leq \delta + \lambda_1^{M+1}\lambda_2(1-\lambda_2)^{-1} < \varepsilon/2,$$

for $\delta > 0$ sufficiently small. On the other hand,

$$q_v'(x) \leq (M+1)x^M\lambda_2(1-\lambda_2)^{-1} \quad \text{and} \quad 1 \leq q_u'(x).$$

Thus,

$$(q_v - q_u)'(x) \leq (M+1)x^M\lambda_2(1-\lambda_2)^{-1} - 1.$$

We can make sure that M is so large that $(q_v - q_u)'(x) \leq -\frac{1}{2}$ in a neighborhood of λ_1. Then it is clear, by an elementary argument, that $(q_v - q_u)(x) = 0$ has a root in $(\lambda_1, \lambda_1 + \varepsilon)$. Taking λ_1' equal to this root, we satisfy all the conditions of the proposition. □

Example 3.5. Let $\lambda_1 = 0.7$, $\lambda_2 = \lambda_3 = 0.4$. Using the algorithm described in Proposition 3.4 we obtained $n_1 = 2$, $m_1 = 5$, $n_2 = 1$, $m_2 = 6$, $n_3 = 1$. Then we stopped, with $y_3 \approx 0.9997$. Then $u = 12^2 1^5 2 1^6 2$ and $v = 21^{12}2^3$. In order to get the coincidence $S_u \equiv S_v$ we replaced λ_1 by a variable and got an algebraic equation of degree 12. Solving the equation numerically we obtained $\lambda_1' \approx .706263$. An approximation to the attractor is shown in Figure 3.

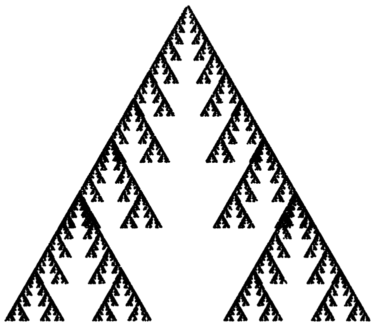

FIGURE 3

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