# Multifractals and the dimension of exceptions

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**Abstract**

We consider a one parameter family of self-similar sets of overlapping construction. We study the exceptional set, that is the set of those parameters for which the correlation dimension is smaller than the similarity dimension. We find some connection between the exceptional set and the multifractal analysis of a measure.

## 1 Introduction

When we compute a certain fractal dimension of a self-similar or self-affine set there is always an easy upper bound for the dimension (see [2]). Although, in many cases it turns out that this most natural upper bound is actually the dimension, it also may occur that the dimension drops compared to its expected value, on a dense set of configurations (see [10, Theorem 2]). There have been lots of efforts trying to understand what causes the drop of dimension but we know very little about the reasons. Obviously, for a self-similar fractal in $\mathbf{R}$ (having similarity dimension smaller than one) if there are two (possibly higher level) cylinders which coincide then the dimension drops. We do not know however, even in this very simple situation, whether there is any other reason for the drop of the dimension. In this paper we

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find a connection between this problem and the multifractal analysis of a measure.

We investigate the simplest possible non-trivial one parameter family of self-similar Iterated Function Systems (IFS) with overlapping cylinders on the real line. For almost all parameters $b$ the correlation dimension $\dim_C$ of the attractor $\Lambda^{(b)}$ is equal to the similarity dimension $s$. The set of those parameters $b$ for which $\dim_C(\Lambda^{(b)}) < s$ is called the exceptional set $E$. Our aim in this paper is to prove that for an exceptional parameter $b$ the correlation dimension of the attractor $\Lambda^{(b)}$ can be expressed as the pointwise dimension of a certain measure $\gamma$ which is a projection of a self-similar measure $\beta$ of the plane. In the last section we discuss the connection between the multifractal analysis of the measure $\gamma$ and the size of the exceptional set $E$.

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# 2 Correlation Dimension

## 2.1 Three equivalent definitions for correlation dimension

Let $\{S_i\}_{i=1}^m$ be a self-similar IFS on $\mathbf{R}^n$. Assume that $0 < \lambda_i < 1$, $i = 1, \ldots, m$ are the ratios of the similarities and $s$ is the similarity dimension, that is $\sum_{i=1}^m \lambda_i^s = 1$. As usual we write $S_{i_0 \ldots i_k} := S_{i_0} \circ \cdots \circ S_{i_k}$. Let $\mu$ be the Bernoulli measure on $\Sigma = \{1, \ldots, m\}^{\mathbf{N}}$ with weights $(\lambda_1^s, \ldots, \lambda_m^s)$. Further, let $I_\alpha(\mu) := \iint_{\Sigma \times \Sigma} |\Pi(\mathbf{i}) - \Pi(\mathbf{j})|^{-\alpha} d\mu(\mathbf{i}) d\mu(\mathbf{j})$, where

$$\Pi(\mathbf{i}) := \lim_{k \to \infty} S_{i_0 \ldots i_k}(0). \tag{1}$$

Following Chin, Hunt and York [1] the correlation dimension of the IFS $\{S_i\}_{i=1}^m$ is defined as

$$\dim_C(\Lambda^{(b)}) := \sup \{\alpha \geq 0 : I_\alpha(\mu) < \infty\}. \tag{2}$$

That is, $\dim_C(\Lambda^{(b)})$ is the correlation dimension of the natural measure $\nu := \mu \circ \Pi^{-1}$. Alternatively, we can define the correlation dimension as follows: Fix a partition $\mathcal{D}_l$ of $\mathbf{R}$ into a grid intervals of length $2l$ for every $l > 0$. Denote $\tau_l := \sum_{Q \in D_l} (\nu(Q))^2$. Peres and Solomyak proved in [7] that the limit

$$D_2(\nu) := \lim_{l \to 0} \frac{\log \tau_l}{\log l}$$

exists. It was proved in [9, Th.18.2] that for $D := \lim_{l \to 0} \frac{\log \int \nu(B_l(x)) d\nu}{\log l}$ we have: $D_2(\nu) = D$, where $B_l(x)$ is the ball of radius $l$ centered at $x$. Further, Sauer and Yorke [12] proved that: $D = \sup \{\alpha \geq 0 : I_\alpha(\mu) < \infty\}$. Thus

(3)

$$\dim_C(\Lambda^{(b)}) = \sup \{\alpha \geq 0 : I_\alpha(\mu) < \infty\} = \lim_{l \to 0} \frac{\log \tau_l}{\log l} = \lim_{l \to 0} \frac{\log \int \nu(B(x,l)) d\nu}{\log l}.$$

## 2.2 Rams' Theorem on correlation dimension

The theorem in this section was proved by M. Rams in his Ph.D. Thesis [11, Wn 6.5] in a much higher generality (in $\mathbf{R}^d$ for self-conformal IFS). For the convenience of the reader we present here Rams' proof of this simplified version of his theorem.

Let $\{S_i\}_{i=1}^m$ be a homogenous self-similar IFS on $\mathbf{R}$, $S_i(x) = \lambda x + t_i$. To keep the notations simple we assume that the smallest interval containing the attractor $\Lambda$ is $[0,1]$. We always write $\omega^n, \tau^n$ for elements of $\sum_n := \{1, \ldots, m\}^n$ and we index them like $\omega^n = (\omega_0 \ldots \omega_{n-1})$. Further, $U_{\omega^n} := S_{\omega^n}(U)$, $U_{\tau^n} := S_{\tau^n}(U)$ if $U \subset \mathbf{R}$. For an $l > 0$ and assuming $U$ is bounded we write $\mathcal{M}_l := \{U_{\omega^n} : \lambda l < |U_{\omega^n}| \leq l\}$, further let

$$A_l(U) := \# \{(\omega^n, \tau^n) \, | U_{\omega^n} \cap U_{\tau^n} \neq \emptyset, \ U_{\omega^n}, \ U_{\tau^n} \in \mathcal{M}_l\}.$$

Observe that $A_l(U) \geq m^n$ if $\lambda l < \lambda^n |U| < l$. Put $s = \frac{\log m}{-\log \lambda}$. We assume that $s \leq 1$.

**Theorem 1 (Rams)** *Let $U$ be a non-empty bounded but not necessarily open interval. For simplicity we suppose that $U \cap [0,1] \neq \emptyset$. Then*

$$\lim_{l \to 0} \frac{\log (A_l(U))}{-\log l} = 2s - \dim_C(\nu).$$

*In particular the limit exists and independent of $U$.*

**Proof.** For an $l > 0$ we call $I_l(x)$ the interval of the $2l$ interval grid $\mathcal{D}_l$ which is centered at $x$. The set of centers of such intervals is called $\mathcal{C}_l$. We assume that $l' > l$. For an $x' \in \mathcal{C}_{l'}$ we define $N_{l',l}(x') := \#\{\omega^n : U_{\omega^n} \cap I_{l'}(x') \neq \emptyset,\ U_{\omega^n} \in \mathcal{M}_l\}$. First we prove that

$$1 < \frac{\sum\limits_{x' \in \mathcal{C}_{l'}} N^2_{l',l}(x')}{A_l(U)} < 3\beta^2 \tag{4}$$

where $\beta := \frac{2(l'+l)}{\lambda l} + 1$. The first inequality is obvious. To see the second one fix an arbitrary $x' \in \mathcal{C}_{l'}$. If $U_{\omega^n} \cap I_{l'}(x') \neq \emptyset$ then $U_{\omega^n} \subset (x' - (l'+l), x' + l' + l)$. Subdivide the interval $(x' - (l'+l), x' + l' + l)$ into subintervals of length $\lambda l$, (for $U_{\omega^n} \in \mathcal{M}_l$, we have $|U_{\omega^n}| > \lambda l$). There are exactly $\beta$ endpoints of such intervals. Thus there is such an interval endpoint which contained in at least $N_{l',l}(x')/\beta$ elements of $\mathcal{M}_l$. These elements of $\mathcal{M}_l$ of course pairwise intersect each other. So there are at least $\frac{N^2_{l',l}(x')}{3\beta^2}$ pairs of elements of $\mathcal{M}_l$ which pairwise intersect each other and which can be associated uniquely with $x'$. This completes the proof of (4).

Next we prove that there exists a $c^* = c^*(U) > 0$ such that

$$(c^*)^{-1} < \frac{\sum\limits_{x \in \mathcal{C}_l} \nu^2(I_l(x))}{l^{2s} A_l(U)} < c^*. \tag{5}$$

For this end, we fix a $c = c(U) < \frac{2}{|U|}$ such that the $c|U|$ neighborhood of $U$, called $B_{c|U|}(U)$ contains $[0,1] \supset \Lambda$. (We assumed that $U \cap [0,1] \neq \emptyset$). Thus $|U| > \frac{1}{2c+1}$. Then for a $U_{\omega^n} \in \mathcal{M}_l$, $\frac{1}{2c+1} < |U| < \frac{2}{c}$ and

$$\frac{1}{m}\frac{c^s}{2^s} l^s < \frac{1}{m^n} < (2c+1)^s l^s.$$

Using that $\mu(\omega^n) = \frac{1}{m^n}$ and $\nu = \mu \circ \Pi^{-1}$ it follows that for an arbitrary $x' \in \mathcal{C}_{l'}$ and for $l' = (1+2c)l$, we have

$$\left(l^s N_{l',l}(x')\right)^2 < \left(\frac{2^s}{c^s \lambda^s}\frac{1}{m^n}\#\{\omega^n | \Lambda_{\omega^n} \subset B_{2l'}(x') \neq \emptyset\}\right)^2 \tag{6}$$

$$< \frac{3 \cdot 2^{2s}}{c^{2s}\lambda^{2s}}\left(\nu^2(I_{l'}(x'_L)) + \nu^2(I_{l'}(x')) + \nu^2(I_{l'}(x'_R))\right) \tag{7}$$

where $x'_L$ and $x'_R$ are the centers of the neighbors of $I_{l'}(x')$ and $s \le 1$ is the similarity dimension.

For an arbitrary $x \in C_l$, let $x'$ be the center of the interval from $\mathcal{D}_{l'}$ which contains $x$ in its interior or as its right endpoint. Further, let $x'_L$ and $x'_R$ are the centers of the two neighbors of $I_{l'}(x')$ in $\mathcal{D}_{l'}$.

$\nu(I_l(x)) \le \frac{1}{m^n} \# \{\omega^n : \Lambda_{\omega^n} \cap I_l(x) \ne \emptyset,\ U_{\omega^n} \in \mathcal{M}_l\}$

$< (2c+1)^s l^s \# \{\omega^n | (I_{l'}(x'_L) \cup I_{l'}(x') \cup I_{l'}(x'_R)) \cap U_{\omega^n} \ne \emptyset, U_{\omega^n} \in \mathcal{M}_l\}$

$\le (2c+1)^s l^s (N_{l',l}(x'_L) + N_{l',l}(x') + N_{l',l}(x'_R))$. Thus using (6) and using twice that $\frac{l'}{l} = 2c+1$

$\sum\limits_{x \in \mathcal{C}_l} \nu^2(I_l(x)) < 3(2c+1)^{1+2s} l^{2s} \sum\limits_{x' \in \mathcal{C}_{l'}} (N^2_{l',l}(x'_L) + N^2_{l',l}(x') + N^2_{l',l}(x'_R))$

$\le 9(2c+1)^{1+2s} l^{2s} \sum\limits_{x' \in \mathcal{C}_{l'}} N^2_{l',l}(x')$

$< 3 \cdot 9(2c+1)^{1+2s} \frac{2^{2s}}{c^{2s}\lambda^{2s}} \sum\limits_{x' \in \mathcal{C}_{l'}} (\nu^2(I_{l'}(x'_L)) + \nu^2(I_{l'}(x')) + \nu^2(I_{l'}(x'_R)))$

$\le 9 \cdot 9(2c+1)^{1+2s} \frac{2^{2s}}{c^{2s}\lambda^{2s}} \sum\limits_{x' \in \mathcal{C}_{l'}} \nu^2(I_{l'}(x')) < 81(2c+1)^{2+2s} \frac{2^{2s}}{c^{2s}\lambda^{2s}} \sum\limits_{x \in \mathcal{C}_l} \nu^2(I_l(x))$.

Thus we obtained that $\left(\frac{c\lambda}{2}\right)^{2s} \frac{1}{9(1+2c)} < \sum\limits_{x \in \mathcal{C}_l} \nu^2(I_l(x)) / l^{2s} \sum\limits_{x' \in \mathcal{C}_{l'}} N^2_{l',l}(x') < 9(2c+1)^{1+2s}$. This and (4) completes the proof of (5). This and (3) immediately implies the statement of Rams' theorem. ■

## 2.3 A Corollary of Rams' Theorem

Since we assumed that the attractor $\Lambda$ spans the interval $J := [0,1]$, thus the left end point of the cylinder interval $J_{\omega_0 \dots \omega_{n-1}} = S_{\omega_0 \dots \omega_{n-1}}(J)$ is $\sum\limits_{k=0}^{n-1} \omega_k \lambda^k$. Therefore if $U := [0,z]$

$$\left| \sum_{k=0}^{n-1} \omega_k \lambda^k - \sum_{k=0}^{n-1} \tau_k \lambda^k \right| \le z\lambda^n \quad \Longleftrightarrow \quad U_{\omega^n} \cap U_{\tau^n} \ne \emptyset. \tag{8}$$

That is $A_{z\lambda^n}(U) = \# \left\{ (\omega^n, \tau^n) : \left| \sum\limits_{k=0}^{n-1} \omega_k \lambda^k - \sum\limits_{k=0}^{n-1} \tau_k \lambda^k \right| \le z\lambda^n \right\}$. So, as a corollary of Rams' Theorem we obtained that

**Lemma 2** *For every* $z > 0$ $\lim_{n\to\infty} \frac{\log \#\left\{(\omega^n,\tau^n):\left|\sum_{k=0}^{n-1}\omega_k\lambda^k-\sum_{k=0}^{n-1}\tau_k\lambda^k\right|\le z\lambda^n\right\}}{-\log\lambda^n} = 2s - \dim_C(\nu)$.

Observe that $A_l(U)$ is constant in $l$ on the interval $l \in [\lambda^n |U|, \lambda^{n-1}|U|)$. We write $A'_n(U)$ for this constant. That is, for every $l > 0$ we choose an $n = n(l)$ such that

$$\frac{\log l}{\log\lambda} - \frac{\log|U|}{\log\lambda} \le n < \frac{\log l}{\log\lambda} - \frac{\log|U|}{\log\lambda} + 1 \tag{9}$$

Put

$$A'_n(U) := A_l(U) = \#\left\{(\omega^n,\tau^n)\left|U_{\omega^n}\cap U_{\tau^n}\ne\emptyset\right.\right\}.$$

and

$$N_n(U) := \#\left\{(\omega^n,\tau^n)\left|\omega_0\ne\tau_0,\ U_{\omega^n}\cap U_{\tau^n}\ne\emptyset, U_{\omega^n}, U_{\tau^n}\in\mathcal{M}_l\right.\right\}.$$

Observe that

$$A'_n(U) = \sum_{k=0}^{n} m^{n-k}N_k(U) + m^n. \tag{10}$$

From the definition it is obvious that

$$\lim_{n\to\infty}\frac{\log A'_n(U)}{n} = \lim_{l\to 0}\frac{\log A_l(U)}{-\log l}\log\frac{1}{\lambda}. \tag{11}$$

Our aim is to prove that in the interesting case (when the exponential growths rate of $A'_n(U)$ is greater than $n$) $A'_n(U)$ growth as fast as $N_n(U)$ at least for $U = [0,z]$ where $1 \le z$. To do this we define $\alpha = \alpha(U), \beta = \beta(U)$ and $\gamma = \gamma(U)$ by

$$\log\alpha := \lim_{n\to\infty}\frac{\log A'_n(U)}{n}, \quad \log\beta := \limsup_{n\to\infty}\frac{\log N_n(U)}{n}$$

and

$$\log\gamma := \liminf_{n\to\infty}\frac{\log N_n(U)}{n}$$

Let $z \ge 1$ be arbitrary. In the rest we assume that $U := [0,z]$.

**Lemma 3** *If $\beta \le m$ then $\alpha = \log m$.*

**Proof.** From the definition $\alpha \ge m$. Let $\varepsilon > 0$ be arbitrary. There exists a $K$ such that for every $k > K$, $N_k(U) \le (m+\varepsilon)^k$. Thus from (10) we obtain that $A'_n(U) \le \sum_{k=0}^{K} m^{n-k} N_k(U) + \sum_{k=K+1}^{n} m^{n-k}(m+\varepsilon)^k + m^n$. That is $A'_n(U) \le const \cdot n \cdot (m+\varepsilon)^n + m^n$. Thus $\alpha \le m$. ∎

**Lemma 4** *If $\beta > m$ then $\beta = \alpha$.*

**Proof.** Obviously $\beta \le \alpha$. On the contrary assume that $\beta < \alpha$. Let $\varepsilon < \alpha - \beta$. Then for all $k$ big enough $N_k(U) < (\beta+\varepsilon)^k$. Then as above we obtain that $\alpha \le \beta + \varepsilon$. Which is a contradiction. ∎

**Lemma 5** *If $\beta > m$ then $\alpha = \beta = \gamma$.*

**Proof.** Since we assumed that $\Lambda \subset [0,1] \subset U$ we have that $S_i(U) \subset U$ $i = 1, \ldots, m$. Therefore $U_{\omega^n} \supset U_{\omega^{n+1}}$ for any $\omega^n \in \Sigma_n$. Thus

$$N_{n+1}(U) < m^2 N_n(U). \tag{12}$$

To get contradiction we assume that $\gamma < \alpha$. Choose $\varepsilon > 0$ so small that the following three requirements are satisfied: $\gamma + \varepsilon < \alpha - \varepsilon$, $m < \alpha - \varepsilon$, and

$$\log \frac{\alpha+\varepsilon}{\alpha-\varepsilon} < \log \frac{\alpha-\varepsilon}{m} \frac{\log \frac{\alpha+\varepsilon}{\gamma+\varepsilon}}{\log m}. \tag{13}$$

If $\varepsilon$ is small enough then (13) holds because for $\varepsilon = 0$ the left hand side is 0 and the right hand side is positive and both sides are continuous in $\varepsilon$. From the definition of $\gamma$ we get that there exists $\{n_k\}_{k=1}^{\infty}$ such that

$$N_{n_k}(U) \le (\gamma+\varepsilon)^{n_k}.$$

Using (12) $k-$times and that for every $j > 0$, $N_j(U) \le const \cdot (\alpha+\varepsilon)^j$ (since $N_j(U) \le A'_j(U)$) it follows from (10) that $A'_{n_i+k}(U) = \sum_{j=0}^{n_i} m^{n_i+k-j} N_j(U) + \sum_{j=n_i+1}^{n_i+k} m^{n_i+k-j} N_j(U) + m^{n_i+k} \le const \cdot n_i m^k (\alpha+\varepsilon)^{n_i} + k(\gamma+\varepsilon)^{n_i} m^{2k} + m^{n_i+k}$ holds for every $k > 0$. By (13), we can choose $\{k_i\}_{i=1}^{\infty}$ such that

$$n_i \frac{\log \frac{\alpha+\varepsilon}{\alpha-\varepsilon}}{\log \frac{\alpha-\varepsilon}{m}} < k_i < n_i \frac{\log \frac{\alpha+\varepsilon}{\gamma+\varepsilon}}{\log m}.$$

For such a $k_i$ we have

$$(14) \quad (\alpha+\varepsilon)^{n_i} m^{k_i} < (\alpha-\varepsilon)^{n_i+k_i} \quad \text{and} \quad (\gamma+\varepsilon)^{n_i} m^{2k_i} < (\alpha+\varepsilon)^{n_i} m^{k_i}.$$

Therefore

$$\lim_{i\to\infty} \frac{\log A'_{n_i+k_i}(U)}{n_i+k_i} \leq \lim_{i\to\infty} \frac{\log(const \cdot (n_i+k_i)(\alpha-\varepsilon)^{n_i+k_i})}{n_i+k_i} = \log(\alpha-\varepsilon).$$

Which contradicts the definition of $\alpha$. ■

Since we know from Rams theorem that $\alpha$ does not depend on $U=[0,z]$, $z \geq 1$ therefore the same is true for $\beta$ and $\gamma$ if $\beta > m$. These lemmas and Rams' theorem imply:

**Proposition 6** *Let $z \geq 1$ be arbitrary. Then for $U=[0,z]$ we obtain that* $\dim_C \Lambda = \begin{cases} s & \textit{if} \quad \limsup_{n\to\infty} \frac{\log N_n(U)}{n} \leq \log m \\ 2s - \lim_{n\to\infty} \frac{\log N_n(U)}{-n\log\lambda} & \textit{otherwise} \end{cases}$

Observe that $U_{\omega^n} \cap U_{\tau^n} \neq \emptyset$ if and only if the left end points $\sum_{k=0}^{n-1} \omega_k \lambda^k$ and $\sum_{k=0}^{n-1} \tau_k \lambda^k$ are closer to each other than the length of $|U_{\omega^n}| = |U_{\tau^n}| = z\lambda^n$ that is $\left|\sum_{k=0}^{n-1} \omega_k\lambda^k - \sum_{k=0}^{n-1} \tau_k\lambda^k\right| \leq z\lambda^n$. Thus

**Corollary 7** *For an arbitrary $z \geq 1$ if*

$$\limsup_{n\to\infty} \frac{\log \# \left\{ (\omega^n, \tau^n) : \quad \omega_0 \neq \tau_0, \left|\sum_{k=0}^{n-1} \omega_k\lambda^k - \sum_{k=0}^{n-1} \tau_k\lambda^k\right| \leq z\lambda^n \right\}}{n} > \log m$$

*then the limit exists and the limit is the same for all $z \geq 1$ further,* $\lim_{n\to\infty} \frac{\log \# \left\{ (\omega^n, \tau^n) : \omega_0 \neq \tau_0, \left|\sum_{k=0}^{n-1} \omega_k\lambda^k - \sum_{k=0}^{n-1} \tau_k\lambda^k\right| \leq z\lambda^n \right\}}{n} = (2s - \dim_C \Lambda)\log\frac{1}{\lambda}.$

This is the corollary of Rams' theorem we are going to use in the proof of our theorem.

# 3 A certain family of fractals with overlaps.

We construct the simplest possible one parameter family of self-similar IFS with overlapping cylinders. This is a simplification of those IFS which appear in M.Keane's so-called '(0,1,3)-problem' (see [4]).

First fix an arbitrary $a \in \left(\frac{1}{4}, \frac{1}{3}\right)$. We define the one parameter family of self-similar IFS $\left\{S_i^{(b)}(x)\right\}_{i \in V}$, where $V = \{0, b, (1-a)\}$, and $S_i^{(b)}(x) := a \cdot x + i$, $i \in V$. The similarity dimension is $s = \frac{\log 3}{-\log a}$. In what follows we always assume that the parameter $b \in (\frac{1-3a}{2}, a)$. ($a$ is not a parameter, $a$ was fixed.) This provides that

$$S_0^{(b)}(\Lambda^{(b)}) \cap S_b^{(b)}(\Lambda^{(b)}) \neq \emptyset \quad \text{and } \Lambda^{(b)} - \Lambda^{(b)} = [-1, 1] \tag{15}$$

holds, where $\Lambda^{(b)} \subset [0,1]$ is the attractor of the IFS $\left\{S_i^{(b)}(x)\right\}_{i \in V}$ and $A - B$ means the arithmetic difference of the sets $A$ and $B$. It follows from the second part of (15) that

$$\Lambda_{i_0 \dots i_m}^{(b)} - \Lambda_{j_0 \dots j_m}^{(b)} = I_{i_0 \dots i_m}^{(b)} - I_{j_0 \dots j_m}^{(b)}, \tag{16}$$

where $\Lambda_{i_0 \dots i_m}^{(b)} := S_{i_0 \dots i_m}^{(b)}(\Lambda^{(b)}) = S_{i_0}^{(b)} \circ \cdots \circ S_{i_m}^{(b)}(\Lambda^{(b)})$ and $I_{i_0 \dots i_m}^{(b)} := S_{i_0 \dots i_m}^{(b)}([0,1])$. Using an argument of Falconer [2] one can easily prove the first part of the next theorem. The proof of the second part is the same as [10, Theorem 2].

**Theorem 8** *1. For Lebesgue-almost all* $b \in (\frac{1-3a}{2}, a)$

$$\dim_H(\Lambda^{(b)}) = \dim_C(\Lambda^{(b)}) = s. \tag{17}$$

*2. There is a dense exceptional subset $E$ of the parameter interval $(\frac{1-3a}{2}, a)$, such that for $b \in E$*

$$\dim_C(\Lambda^{(b)}) < s. \tag{18}$$

As usual, we denote the symbolic space by $\Sigma$. That is $\Sigma = \{(i_0, i_1, i_2, \dots) : i_k \in V,\, k \geq 0\}$. Note that the indices start with zero. Let $\mu$ be the $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ equally weighted Bernoulli measure on $\Sigma$. We denote the product set $\Sigma \times \Sigma$, the product measure $\mu \times \mu$ and the product of the metric by $\Sigma_2$, $\mu_2$ and $\rho_2$ respectively.

## 3.1 The construction of the measure $\gamma$

First we construct a self-similar measure $\beta$ on the plane. Let

$$\mathcal{I} := V - V = \{\pm(1-a), \pm b, \pm(1-a-b), 0\} \tag{19}$$

We define a self-similar IFS $\{R_w\}_{w\in\mathcal{I}}$ on the plane as follows:

$$R_{\pm(1-a)}(x,y) = (ax, ay) + (0, \pm a),\ R_{\pm b}(x,y) = (ax, ay) + (\mp a, 0),$$

$$R_{\pm(1-a-b)}(x,y) = (ax, ay) + (\pm a, \pm a),\ R_0(x,y) = (ax, ay).$$

Write $\Lambda'$ for the attractor of $\{R_w\}_{w\in\mathcal{I}}$. In fact what we need is a translation of $\Lambda'$. Let $\tilde{\Lambda} = \Lambda' + (1,0)$. Let $\tilde{\Sigma} := \{(\tau_1, \tau_{2,\dots}) : \tau_k \in \mathcal{I},\ k \geq 1\}$ (the indices of the symbolic sequences start with 1). We denote the natural projection from $\tilde{\Sigma}$ to $\tilde{\Lambda}$ by

$$\tilde{\Pi}(\tau) := \lim_{k\to\infty} R_{\tau_1,\dots\tau_k}(0,0) + (1,0), \tag{20}$$

for $\tau \in \tilde{\Sigma}$. We call

$\tilde{\Lambda}_{\tau_1\dots\tau_m} := \tilde{\Pi}(\tau_1,\dots,\tau_m) = \left\{x \in \tilde{\Lambda} : x = \tilde{\Pi}(\tau), \text{ where } \tau|_m = \tau_1\dots\tau_m\right\}$ an $m$-cylinder of $\tilde{\Lambda}$, where $\tau_k \in \mathcal{I}$, $k = 1,\dots,m$.

Define the Bernoulli measure $\tilde{\beta}$ on the symbolic space $\tilde{\Sigma}$ as follows: The weight of $0$ is $\frac{1}{3}$ (we get 0 in $V - V$ in three different ways), and the weight of all other elements of $\mathcal{I}$ are $\frac{1}{9}$. In this way for $\bar{\imath}_m = i_1\dots i_m$ and $\bar{\jmath}_m = j_1\dots j_m$, where $i_k, j_k \in V$ for $k = 1,\dots,m$ we get

$$\tilde{\beta}(\bar{\imath}_m - \bar{\jmath}_m) = \frac{1}{9^m}\#\{(\bar{\imath}'_m, \bar{\jmath}'_m) : \bar{\imath}_m - \bar{\jmath}_m = \bar{\imath}'_m - \bar{\jmath}'_m\}. \tag{21}$$

The push down measure of $\tilde{\beta}$ is called $\beta$. Since $a < \frac{1}{3}$, the cylinders of $\tilde{\Lambda}$ are **disjoint**. So $\beta$ is a nice self-similar measure on the plane which does not depend on $b$. Via projections with rays through the origin $\beta$ induces a measure $\gamma$ on the real line with compact support. Namely, consider the cone $C(c,\varepsilon) := \left\{(x,y) : c - \varepsilon < \frac{y}{x} < c + \varepsilon\right\}$. We define the measure $\gamma$ as follows:

(22) $$\gamma(c-\varepsilon, c+\varepsilon) := \beta(C(c,\varepsilon)).$$

The pointwise dimension of $\gamma$ at $x$ is denoted by $d\gamma(x)$. That is $d\gamma(x) := \lim_{r\to 0} \frac{\log \gamma(x-r,x+r)}{\log r}$.

# 4 The main result

**Theorem 9** $\dim_C(\Lambda^{(b)}) = \min\left\{d\gamma(\frac{b}{1-a}), s\right\}$.

Before we prove this Theorem, we need some observations stated in the following Lemmas.

We know from Proposition 6 and Corollary 7 that $\dim_C \Lambda^{(b)} < s = \frac{\log 3}{-\log \lambda}$ if and only if for $\omega_k, \tau_k \in \{0, b, 1-a\}$

(23)
$$\limsup_{m\to\infty} \frac{\log \# \left\{(\omega^{m+1}, \tau^{m+1}) : \ \omega_0 \neq \tau_0, \left|\sum_{k=0}^{m} \omega_k a^k - \sum_{k=0}^{m} \tau_k a^k\right| \leq a^{m+1}\right\}}{m+1} > \log 3.$$

First we observe that for $i_0 \neq j_0$ $I^{(b)}_{i_0\dots i_m} \cap I^{(b)}_{j_0\dots j_m} \neq \emptyset$ holds if and only if either $i_0 = 0$ and $j_0 = b$ or vice versa. (We remind the reader that $I^{(b)}_{i_0\dots i_m} := S^{(b)}_{i_0\dots i_m}([0,1])$ was defined previously.) We write

$$\bar{\imath}_m := (i_1, \dots i_m) = \{\mathbf{j} \in \mathbf{\Sigma} : j_0 = i_1, \dots, j_{m-1} = i_m\}$$

that is the $k-$th coordinate of an element of $\bar{\imath}_m$ is $i_{k+1}$ for $0 \leq k \leq m-1$. Moreover $(i_0, \bar{\imath}_m) := \{\mathbf{j} : j_k = i_k, \ 0 \leq k \leq m\}$.

Put
$$\mathcal{B}_m := \left\{(\bar{\imath}_m, \bar{\jmath}_m) : I^{(b)}_{0\dots i_m} \cap I^{(b)}_{b\dots j_m} \neq \emptyset\right\}.$$

The cardinality of $\mathcal{B}_m$ is called $N^{(b)}_m$ which is just the half of the cardinality which appears in the numerator of (23) since $\sum_{k=1}^{m} i_k a^k$ and $b + \sum_{k=1}^{m} j_k a^k$ are the left end points of the intervals $I^{(b)}_{0\dots i_m}$, $I^{(b)}_{b\dots j_m}$ respectively, and $a^{m+1}$ is the length of these intervals. To estimate $N^{(b)}_m$ we observe that

$$(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m \Longleftrightarrow \Lambda^{(b)}_{(0,\bar{\imath}_m)} \cap \Lambda^{(b)}_{(b,\bar{\jmath}_m)} \neq \emptyset. \tag{24}$$

Since $I^{(b)}_{i_0 i_1 \dots i_m} = \left[\sum_{k=0}^{m} i_k a^k, \sum_{k=0}^{m} i_k a^k + a^{m+1}\right]$ we get $(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m \Longleftrightarrow$

$$|\sum_{k=1}^{m} (i_k - j_k) a^k - b| \leq a^{m+1} \tag{25}$$

holds for every $m \geq 1$.

Fix arbitrary $\bar{\imath}_m = (i_1, \dots, i_m), \bar{\jmath}_m = (j_1, \dots, j_m)$ such that $i_k, j_k \in V$ for $1 \leq k \leq m$. Observe that

$$\sum_{k=1}^{m} (i_k - j_k) a^k - b = b(q_m - 1) + (1-a) p_m \tag{26}$$

where using the notation $T_u(m) = \{1 \leq k \leq m : i_k - j_k = u\}$ $(u \in \mathcal{I})$, $p_m = p_m(\bar{\imath}_m, \bar{\jmath}_m)$ and $q_m = q_m(\bar{\imath}_m, \bar{\jmath}_m)$ are defined as follows:

$$p_m = \sum_{k \in T_{1-a}(m)} a^k - \sum_{k \in T_{-(1-a)}(m)} a^k + \sum_{k \in T_{(1-a-b)}(m)} a^k - \sum_{k \in T_{-(1-a-b)}(m)} a^k \tag{27}$$

and

$$q_m = \sum_{k \in T_b(m)} a^k - \sum_{k \in T_{-b}(m)} a^k + \sum_{k \in T_{-(1-a-b)}(m)} a^k - \sum_{k \in T_{1-a-b}(m)} a^k. \tag{28}$$

Let $c_m := \frac{1}{(1-a)(1-q_m)}$ then

$$1 < c_m < a^{-1}. \tag{29}$$

It follows from (25) and (26) that

$$(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m \Longleftrightarrow \left| \frac{p_m}{1-q_m} - \frac{b}{1-a} \right| < c_m a^{m+1}. \tag{30}$$

Thus we have proved that:

**Lemma 10**

$$\text{if } (\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m \text{ then } \left| \frac{p_m}{1-q_m} - \frac{b}{1-a} \right| < a^m \tag{31}$$

*and*

$$\text{if } \left| \frac{p_m}{1-q_m} - \frac{b}{1-a} \right| < a^{m+1} \text{ then } (\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m. \tag{32}$$

Using (20) for $(\tau_1, \ldots, \tau_m) = (i_1, \ldots, i_m) - (j_1, \ldots, j_m)$ the center of the $m-$th cylinder of $\tilde{\Lambda}_{\bar{\tau}_m} = \tilde{\Lambda}_{\tau_1 \ldots \tau_m}$ is

$$\text{center}\left(\tilde{\Lambda}_{\bar{\tau}_m}\right) = R_{\tau_1, \ldots \tau_m}(0,0) + (1,0) = (1-q_m, p_m). \tag{33}$$

Roughly speaking, $(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m$ means that the slope of the center of $\tilde{\Lambda}_{\bar{\tau}_m}$ is $c_m a^m$-close to $\frac{b}{1-a}$, where $\bar{\tau}_m = \bar{\imath}_m - \bar{\jmath}_m$. Let

$$\mathcal{U}^{(b)}(m) := \left\{ (\bar{\imath}_{m+4}, \bar{\jmath}_{m+4}) \,|\tilde{\Lambda}_{\bar{\tau}_{m+4}} \subset C\left(\frac{b}{1-a}, a^{m+2}\right) \right\}$$

where $\bar{\tau}_{m+4} = \bar{\imath}_{m+4} - \bar{\jmath}_{m+4}$. Denote the cardinality of $\mathcal{U}^{(b)}(m)$ by $u^{(b)}(m)$.

We need two simple geometric observations:

**Lemma 11** *If center* $\left(\tilde{\Lambda}_{\bar{\tau}_m}\right) \in C\left(\frac{b}{1-a}, a^m\right)$ *then* $\tilde{\Lambda}_{\bar{\tau}_m} \subset C\left(\frac{b}{1-a}, a^{m-3}\right)$.

**Lemma 12** *If* $\tilde{\Lambda}_{\bar{\tau}_{m+4}} \cap C\left(\frac{b}{1-a}, a^{m+3}\right) \neq \emptyset$ *then* $\tilde{\Lambda}_{\bar{\tau}_{m+4}} \subset C\left(\frac{b}{1-a}, a^{m+2}\right)$. *That is, if* $\bar{\imath}_{m+4} - \bar{\jmath}_{m+4} = \tau_{m+4}$ *and* $\tilde{\Lambda}_{\bar{\tau}_{m+4}} \cap C\left(\frac{b}{1-a}, a^{m+3}\right) \neq \emptyset$ *then* $(\bar{\imath}_{m+4}, \bar{\jmath}_{m+4}) \in \mathcal{U}^{(b)}(m)$.

Since their proofs are almost identical we should see the proof of Lemma 12 only.

**Proof.** (of lemma 12) Observe, that $\tilde{\Lambda}_{\bar{\tau}_{m+4}}$ lies in the $x > \frac{1}{2}$ half plane and $\tilde{\Lambda}_{\bar{\tau}_{m+4}} \subset C\left(0, \frac{\pi}{4}\right)$, further $\tilde{\Lambda}_{\bar{\tau}_{m+4}}$ is contained in a square parallel to the coordinate axes, of sides $\frac{2a^{m+5}}{1-a}$ called $Q_{\bar{\tau}_{m+4}}$. Assume that *center* $\left(\tilde{\Lambda}_{\bar{\tau}_{m+4}}\right)$ is not above the line $y = \frac{b}{1-a}x$. (The opposite case is similar.) Then we have to prove that the right bottom corner of $Q_{\bar{\tau}_{m+4}}$ is contained in

$C\left(\frac{b}{1-a}, a^{m+2}\right)$. From the geometric position of $Q_{\bar{\tau}_{m+4}}$, this would imply that $Q_{\bar{\tau}_{m+4}} \subset C\left(\frac{b}{1-a}, a^{m+2}\right)$. Let $(x_0, y_0)$ be the left upper corner of $Q_{\bar{\tau}_{m+4}}$. Then it is enough to show that $y_0 - 2 \cdot side(Q_{m+4}) > x_0\left(\frac{b}{1-a} - a^{m+2}\right)$ since $\tilde{\Lambda}_{\bar{\tau}_m} \subset C\left(0, \frac{\pi}{4}\right)$. From the assumption of the Lemma: $y_0 \geq \left(\frac{b}{1-a} - a^{m+3}\right) x_0$. Thus we have to show that $\left(\frac{b}{1-a} - a^{m+3}\right) x_0 - \frac{4a^{m+5}}{1-a} > x_0\left(\frac{b}{1-a} - a^{m+2}\right)$, what is obvious since $x_0 > \frac{1}{2}$ and $0 < a < \frac{1}{3}$. ■

Using (31) and (33), Lemma (11) immediately implies that

**Lemma 13** *For $\bar{\tau}_m = \bar{\imath}_m - \bar{\jmath}_m$ if $(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m$ then $\tilde{\Lambda}_{\bar{\tau}_m} \subset C\left(\frac{b}{1-a}, a^{m-3}\right)$.*

As a consequence of Lemma 13 we can see that

**Lemma 14** $\frac{N_m^{(b)}}{9^m} \leq \beta\left(C\left(\frac{b}{1-a}, a^{m-3}\right)\right)$.

**Proof.** Using (21) first we observe that for $(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m$ $\beta\left(\tilde{\Lambda}_{\bar{\imath}_m - \bar{\jmath}_m}\right) = \frac{1}{9^m} \#\left\{(\bar{\imath}'_m, \bar{\jmath}'_m) : \bar{\imath}'_m - \bar{\jmath}'_m = \bar{\imath}_m - \bar{\jmath}_m\right\} = \frac{1}{9^m} \#\left\{(\bar{\imath}'_m, \bar{\jmath}'_m) \in \mathcal{B}_m : \bar{\imath}'_m - \bar{\jmath}'_m = \bar{\imath}_m - \bar{\jmath}_m\right\}$. This is so, because by (25) if $\bar{\imath}'_m - \bar{\jmath}'_m = \bar{\imath}_m - \bar{\jmath}_m$ and $(\bar{\imath}_m, \bar{\jmath}_m) \in \mathcal{B}_m$ then $(\bar{\imath}'_m, \bar{\jmath}'_m) \in \mathcal{B}_m$ either. Using this and Lemma (13) we obtain the statement of the lemma. ■

As a trivial consequence of Lemma 12 we obtain

**Lemma 15** *$C\left(\frac{b}{1-a}, a^{m+3}\right) \cap \tilde{\Lambda} \subset \bigcup_{(\bar{\imath}_{m+4}, \bar{\jmath}_{m+4}) \in \mathcal{U}^{(b)}(m)} \tilde{\Lambda}_{\bar{\tau}_{m+4}}$, where $\bar{\tau}_{m+4} = \bar{\imath}_{m+4} - \bar{\jmath}_{m+4}$ as usual.*

As a consequence of Lemmas 14,15 we get:

$$\frac{N_m^{(b)}}{9^m} \leq \beta\left(C\left(\frac{b}{1-a}, a^{m-3}\right)\right) = \gamma\left(\frac{b}{1-a} - a^{m-3}, \frac{b}{1-a} + a^{m-3}\right) \tag{34}$$

and

$$\gamma\left(\frac{b}{1-a} - a^{m+3}, \frac{b}{1-a} + a^{m+3}\right) = \beta\left(C\left(\frac{b}{1-a}, a^{m+3}\right)\right) \leq \frac{u^{(b)}(m)}{9^{m+4}} \tag{35}$$

respectively. To get asymptotic for $N_m^{(b)}$ and $u^{(b)}(m)$ we prove some further lemmas:

**Lemma 16** *If*

$$\left|\sum_{k=1}^{m+4}(i_k-j_k)\,a^k-b\right|\leq a^{m+4} \tag{36}$$

*then* $(\bar{\imath}_{m+4},\bar{\jmath}_{m+4})\in\mathcal{U}^{(b)}(m)$.

**Proof.** Using (26) and (29) we see that (36) implies $|(1-a)\,p_{m+4}-b\,(1-q_{m+4})|<a^{m+3}\,(1-a)\,(1-q_{m+4})$. Therefore, $\left|\frac{p_{m+4}}{1-q_{m+4}}-\frac{b}{1-a}\right|<a^{m+3}$. From (33) we obtain that *center* $\left(\tilde{\Lambda}_{\bar{\tau}_{m+4}}\right)\in C\left(\frac{b}{1-a},a^{m+3}\right)$. Using Lemma 12 we obtain the statement of our lemma. ■

Using the Corollary of Rams' Theorem we shall prove that for those $b$ for which

$$\limsup_{m\to\infty}\frac{\log N_m^{(b)}}{m}>\log 3 \tag{37}$$

the exponential growth rates of $u^{(b)}(m)$ and $N_m^{(b)}$ are the same.

**Lemma 17** *If (37) holds then*

$$\lim_{m\to\infty}\frac{\log N_m^{(b)}}{m}=\lim_{m\to\infty}\frac{\log u^{(b)}(m)}{m}=(2s-\ \dim_C\Lambda)\log\frac{1}{a}$$

*in particular the second limit exists.*

**Proof.** If two cylinders $\Lambda_{i_0\dots i_n}^{(b)}$ and $\Lambda_{j_0\dots j_n}^{(b)}$ of $\Lambda^{(b)}$ with different first digits $i_0\neq j_0$, are close to each other then either $i_0=0$ and $j_0=b$ or vice versa. Thus it follows from Lemma 16 that for $\tau_k,\omega_k\in V$

$$u^{(b)}(m)\geq\frac{1}{2}\#\left\{(\omega^n,\tau^n):\omega_0\neq\tau_0,\left|\sum_{k=0}^{n-1}\omega_k a^k-\sum_{k=0}^{n-1}\tau_k a^k\right|\leq za^n\right\} \tag{38}$$

where $n=m+5$, $z=a^{-1}$. Using (25), we obtain that for $\tau_k,\omega_k\in V$ and $n=m+1$, and $z=1$

$$N_m^{(b)}=\frac{1}{2}\#\left\{(\omega^n,\tau^n):\omega_0\neq\tau_0,\left|\sum_{k=0}^{n-1}\omega_k a^k-\sum_{k=0}^{n-1}\tau_k a^k\right|\leq za^n\right\} \tag{39}$$

Finally, if $(\bar{\imath}_{m+4}, \bar{\jmath}_{m+4}) \in \mathcal{U}^{(b)}(m)$ then by definition $\tilde{\Lambda}_{\bar{\tau}_{m+4}} \subset C\left(\frac{b}{1-a}, a^{m+2}\right)$. So, in particular, $center(\tilde{\Lambda}_{\bar{\tau}_{m+4}}) \in C\left(\frac{b}{1-a}, a^{m+2}\right)$. Then by an argument parallel to the one in the proof of Lemma 16, we obtain that $\left|\sum_{k=1}^{m+4}(i_k - j_k)\, a^k - b\right| < a^{m+2}(1-a)(1-q_{m+4}) < a^{m+2}$. Thus, for $\tau_k, \omega_k \in V$ and $n = m+5$, and $z = a^{-3}$

$$u^{(b)}(m) \le \frac{1}{2}\#\left\{(\omega^n, \tau^n) : \omega_0 \ne \tau_0, \left|\sum_{k=0}^{n-1}\omega_k a^k - \sum_{k=0}^{n-1}\tau_k a^k\right| \le z a^n\right\} \tag{40}$$

Now, putting together (38), (39) and (40), Corollary 7 immediately implies the statement of our lemma. ■

Now we are ready to prove our main Theorem.

**Proof of the Main Theorem.** Assume that (37) holds. Then from (34) and (35) we get that

$$\frac{N^{(b)}(m)}{9^m} \le \gamma\left(\frac{b}{1-a} - a^{m-3}, \frac{b}{1-a} + a^{m-3}\right) < \frac{u^{(b)}(m-6)}{9^{m-2}}.$$

From Lemma 17 we get that $\lim_{m\to\infty} \frac{\log \beta\left(C\left(\frac{b}{1-a}, a^m\right)\right)}{m \log a} = -2s + \dim_C \Lambda^{(b)} - 2\frac{\log 3}{\log a} = \dim_C \Lambda^{(b)}$. If (37) does not hold then it follows Proposition 6 that $\dim_C \Lambda^{(b)} = s$. This completes the proof of the main theorem. ■

# 5 Connection with multifractal analysis

It follows from our result above that if $b \in E$, that is, $b$ is exceptional ($\dim_C \Lambda^{(b)} < s$) then the correlation dimension is given by the lower pointwise dimension of $\gamma$. So to understand how big the exceptional set is, we have to understand, how big is the set on which the pointwise dimension of $\gamma$ is smaller than $s$. This so because if the lower pointwise dimension of $\gamma$ at $\frac{b}{1-a}$, $\underline{d}\gamma\left(\frac{b}{1-a}\right) < s$ then (37) holds. Therefore in this case $\underline{d}\gamma\left(\frac{b}{1-a}\right) = d\gamma\left(\frac{b}{1-a}\right) = \dim_C \Lambda^{(b)}$.

For this reason the multifractal analysis of $\gamma$ may be useful. Let $f(\alpha) = \dim_H\{x | d\gamma(x) = \alpha\}$ and $\alpha_{\min} := \inf\{\alpha | f(\alpha) > 0\}$.

Since the measure $\gamma$ is not a self-similar measure, it is not trivial to find its multifractal analysis. However, $\gamma$ is the projection via rays through the origin of a very nice (no overlaps) self-similar measure $\beta$.

In the literature there are estimates on $E$ from above (see e.g. [6] or [10]) but there are no estimates on $E$, even at special cases, from below. If $\alpha_{\min} < s$ then it implies that the exceptional set $E$ has positive Hausdorff dimension, and in this way it would prove that the dimension drops not only in case of having two cylinders which coincide. This could be a partial answer on the problem mentioned in the introduction.

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