ABSOLUTE CONTINUITY FOR RANDOM ITERATED FUNCTION SYSTEMS WITH OVERLAPS

YUVAL PERES, KÁROLY SIMON, AND BORIS SOLOMYAK

ABSTRACT. We consider linear iterated function systems with a random multiplicative error on the real line. Our system is $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed and $Y > 0$ is a random variable with an absolutely continuous distribution. The iterated maps are applied randomly according to a stationary ergodic process, with the sequence of i.i.d. errors $y_1, y_2, \ldots$, distributed as Y, independent of everything else. Let h be the entropy of the process, and let $\chi = \mathbb{E}[\log(\lambda Y)]$ be the Lyapunov exponent. Assuming that $\chi < 0$, we obtain a family of conditional measures $\nu_{\mathbf{y}}$ on the line, parametrized by $\mathbf{y} = (y_1, y_2, \ldots)$, the sequence of errors. Our main result is that if $h > |\chi|$, then $\nu_{\mathbf{y}}$ is absolutely continuous with respect to the Lebesgue measure for a.e. $\mathbf{y}$. We also prove that if $h < |\chi|$, then the measure $\nu_{\mathbf{y}}$ is singular and has dimension $h/|\chi|$ for a.e. $\mathbf{y}$. These results are applied to a randomly perturbed IFS suggested by Y. Sinai, and to a class of random sets considered by R. Arratia, motivated by probabilistic number theory.

1. INTRODUCTION

Let $\{f_1, \ldots, f_m\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities $(p_1, \ldots, p_m)$, with the choice of the map random and independent at each step. We assume that the system is contracting on average, that is, the Lyapunov exponent χ (appropriately defined) is negative. In this paper the maps will be linear, $f_i(x) = \lambda_i x + d_i$, and then $\chi := \sum_{i=1}^m p_i \log \lambda_i$. If $\chi < 0$, then there is a well-defined invariant probability

Date: January 9, 2006.

2000 *Mathematics Subject Classification.* Primary 37C45 Secondary 28A80, 60D05

Key words and phrases. Hausdorff dimension, Contracting on average

Research of Peres was partially supported by NSF grants #DMS-0104073 and #DMS-0244479. Part of this work was done while he was visiting Microsoft Research. Research of Solomyak was partially supported by NSF grants #DMS-0099814 and #DMS-0355187. Research of Simon was partially supported by OTKA Foundation grant #T42496. The collaboration of K.S. and B.S. was supported by NSF-MTA-OTKA grant #77.

measure ν on $\mathbb{R}$ (see [3]). It is of interest to determine whether this measure is singular or absolutely continuous, and if it is singular, to compute its Hausdorff dimension

$$\dim_H(\nu) = \inf\{\dim_H(Y) : \ \nu(\mathbb{R} \setminus Y) = 0\}.$$

Let $h = -\sum_{i=1}^m p_i \log p_i$ be the entropy of the underlying Bernoulli process. It was proved in [15] for non-linear contracting on average IFS (and later extended in [6]) that

$$\dim_H(\nu) \le h/|\chi|.$$

A question arises what happens when the entropy is greater than the absolute value of the Lyapunov exponent. One can expect that, at least "typically," the measure ν is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is "transversality" (see [17, 16, 18]), but it has its limitations. Even in the case of two contractions, there remains a domain of parameters which is not covered by existing results [13, 14]. If the system is not uniformly contracting (but contracting on average), the situation gets even more complicated. We note that there is another direction in the study of IFS with overlaps, which is concerned with concrete, but non-typical systems, often of arithmetic nature, for which there is a dimension drop, see e.g. [11].

In this paper we study a modification of the problem which makes it more tractable, namely we consider linear IFS with a random multiplicative error. Our system is $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed and $Y \ge 0$ is a random variable with an absolutely continuous distribution. The iterated maps are applied randomly according to a stationary ergodic process, with the sequence of i.i.d. errors $y_1, y_2, \ldots$, distributed as Y, independent of everything else. Let h be the entropy of the process, and let $\chi = \mathbb{E}\,[\log(\lambda Y)]$ be the Lyapunov exponent (the symbol $\mathbb{E}$ denotes expectation). Assuming that $\chi < 0$, we obtain a family of conditional measures $\nu_{\mathbf{y}}$ on the line, parametrized by $\mathbf{y} = (y_1, y_2, \ldots)$, the sequence of errors. Our main result is that if $h > |\chi|$, then $\nu_{\mathbf{y}}$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$ for a.e. $\mathbf{y}$. We also prove that if $h < |\chi|$, then the measure $\nu_{\mathbf{y}}$ is singular and has dimension $h/|\chi|$ for a.e. $\mathbf{y}$. Random IFS are quite well understood under separation conditions (see [9, 10]), and the novelty here is that overlaps are allowed.

Two applications that motivated our work, one of them a random perturbation of a problem of Sinai, and the other a problem of Arratia, are described in Section 3.

2. Statement of results

Consider a random variable Y with an absolutely continuous distribution η on $(0,\infty)$, such that for some $C_1 > 0$ we have

$$\frac{d\eta}{dx} \le C_1 x^{-1},\ \forall\, x > 0. \tag{2.1}$$

Let $\mathbb{R}^{\mathbb{N}}$ be the infinite product equipped with the product measure $\eta_\infty := \eta^{\mathbb{N}}$. Let μ be an ergodic σ-invariant measure on $\Sigma = \{1,\dots,m\}^{\mathbb{N}}$, where σ is the left shift. Denote by $h(\mu)$ the entropy of the measure μ. We consider linear IFS with a random multiplicative error $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed. The iterated maps are applied randomly according to the stationary measure μ, with the sequence of i.i.d. errors $y_1, y_2, \dots$, distributed as Y, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu,\eta) := \mathbb{E}\left[\log(\lambda Y)\right] = \mathbb{E}\left[\log Y\right] + \int_\Sigma \log(\lambda_{i_1})\, d\mu(\mathbf{i}).$$

Throughout the paper, we assume that

$$\chi(\mu,\eta) < 0, \tag{2.2}$$

which means that the IFS is contracting on average. The natural projection $\Pi : \Sigma \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$\Pi(\mathbf{i},\mathbf{y}) := d_{i_1} + d_{i_2}\lambda_{i_1} y_1 + \cdots + d_{i_{n+1}}\lambda_{i_1\dots i_n} y_{1\dots n} + \dots, \tag{2.3}$$

where $\mathbf{i} = (i_1, i_2, \dots)$, $\lambda_{i_1\dots i_n} := \lambda_{i_1}\cdots\lambda_{i_n}$, and $y_{1\dots n} = y_1 \cdots y_n$. Note that $\Pi(\mathbf{i},\mathbf{y})$ is a Borel map defined $\mu \times \eta_\infty$ a.e., since $n^{-1}\log(\lambda_{1\dots n} y_{1\dots n}) \to \mathbb{E}\left[\log(\lambda Y)\right] < 0$ a.e., by the Birkhoff Ergodic Theorem. For a fixed $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ we define $\Pi_{\mathbf{y}} : \Sigma \to \mathbb{R}$ and the measure $\nu_{\mathbf{y}}$ on $\mathbb{R}$ as

$$\Pi_{\mathbf{y}}(\mathbf{i}) := \Pi(\mathbf{i},\mathbf{y}) \quad \text{and} \quad \nu_{\mathbf{y}} := (\Pi_{\mathbf{y}})_* \mu. \tag{2.4}$$

We need to impose a condition which guarantees that the maps of the IFS are sufficiently different. We consider two cases which cover the interesting examples that we know of. We assume that either all the digits are distinct:

$$d_i \ne d_j, \quad \text{for all } i \ne j, \tag{2.5}$$

or all the digits d_i are equal to some $d \neq 0$ (which we can assume to be 1, without loss of generality), but the average contraction ratios are all distinct:

$$d_i = 1, \quad \lambda_i \neq \lambda_j, \quad \text{for all } i \neq j. \tag{2.6}$$

Theorem 2.1. *Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \ldots)$. We assume that (2.1), (2.2) hold, and either (2.5) or (2.6) is satisfied.*

(a) *If $h(\mu) > |\chi(\mu,\eta)|$, then*

$$\nu_{\mathbf{y}} \ll \mathcal{L} \ \ \textit{for } \eta_\infty \textit{ a.e. } \mathbf{y}. \tag{2.7}$$

(b) *If $h(\mu) \leq |\chi(\mu,\eta)|$, then*

$$\dim_H(\nu_{\mathbf{y}}) = \frac{h(\mu)}{|\chi(\mu,\eta)|} \ \ \textit{for } \eta_\infty \textit{ a.e. } \mathbf{y}. \tag{2.8}$$

Assuming that μ is a product (Bernoulli) measure, that is, $\mu = (p_1, \ldots, p_m)^{\mathbb{N}}$ and $h(\mu) = |\chi(\mu,\eta)|$, we can show that the measure $\nu_{\mathbf{y}}$ is singular for a.e. $\mathbf{y}$.

Proposition 2.2. *Suppose that $\mu = (p_1, \ldots, p_m)^{\mathbb{N}}$ and $Y > 0$ is any random variable, such that (2.2) holds and $(\mathbf{i}, \mathbf{y}) \mapsto p_{i_1}\lambda_{i_1}^{-1} y_1^{-1}$ is non-constant on $\Sigma \times \mathbb{R}^{\mathbb{N}}$. If $h(\mu) = -\chi(\mu,\eta)$, then $\nu_{\mathbf{y}} \perp \mathcal{L}$ for η_∞ a.e. $\mathbf{y}$.*

Note that in the proposition we do not make any assumptions on the distribution of Y. If Y is any non-constant random variable, then the proposition applies. On the other hand, it includes the case when Y is constant (in other words, this is a usual IFS with no randomness), but λ_j/p_j is not constant. Then of course Y can be eliminated altogether and there is no a.e. $\mathbf{y}$ in the statement. In the special case $Y \equiv 1$ and $\lambda_i < 1$ for all $i \leq m$, our statement is contained in [14, Th.1.1(ii)].

We should emphasize that Proposition 2.2, as well as the upper dimension estimate in (2.8), are rather standard; they are included for completeness, in order to indicate that our results are sharp.

2.1. **Homogeneous case: random measures.** Suppose that $\lambda_i = \lambda$ for all $i \leq m$, so we have $d_i \neq d_j$ for $i \neq j$ by (2.5). Consider the random sums

$$X = \sum_{i=1}^{\infty} J_i Y_1 \cdots Y_{i-1} \lambda^{i-1}, \tag{2.9}$$

where Y_i are i.i.d. with the absolutely continuous distribution η, and J_i take the values in $\{d_1, \ldots, d_m\}$, are independent of $\{Y_j\}$, and are chosen according to an ergodic σ-invariant measure μ. Then Theorem 2.1 applies to $\nu_{\mathbf{y}}$, the conditional distribution of X given $\mathbf{y} = (y_1, y_2, \ldots)$, a realization of the process $\{Y_j\}$. The Lyapunov exponent $\chi(\eta) = \log\lambda + \mathbb{E}[\log Y]$ does not depend on μ.

When μ is Bernoulli, that is, $\mu = (p_1, \ldots, p_m)^{\mathbb{N}}$, there is an alternative method to study $\nu_{\mathbf{y}}$ which goes back to the work of Kahane and Salem [7] and uses Fourier transform. It requires a stronger assumption on the distribution η, namely that

$$\eta \text{ has compact support and } d\eta/dx \text{ is of bounded variation.} \tag{2.10}$$

Theorem 2.3. *Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \ldots)$, defined by (2.4). We assume that (2.10) and (2.5) are satisfied, $\mu = (p_1, \ldots, p_m)^{\mathbb{N}}$ and $\lambda_i = \lambda$ for all $i \le m$. Suppose that $\chi(\eta) = \log\lambda + \mathbb{E}[\log Y] < 0$.*

(a) *If $|\log(\sum_{i=1}^m p_i^2)| > |\chi(\eta)|$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$ for η_∞ a.e. $\mathbf{y}$.*

(b) *If $|\log(\sum_{i=1}^m p_i^2)| > 2|\chi(\eta)|$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a continuous density for η_∞ a.e. $\mathbf{y}$.*

Observe that in the uniform case, when $p_i = \frac{1}{m}$ for $i \le m$, we get the same threshold $|\chi(\eta)| = \log m$ for absolute continuity in Theorem 2.1(a) and for absolute continuity with a density in L^2, in Theorem 2.3(a).

2.2. **Homogeneous case: random sets.** The results on random measures yield information on random sets. Recall that $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$, and let $\Gamma \subset \Sigma$ be a closed σ-invariant subset. For a digit set $\{d_1, \ldots, d_m\}$ and $\mathbf{y} \in (0, \infty)^{\mathbb{N}}$ consider

$$S_\Gamma(\mathbf{y}) = \left\{ \sum_{i=1}^{\infty} d_{a_i} y_{1\ldots(i-1)} : \{a_i\}_1^\infty \in \Gamma \right\}.$$

We let $S(\mathbf{y}) = S_\Sigma(\mathbf{y})$. Denote by $h_{\mathrm{top}}(\Gamma)$ the topological entropy of (Γ, σ). In the next three corollaries we consider a digit set $\{d_1, \ldots, d_m\}$ satisfying (2.5), that is, all the digits are assumed to be distinct. For a random variable η we let $\chi(\eta) := \int \log\eta\, d\eta < 0$.

Corollary 2.4. *Suppose that η satisfies (2.1) and $\chi(\eta) < 0$.*

(a) *If $h_{\mathrm{top}}(\Gamma) > |\chi(\eta)|$, then $\mathcal{L}(S_\Gamma(\mathbf{y})) > 0$ for η_∞ a.e. $\mathbf{y}$.*

(b) *If $h_{\mathrm{top}}(\Gamma) \le |\chi(\eta)|$, then $\dim_H(S_\Gamma(\mathbf{y})) = h_{\mathrm{top}}(\Gamma)/|\chi(\eta)|$ for η_∞ a.e. $\mathbf{y}$.*

Corollary 2.5. *Suppose that η satisfies (2.10) and $\chi(\eta) < 0$. We consider $\Gamma = \Sigma$.*

(a) *If $\log m > 2|\chi(\eta)|$, then $S(\mathbf{y})$ contains an interval for η_∞ a.e. $\mathbf{y}$.*

(b) *If $\log m \le |\chi(\eta)|$, then $\dim_H(S(\mathbf{y})) = \log m/|\chi(\eta)|$ for η_∞ a.e. $\mathbf{y}$.*

Corollary 2.6. *Suppose that Y is any non-constant random variable on $(0,\infty)$ such that $\chi(\eta) < 0$. If $\log m = |\chi(\eta)|$, then $\mathcal{L}(S(\mathbf{y})) = 0$ for η_∞ a.e. $\mathbf{y}$.*

The rest of the paper is organized as follows. In Section 3 we consider the applications that motivated this work. Theorem 2.1(a) is proved in Sections 4 and 5; the latter also contains a key "transversality lemma," which is used in the proof of both Theorem 2.1(a) and the lower estimate in Theorem 2.1(b). Then Theorem 2.1(b) is derived in Section 6. Section 7 is devoted to the proofs of other results, especially Theorem 2.3, following the method of Kahane and Salem. Finally, Section 8 contains some open questions.

3. Applications

3.1. **A random perturbation of a problem of Sinai.** Consider the random series

$$X = 1 + Z_1 + Z_1 Z_2 + \ldots + Z_1 Z_2 \cdots Z_n + \ldots \tag{3.1}$$

where Z_i are i.i.d. random variables, taking the values in $\{1-a, 1+a\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$, for a fixed parameter $a \in (0,1)$. The series converges almost surely, since $\chi = \mathbb{E}[\log Z] = \frac{1}{2}\log(1-a^2) < 0$. Let ν^a denote the distribution of X. Sinai [personal communication], motivated by a statistical analog of the well-known open "$3n+1$ problem," asked for which a the distribution of X is absolutely continuous. Observe that ν^a is supported on $[\frac{1}{a}, \infty)$ for all $a > 0$. Note that ν^a is the invariant measure for the IFS $\{1+(1-a)x, 1+(1+a)x\}$, with probabilities $(\frac{1}{2}, \frac{1}{2})$. If $h < |\chi|$, that is, $\log 2 < -\frac{1}{2}\log(1-a^2)$, or $a > \sqrt{3}/2$, then the measure is singular. It is natural to predict that $\nu^a \ll \mathcal{L}$ for a.e. $a \in (0, \sqrt{3}/2)$. While this question remains open, here we resolve a "randomly perturbed" version. The following corollary is a special case of Theorem 2.1 and Proposition 2.2.

Corollary 3.1. *Consider the random sum (3.1), with $Z_i = \lambda_i Y$, where $\lambda_i \in \{1-a, 1+a\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$ and Y has an absolutely continuous distribution on $(1-\varepsilon_1, 1+\varepsilon_2)$ for small ε_1 and ε_2, with a bounded density, such that $\mathbb{E}[\log Y] = 0$. The "errors" y_i at each step are i.i.d. with the distribution of Y, and are independent of everything else. Let $\nu^a_{\mathbf{y}}$ be the conditional distribution of Z, given a sequence of errors $\mathbf{y} = (y_1, y_2, \ldots)$.*

(a) *If $a \in (0, \sqrt{3}/2)$, then $\nu^a_{\mathbf{y}} \ll \mathcal{L}$ for a.e. $\mathbf{y}$;*

(b) *if $a \geq \sqrt{3}/2$, then $\nu^a_{\mathbf{y}} \perp \mathcal{L}$ and $\dim_H(\nu^a_{\mathbf{y}}) = 2\log 2 / \log\frac{1}{1-a^2}$ for a.e. $\mathbf{y}$.*

3.2. **A problem of Arratia.** Our second example is probabilistic in its origin (rather than a random perturbation of a deterministic one, as above). It comes from a question of Arratia (see Section 22 in [2]), who considered the following distributions, motivated by some questions in probabilistic number theory. Let $Y = U^{1/\theta}$, where U has the uniform distribution on $[0,1]$, and consider $X_i = Y_1 \cdots Y_i$, where Y_i are i.i.d. with the distribution of Y. The process $\{X_i\}$ is known as the scale-invariant Poisson process with intensity $\theta x^{-1}\,dx$. Consider the random sum $Z = \sum_{i\ge 1} J_i X_i$ where $J_1, J_2, \dots$ are "fair coins" with values in $\{0,1\}$, independent of each other and of everything else. One is interested in the conditional distribution $\nu_{\mathbf{y}}^{\theta}$ of Z given the process $\{Y_i\}$, and in its support, $S_{\mathbf{y}}^{\theta} = \{\sum_{i=1}^{\infty} a_i X_i :\ a_i \in \{0,1\}\}$. Observe that this fits into our class of IFS $\{d_i + \lambda_i Y x\}_{i=1}^m$, by taking $m = 2$, $d_1 = 0, d_2 = 1, \lambda_1 = \lambda_2 = 1$. The distribution of $U^{1/\theta}$ has the density $\theta x^{\theta-1}\mathbf{1}_{[0,1]}$, so we have $\chi = \mathbb{E}[\log \lambda Y] = \mathbb{E}[\log Y] = -\theta^{-1}$. The entropy of the "fair coins" process is $h = \log 2$, so $h/|\chi| = \theta \log 2$.

Corollary 3.2. *Let $Z = \sum_{i\ge 1} J_i X_i$ as above. Let $\nu_{\mathbf{y}}^{\theta}$ be the conditional distribution of Z given the process $\{Y_i\}$, and let $S_{\mathbf{y}}^{\theta}$ be the support of $\nu_{\mathbf{y}}^{\theta}$.*

(a) *If $\theta > 1/\log 2$, then $\nu_{\mathbf{y}}^{\theta} \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$, hence $\mathcal{L}(S_{\mathbf{y}}^{\theta}) > 0$, for a.e. $\mathbf{y}$.*

(b) *If $\theta > 2/\log 2$, then $\nu_{\mathbf{y}}^{\theta} \ll \mathcal{L}$ with a continuous density, hence $S_{\mathbf{y}}^{\theta}$ contains an interval, for a.e. $\mathbf{y}$.*

(c) *If $\theta \in (0, 1/\log 2]$, then $\mathcal{L}(S_{\mathbf{y}}^{\theta}) = 0$, and $\dim_H(S_{\mathbf{y}}^{\theta}) = \dim_H(\nu_{\mathbf{y}}^{\theta}) = \theta \log 2$ for a.e. $\mathbf{y}$.*

Proof. Parts (a) and (b) follow from Theorem 2.3. Part (c) follows from Theorem 2.1(b) and Corollary 2.6. □

An intriguing open problem is whether $S_{\mathbf{y}}^{\theta}$ contains intervals for a.e. $\mathbf{y}$ when $\theta \in (\frac{1}{\log 2}, \frac{2}{\log 2})$. The proof of Corollary 3.2 is easily adapted to the Poisson-Dirichlet distributions where the variables $X_i = Y_1 \cdots Y_i$ are replaced by $\widetilde{X}_i = Y_1 \cdots Y_{i-1}(1 - Y_i)$, see the equations (6.2) and (8.2) in [2].

Our results yield many variants of Corollary 3.2. For example, let $\widetilde{S}_{\mathbf{y}}^{\theta} = \{\sum_{i=1}^{\infty} a_i X_i :\ a_i \in \{0,1\}, a_i a_{i+1} = 0, i \ge 1\}$; in other words, we consider only the sums corresponding to the "Fibonacci" shift of finite type. Let $\tau = (1+\sqrt{5})/2$.

Corollary 3.3. **(a)** *If $\theta > 1/\log \tau$, then $\mathcal{L}(\widetilde{S}_{\mathbf{y}}^{\theta}) > 0$, for a.e. $\mathbf{y}$.*

(b) *If $\theta \in (0, 1/\log \tau)$, then $\mathcal{L}(\widetilde{S}_{\mathbf{y}}^{\theta}) = 0$ and $\dim_H(\widetilde{S}_{\mathbf{y}}^{\theta}) = \theta \log \tau$ for a.e. $\mathbf{y}$.*

Proof. This follows from Corollary 2.4, since $\Gamma = \{(a_i)_1^\infty \in \{0,1\}^{\mathbb{N}} : \ a_i a_{i+1} = 0,\ i \geq 1\}$ has topological entropy $\log \frac{1+\sqrt{5}}{2}$. □

4. Preliminaries and the Proof of Theorem 2.1(a)

Notation. For $\omega \in \{1,\dots,m\}^n$ we denote by $[\omega]$ the cylinder set of $\mathbf{i} \in \Sigma$ which start with ω. For $\mathbf{i} \in \Sigma$ let $[\mathbf{i}, n] = [i_1 \dots i_n]$. For $\mathbf{i}, \mathbf{j} \in \Sigma$ we denote by $\mathbf{i} \wedge \mathbf{j}$ their common initial segment.

By adding the constant $\mathbb{E}[\log Y]$ to $\log\lambda$ and subtracting it from $\log Y$, we can assume without loss of generality that $\mathbb{E}[\log Y] = 0$, so that $\chi(\mu,\eta) = \chi(\mu) = \mathbb{E}[\log\lambda]$. In order to prove Theorem 2.1, we need to make a certain "truncation" both in $\mathbb{R}^{\mathbb{N}}$ and in Σ. By the Law of Large Numbers,

$$n^{-1}\log(y_{1\dots n}) \to 0 \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y}. \tag{4.1}$$

By Egorov's Theorem, for any $\varepsilon > 0$ there exists $F_\varepsilon \subset \mathbb{R}^{\mathbb{N}}$, with $\eta_\infty(F_\varepsilon) > 1-\varepsilon$, such that $(y_{1\dots n})^{1/n} \to 1$ uniformly on F_ε.

Next we do the truncation in Σ. By the Shannon-McMillan-Breiman Theorem,

$$n^{-1}\log(\mu[\mathbf{i}, n]) \to -h(\mu) \quad \text{for } \mu \text{ a.e. } \mathbf{i} \in \Sigma. \tag{4.2}$$

By the Birkhoff Ergodic Theorem,

$$n^{-1}\log(\lambda_{i_1\dots i_n}) \to \chi(\mu) \quad \text{for } \mu \text{ a.e. } \mathbf{i} \in \Sigma. \tag{4.3}$$

Applying Egorov's Theorem, we can find $G_\varepsilon \subset \Sigma$, with $\mu(G_\varepsilon) > 1-\varepsilon$, such that the convergence in (4.2) and (4.3) is uniform on G_ε.

Define $\mu_\varepsilon = \mu|_{G_\varepsilon}$ and let $\nu_{\mathbf{y}}^\varepsilon = (\Pi_{\mathbf{y}})_*\mu_\varepsilon$. If $\nu_{\mathbf{y}}^\varepsilon \ll \mathcal{L}$ for all $\varepsilon > 0$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$, and $\dim_H(\nu_{\mathbf{y}}) = \sup_{\varepsilon>0}\dim_H(\nu_{\mathbf{y}}^\varepsilon)$. Since we can obviously assume that $F_\varepsilon \subset F_{\varepsilon'}$ for $\varepsilon' < \varepsilon$, (2.7) will follow if we prove that

$$\forall\,\varepsilon > 0,\ \nu_{\mathbf{y}}^\varepsilon \ll \mathcal{L} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon. \tag{4.4}$$

Similarly, (2.8) will follow if we prove that

$$\forall\,\varepsilon > 0,\ \dim_H(\nu_{\mathbf{y}}^\varepsilon) = \frac{h(\mu)}{|\chi(\mu)|} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon. \tag{4.5}$$

Thus, both parts of Theorem 2.1 will be established if the statements are proved for the measures $\nu_{\mathbf{y}}^\varepsilon$ for all $\varepsilon > 0$ instead of $\nu_{\mathbf{y}}$.

Beginning of the Proof of Theorem 2.1(a). Fix $\varepsilon \in (0,1)$; our goal is to prove (4.4) assuming that $-h(\mu) < \chi(\mu) < 0$. We can fix positive $\theta < \rho$ such that

$$-h(\mu) < \log\theta < \log\rho < \chi(\mu) < 0. \tag{4.6}$$

Next fix $\delta > 0$ such that

$$(1+\delta)\theta < \rho. \tag{4.7}$$

Using the uniform convergence on F_ε and G_ε, we can find $N = N(\varepsilon, \delta, \theta, \rho) \in \mathbb{N}$ such that, in view of (4.1),

$$(1+\delta)^{-n} \le y_1 \cdots y_n \le (1+\delta)^n \quad \text{for all } \ n \ge N,\ \mathbf{y} \in F_\varepsilon, \tag{4.8}$$

and, in view of (4.2), (4.3) and (4.6),

$$\mu[\mathbf{i}, n] < \theta^n < \rho^n < \lambda_{i_1 \dots i_n}, \quad \text{for all } \ n \ge N,\ \mathbf{i} \in G_\varepsilon. \tag{4.9}$$

We can decompose the measure into the sum of measures on cylinders:

$$\nu_{\mathbf{y}}^{\varepsilon} = \sum_{|\omega|=N} \nu_{\mathbf{y},\omega}^{\varepsilon}, \quad \text{where } \ \nu_{\mathbf{y},\omega}^{\varepsilon} := (\mu|_{[\omega]\cap G_\varepsilon}) \circ \Pi_{\mathbf{y}}^{-1}. \tag{4.10}$$

Thus it is enough to show that

$$\nu_{\mathbf{y},\omega}^{\varepsilon} \ll \mathcal{L} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon,\ \forall\, \omega,\ |\omega| = N. \tag{4.11}$$

Let $\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) := |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})|$ and

$$g_r(\mathbf{i}, \mathbf{j}) := \eta_\infty\{\mathbf{y} \in F_\varepsilon :\ \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r\}. \tag{4.12}$$

Let $P := G_\varepsilon \times G_\varepsilon$ and $\mu_2 := \mu_\varepsilon \times \mu_\varepsilon$. Denote $P^N := \{(\mathbf{i}, \mathbf{j}) \in P :\ |\mathbf{i} \wedge \mathbf{j}| \ge N\}$.

Proposition 4.1. *There exists $C = C(\varepsilon) > 0$, such that for all $r > 0$,*

$$A(r) := \iint_{P^N} g_r(\mathbf{i}, \mathbf{j}) d\mu_2(\mathbf{i}, \mathbf{j}) \le Cr. \tag{4.13}$$

We will prove the proposition in the next section. Before that, using the proposition we prove Theorem 2.1(a).

Conclusion of the Proof of Theorem 2.1(a). In order to prove (4.11), it is enough to verify that

$$\mathcal{I} := \sum_{|\omega|=N} \int_{\mathbf{F}_\varepsilon} \int_{\mathbb{R}} \underline{D}(\nu_{\mathbf{y},\omega}^{\varepsilon}, x) d\nu_{\mathbf{y},\omega}^{\varepsilon}(x) d\eta_\infty(\mathbf{y}) < \infty,$$

where

$$\underline{D}(\nu, x) := \liminf_{r \to 0} \frac{\nu\left([x-r, x+r]\right)}{2r},$$

is the lower derivative of a measure ν, see [12, 2.12]. Observe that

$$\sum_{|\omega|=N} \int_{\mathbb{R}} \nu^{\varepsilon}_{\mathbf{y},\omega}\left([x-r,x+r]\right) d\nu^{\varepsilon}_{\mathbf{y},\omega}(x) = \iint_{P^N} \mathbf{1}_{\{(\mathbf{i},\mathbf{j}):|\Pi_{\mathbf{y}}(\mathbf{i})-\Pi_{\mathbf{y}}(\mathbf{j})|\le r\}} d\mu_2(\mathbf{i},\mathbf{j})$$

by the definition of $\nu^{\varepsilon}_{\mathbf{y},\omega}$. Using this with Fatou's Lemma, and exchanging the order of integration, we obtain that

$$\mathcal{I} \le \liminf_{r\to 0} (2r)^{-1} A(r), \tag{4.14}$$

where $A(r)$ was defined in (4.13). Thus $\mathcal{I} < \infty$ follows immediately from Proposition 4.1. □

5. Transversality lemma and the proof of Proposition 4.1

We begin with a technical lemma, which is a key for the proof of both parts of Theorem 2.1. We are assuming all the conditions of Theorem 2.1, in particular, that either (2.5) or (2.6) holds.

By the definition of φ and Π we have

$$\varphi(\mathbf{y},\mathbf{i},\mathbf{j}) = |d_{i_1} - d_{j_1} + y_1\Phi(\mathbf{y},\mathbf{i},\mathbf{j})|, \tag{5.1}$$

where

$$\Phi(\mathbf{y},\mathbf{i},\mathbf{j}) = \lambda_{i_1}d_{i_2} - \lambda_{j_1}d_{j_2} + \sum_{\ell=2}^{\infty} y_{2\dots\ell}(\lambda_{i_1\dots i_\ell}d_{i_{\ell+1}} - \lambda_{j_1\dots j_\ell}d_{j_{\ell+1}}). \tag{5.2}$$

Note that $\Phi(\mathbf{y},\mathbf{i},\mathbf{j})$ does not depend on y_1. If $|\mathbf{i}\wedge\mathbf{j}| = k \ge 1$ then

$$\varphi(\mathbf{y},\mathbf{i},\mathbf{j}) = \lambda_{i_1\dots i_k}\cdot y_{1\dots k}\cdot \varphi\left(\sigma^k\mathbf{y},\sigma^k\mathbf{i},\sigma^k\mathbf{j}\right). \tag{5.3}$$

Lemma 5.1. *Let $\delta > 0$, $\rho \in (0,1)$, and $N \in \mathbb{N}$. Consider*

$$F = \{\mathbf{y}\in\mathbb{R}^{\mathbb{N}} :\ y_{1\dots n} \ge (1+\delta)^{-n},\ \forall n \ge N\},$$

$$G = \{\mathbf{i}\in\Sigma :\ \lambda_{i_1\dots i_n} \ge \rho^n,\ \forall n\ge N\}.$$

There exists $C_2 > 0$ such that for all $k \ge N$, for all $\mathbf{i},\mathbf{j}\in G$, with $|\mathbf{i}\wedge\mathbf{j}| = k$,

$$\eta_\infty\{\mathbf{y}\in F :\ \varphi(\mathbf{y},\mathbf{i},\mathbf{j}) < r\} \le C_2(1+\delta)^k\rho^{-k}r \quad \textit{for all } r > 0. \tag{5.4}$$

Proof. First suppose that the condition (2.5) holds. Then $b := \min_{\ell\ne s}|d_\ell - d_s| > 0$. Since $\mathbf{i}\in G$ and $k\ge N$, we have for $\mathbf{y}\in F$ by (5.1) and (5.3):

$$\varphi(\mathbf{y},\mathbf{i},\mathbf{j}) < r \ \Rightarrow\ |d_{i_{k+1}} - d_{j_{k+1}} + y_{k+1}\Phi| < (1+\delta)^k\rho^{-k}r, \tag{5.5}$$

where $\Phi = \Phi(\sigma^k \mathbf{y}, \sigma^k \mathbf{i}, \sigma^k \mathbf{j})$ does not depend on y_{k+1}. Denote $\Delta_{k+1} := d_{i_{k+1}} - d_{j_{k+1}}$; we have $|\Delta_{k+1}| \geq b$ since $i_{k+1} \neq j_{k+1}$. We can assume that $\Delta_{k+1} < 0$; the case $\Delta_{k+1} > 0$ is similarly proved. Since the left-hand side of (5.4) is always bounded above by one, (5.4) holds for $r \geq (1+\delta)^{-k}\rho^k b/2$ with the constant $C_2 = 2/b$. If $r < (1+\delta)^{-k}\rho^k b/2$ then

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \;\Rightarrow\; |\Delta_{k+1} + y_{k+1}\Phi| < (1+\delta)^k \rho^{-k} r < b/2, \tag{5.6}$$

and this implies $\Phi > 0$, in view of y_{k+1} being positive and the fact that $\Delta_{k+1} \leq -b$. Moreover, the right-hand side of (5.6) implies

$$y_{k+1} \in B, \quad \text{where } \; B := \left[\frac{-\Delta_{k+1} - (1+\delta)^k \rho^{-k} r}{\Phi}, \; \frac{-\Delta_{k+1} + (1+\delta)^k \rho^{-k} r}{\Phi} \right].$$

Note that B depends on $y_{k+2}, y_{k+3}, \ldots$ but not on y_{k+1}. We have $(1+\delta)^k \rho^{-k} r < b/2 \leq -\Delta_{k+1}/2$, so

$$B \subset [-\Delta_{k+1}/(2\Phi), \infty) \subset [b/(2\Phi), \infty).$$

By (2.1), we obtain that for any $y_{k+2}, y_{k+3}, \ldots$,

$$\eta\{y_{k+1} \in B\} \leq C_1 (2\Phi/b) \mathcal{L}(B) = C_1 (4/b)(1+\delta)^k \rho^{-k} r.$$

This implies the desired inequality (5.4) by Fubini Theorem, since y_{k+1} is independent of $y_{k+2}, y_{k+3}, \ldots$

Now suppose that the condition (2.6) holds. Then by (5.1) and (5.2),

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) = \lambda_{i_1 \ldots i_k} \cdot y_{1\ldots k,(k+1)} \cdot |\lambda_{i_{k+1}} - \lambda_{j_{k+1}} + y_{k+2}\Psi|,$$

where

$$\Psi = \lambda_{i_{k+1} i_{k+2}} - \lambda_{j_{k+1} j_{k+2}} + \sum_{\ell=3}^{\infty} y_{(k+3)\ldots(k+\ell)} (\lambda_{i_{k+1} \ldots i_{k+\ell}} - \lambda_{j_{k+1} \ldots j_{k+\ell}})$$

does not depend on y_{k+2}. Since $\mathbf{i} \in G$ and $k \geq N$, we have for $\mathbf{y} \in F$:

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \;\Rightarrow\; |\lambda_{i_{k+1}} - \lambda_{j_{k+1}} + y_{k+2}\Psi| < (1+\delta)^{k+1} \rho^{-k} r.$$

Here $|\lambda_{i_{k+1}} - \lambda_{j_{k+1}}| \geq b' := \min_{\ell \neq s} |\lambda_\ell - \lambda_s| > 0$ by (2.6), and we argue similarly to the first case to obtain (5.4), with $C_2 = (1+\delta) \max\left\{ \frac{2}{b'}, \frac{4C_1}{b'} \right\}$. □

Proof of Proposition 4.1. Let

$$P_\omega := \{(\mathbf{i}, \mathbf{j}) \in P : \mathbf{i} \wedge \mathbf{j} = \omega\},$$

where $\omega = (\omega_1, \ldots, \omega_k) \in \{1, \ldots, m\}^k$ for some k. Denote

$$A_\omega(r) := \iint\limits_{P_\omega} g_r(\mathbf{i}, \mathbf{j}) d\mu_2(\mathbf{i}, \mathbf{j}),$$

so that

$$A(r) = \sum_{k=N}^{\infty} \sum_{|\omega|=k} A_\omega(r). \tag{5.7}$$

We can apply Lemma 5.1 with $N = N(\varepsilon)$. Then $F_\varepsilon \subset F$ and $G_\varepsilon \subset G$ by (4.8) and (4.9), so for $\mathbf{i}, \mathbf{j} \in G_\varepsilon$, with $|\mathbf{i} \wedge \mathbf{j}| = k \geq N$, we have

$$g_r(\mathbf{i}, \mathbf{j}) = \eta_\infty\{\mathbf{y} \in F_\varepsilon : \ \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r\} \leq C_2(1+\delta)^k \rho^{-k} r.$$

Thus for $|\omega| = k \geq N$,

$$A_\omega(r) \leq C_2(1+\delta)^k \rho^{-k} r \cdot (\mu \times \mu)\{(\mathbf{i}, \mathbf{j}) : \ \mathbf{i} \wedge \mathbf{j} = \omega\}. \tag{5.8}$$

On the other hand,

$$(\mu \times \mu)\{(\mathbf{i}, \mathbf{j}) : \ \mathbf{i} \wedge \mathbf{j} = \omega\} \leq \mu([\omega])^2 = \mu[\mathbf{i}, k] \cdot \mu([\omega]) \leq \theta^k \mu([\omega]), \tag{5.9}$$

in view of (4.9). Combining this with (5.8) and (5.7) we obtain

$$A(r) \leq C_2 \sum_{k \geq N} \sum_{|\omega|=k} (1+\delta)^k \theta^k \rho^{-k} \mu([\omega]) \cdot r < \text{const} \cdot r,$$

where we used that $\sum_{|\omega|=k} \mu([\omega]) = 1$ and (4.7). The proof is complete. □

6. Proof of Theorem 2.1(b)

Fix $\varepsilon \in (0,1)$; our goal is to prove (4.5) assuming that $-h(\mu) \geq \chi(\mu)$.

Estimate from below. Fix an arbitrary $\alpha < h(\mu)/|\chi(\mu)|$; it is enough to prove that

$$\dim_H(\nu^\varepsilon_{\mathbf{y}}) \geq \alpha \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon. \tag{6.1}$$

We can find $\theta, \rho, \delta > 0$ such that

$$\alpha < \frac{\log \theta}{\log((1+\delta)^{-1}\rho)}, \quad |\chi(\mu)| < -\log \rho, \quad \text{and} \quad h(\mu) > -\log \theta. \tag{6.2}$$

Similarly to the proof of Theorem 2.1(a), we can find $N = N(\varepsilon, \delta, \theta, \rho)$ such that

$$(1+\delta)^{-n} \leq y_1 \cdots y_n \leq (1+\delta)^n \quad \text{for all} \quad n \geq N, \ \mathbf{y} \in F_\varepsilon$$

and

$$\mu[\mathbf{i}, n] < \theta^n \quad \text{and} \quad \rho^n < \lambda_{i_1 \ldots i_n}, \quad \text{for all} \quad n \geq N, \ \mathbf{i} \in G_\varepsilon.$$

We will use the decomposition (4.10) again.

By Frostman's Theorem, see [4, Theorem 4.13], for any Borel measure ν on the line,

$$\dim_H(\nu) \geq \sup\left\{\alpha > 0 : \iint\limits_{\mathbb{R}^2} \frac{d\nu(\xi)\, d\nu(\zeta)}{|\xi - \zeta|^\alpha} < \infty\right\}.$$

Thus the desired estimate (6.1) will follow by Fubini's Theorem, if we show that

$$\mathcal{S} := \sum_{|\omega|=N} \int\limits_{F_\varepsilon} \iint\limits_{\mathbb{R}^2} |\xi - \zeta|^{-\alpha}\, d\nu^\varepsilon_{\mathbf{y},\omega}(\xi)\, d\nu^\varepsilon_{\mathbf{y},\omega}(\zeta)\, d\eta_\infty(\mathbf{y}) < \infty.$$

After changing the variables and reversing the order of integration we obtain

$$\mathcal{S} = \sum_{k=N}^{\infty} \sum_{|\omega|=k} \iint\limits_{P_\omega} \int\limits_{F_\varepsilon} \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j})^{-\alpha}\, d\eta_\infty(\mathbf{y})\, d\mu_2(\mathbf{i}, \mathbf{j}), \tag{6.3}$$

where again

$$P_\omega = \{(\mathbf{i}, \mathbf{j}) \in G_\varepsilon \times G_\varepsilon :\ \mathbf{i} \wedge \mathbf{j} = \omega\}.$$

Suppose that $|\mathbf{i} \wedge \mathbf{j}| = k$. The inner integral in (6.3) is equal to

$$\alpha \int_0^\infty \eta_\infty\{\mathbf{y} \in F_\varepsilon :\ \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) \leq r\}\, r^{-\alpha-1}\, dr = \int_0^{(1+\delta)^{-k}\rho^k} + \int_{(1+\delta)^{-k}\rho^k}^\infty.$$

The first integral in the right-hand side is estimated by Lemma 5.1, and the second integral is estimated by the trivial estimate $\eta\{\cdot\} \leq 1$ yielding the inequality

$$\int_{F_\varepsilon} \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j})^{-\alpha}\, d\eta_\infty(\mathbf{y}) \leq \text{const} \cdot [(1+\delta)\rho^{-1}]^{\alpha k}.$$

Substituting this into (6.3) we obtain

$$\mathcal{S} \leq \text{const} \cdot \sum_{k=N}^{\infty} \sum_{|\omega|=k} [(1+\delta)\rho^{-1}]^{\alpha k} \mu_2\{(\mathbf{i}, \mathbf{j}) :\ \mathbf{i} \wedge \mathbf{j} = \omega\}.$$

Now we can apply (5.9) to get

$$\mathcal{S} \leq \text{const} \cdot \sum_{k=N} [(1+\delta)\rho^{-1}]^{\alpha k} \theta^k < \infty,$$

by (6.2).

Estimate from above. Dimension estimates from above are fairly standard. This is also the case here, although there are technical complications because of the generality of our set-up. Note that we obtain the upper bound for all, rather than almost all, $\mathbf{y} \in F_\varepsilon$, and the distribution of y_i's is irrelevant here. (Recall that F_ε was defined at the beginning of Section 4.) Similar upper bounds for (possibly nonlinear, but non-random) contracting on average IFS were obtained in [15, 6].

Fix an arbitrary $\alpha > h(\mu)/|\chi(\mu)|$; it is enough to prove that

$$\forall\, \varepsilon > 0,\ \dim_H(\nu_{\mathbf{y}}) \le \alpha \ \text{ for all } \mathbf{y} \in F_\varepsilon.$$

We fix $\varepsilon > 0$ and $\mathbf{y} \in F_\varepsilon$ for the rest of this proof.

Now let $\gamma > 0$ and consider G_γ, with $\mu(G_\gamma) > 1 - \gamma$, such that the convergence in (4.2) and (4.3) is uniform on G_γ. Further, let $C_3 > 0$ be such that

$$\mu(\Omega_\gamma) \ge 1 - \gamma, \quad \text{where } \ \Omega_\gamma := \{\mathbf{i} \in \Sigma : \ |\Pi_{\mathbf{y}}(\mathbf{i})| \le C_3\}. \tag{6.4}$$

Consider

$$\mathcal{E}_\gamma^n := \left\{\mathbf{i} \in \Sigma : \ \mu([\mathbf{i}, n] \cap \sigma^{-n}\Omega_\gamma) \ge 0.5 \cdot \mu([\mathbf{i}, n])\right\},$$

and let

$$A_\gamma^n := G_\gamma \cap \sigma^{-n}\Omega_\gamma \cap \mathcal{E}_\gamma^n.$$

We claim that $\mu(A_\gamma^n) \ge 1 - 4\gamma$ for all $n \in \mathbb{N}$. Indeed, $(\mathcal{E}_\gamma^n)^c$ is the disjoint union of cylinders $[\omega]$ of length n satisfying $\mu([\omega]) < 2\mu([\omega] \cap (\sigma^{-n}\Omega_\gamma)^c)$. Therefore,

$$\mu((\mathcal{E}_\gamma^n)^c) < 2\mu((\sigma^{-n}\Omega_\gamma)^c) = 2(1 - \mu(\Omega_\gamma)) \le 2\gamma,$$

by the σ-invariance of μ and (6.4). Thus,

$$\begin{aligned} \mu(A_\gamma^n) &\ge \mu(G_\gamma \cap \sigma^{-n}\Omega_\gamma) + \mu(\mathcal{E}_\gamma^n) - 1 \\ &\ge \mu(G_\gamma) + \mu(\sigma^{-n}\Omega_\gamma) - 1 + \mu(\mathcal{E}_\gamma^n) - 1 \\ &\ge (1-\gamma) + (1-\gamma) - 1 + (1 - 2\gamma) = 1 - 4\gamma. \end{aligned}$$

It follows that

$$\mu(H_\gamma) \ge 1 - 4\gamma, \quad \text{where } \ H_\gamma := \limsup(A_\gamma^n) = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_\gamma^n. \tag{6.5}$$

Recall that our goal is to prove $\dim_H(\nu_{\mathbf{y}}) \le \alpha$. Billingsley's Theorem (see [5, p.171]) states that

$$\dim_H(\nu_{\mathbf{y}}) = \nu_{\mathbf{y}}\text{-ess}\sup\left\{\liminf_{r \downarrow 0} \frac{\log \nu_{\mathbf{y}}[x - r, x + r]}{\log(2r)}\right\}.$$

Thus it is enough to verify that

$$\liminf_{r \downarrow 0} \frac{\log \nu_{\mathbf{y}}[x - r, x + r]}{\log(2r)} \le \alpha \tag{6.6}$$

for $\nu_{\mathbf{y}}$ a.e. x. Since $\nu_{\mathbf{y}} = \mu \circ \Pi_{\mathbf{y}}^{-1}$ and in view of (6.5), this will follow if we prove (6.6) for all $x \in \Pi_{\mathbf{y}}(H_\gamma)$, for every $\gamma > 0$. To this end, let us fix $\gamma > 0$ and

$x = \Pi_{\mathbf{y}}(\mathbf{i})$ for some $\mathbf{i} \in H_\gamma$. Since $\mathbf{i} \in H_\gamma$, there exists a sequence $n_k \to \infty$ such that

$$\mu([\mathbf{i}, n_k] \cap \sigma^{-n_k}\Omega_\gamma) \ge 0.5 \cdot \mu[\mathbf{i}, n_k], \quad \forall\, k \in \mathbb{N}. \tag{6.7}$$

Since $\alpha > h(\mu)/|\chi(\mu)|$, we can find $\theta, \rho, \delta > 0$ such that

$$\alpha > \frac{\log\theta}{\log((1+\delta)\rho)}, \quad |\chi(\mu)| > -\log\rho, \quad \text{and} \quad h(\mu) < -\log\theta. \tag{6.8}$$

Similarly to the proof of Theorem 2.1(a), we can find N such that (4.8) holds and

$$\mu[\mathbf{i}, n] > \theta^n \quad \text{and} \quad \rho^n > \lambda_{i_1\ldots i_n}, \quad \text{for all} \;\; n \ge N,\ \mathbf{i} \in G_\gamma. \tag{6.9}$$

Let $r_k = 2C_3\rho^{n_k}(1+\delta)^{n_k}$. We claim that for all k sufficiently large,

$$\nu_{\mathbf{y}}[x - r_k, x + r_k] = \mu\{\mathbf{j} :\ |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| \le r_k\} \ge 0.5 \cdot \mu[\mathbf{i}, n_k] \tag{6.10}$$

(the equality here is by definition; the claim is the inequality). Indeed, let $\mathbf{j} \in [\mathbf{i}, n_k] \cap \sigma^{-n_k}\Omega_\gamma$. Then for k sufficiently large (so that $n_k \ge N$), we have

$$|\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| = \lambda_{i_1\ldots i_{n_k}} y_{1\ldots n} \cdot |\Pi_{\mathbf{y}}(\sigma^{n_k}\mathbf{i}) - \Pi_{\mathbf{y}}(\sigma^{n_k}\mathbf{j})| \le 2C_3\rho^{n_k}(1+\delta)^{n_k} = r_k,$$

using (6.9), (4.8) and the fact that $\sigma^{n_k}\mathbf{i}, \sigma^{n_k}\mathbf{j} \in \Omega_\gamma$, where Ω_γ is defined by (6.4). This, combined with (6.7), proves (6.10). Now, keeping in mind that the numerator and denominator below are negative, we obtain

$$\begin{aligned}\liminf_{k\to\infty} \frac{\log\nu[x - r_k, x + r_k]}{\log(2r_k)} &\le \liminf_{k\to\infty} \frac{\log\mu[\mathbf{i}, n_k] - \log 2}{\log(2C_3) + n_k\log((1+\delta)\rho)} \\ &\le \lim_{k\to\infty} \frac{n_k\log\theta}{n_k\log((1+\delta)\rho)} < \alpha,\end{aligned}$$

where we used (6.9) and (6.8). The proof is complete. □

7. Proofs of other results

7.1. **Method of Kahane-Salem.** Here we prove Theorem 2.3 using a variant of the approach from [7]. Recall that for a finite measure ν on $\mathbb{R}$ its Fourier transform is defined by $\widehat{\nu}(\xi) = \int_{\mathbb{R}} e^{it\xi}\, d\nu(t)$.

Definition 7.1. (See [16].) For a finite measure ν on $\mathbb{R}$, its *Sobolev dimension* is defined as

$$\dim_s(\nu) = \sup\left\{\alpha \in \mathbb{R} :\ \mathcal{E}_\alpha(\nu) = \int_{\mathbb{R}} |\widehat{\nu}(\xi)|^2(1+|\xi|)^{\alpha-1}\, d\xi < \infty\right\}.$$

Remark 7.2. If $\dim_s(\nu) < 1$, then $\dim_s(\nu)$ is also known as the correlation dimension of the measure ν. If $\mathcal{E}_\alpha(\nu) < \infty$ for $\alpha > 1$, then $\nu \ll \mathcal{L}$, and its density is said to have the fractional derivative of order $(\alpha-1)/2$ in $L^2(\mathbb{R})$. If $\mathcal{E}_1(\nu) < \infty$, then ν has a density in $L^2(\mathbb{R})$ (this is just Plancherel's Theorem), and if $\dim_s(\nu) > 2$, then ν has a continuous density, see e.g. [1, Th. 1.2.4].

Theorem 7.3. *Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \ldots)$, defined by (2.4). We assume that (2.10) and (2.5) are satisfied, $\mu = (p_1, \ldots, p_m)^{\mathbb{N}}$ and $\lambda_i = \lambda$ for all $i \le m$. Suppose that $\chi(\eta) = \log\lambda + \mathbb{E}[\log Y] < 0$ and denote $\beta = \sum_{j=1}^m p_j^2$. Then*

$$\dim_s(\nu_{\mathbf{y}}) \ge |\log\beta|/|\chi(\eta)| \quad \text{for} \;\; \eta_\infty \text{ a.e. } \mathbf{y}.$$

In particular, if $0 > \chi(\eta) > \log\beta$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$ for η_∞ a.e. $\mathbf{y}$. If $0 > \chi(\eta) > \frac{1}{2}\log\beta$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a continuous density for η_∞ a.e. $\mathbf{y}$.

In view of Remark 7.2, Theorem 2.3 is contained in Theorem 7.3.

Proof of Theorem 7.3. Since $\lambda_i = \lambda$ for all $i \le m$, we can assume without loss of generality that $\lambda = 1$ (just replace Y with λY). Then $\chi(\eta) = \mathbb{E}[\log Y]$. Our goal is to prove that for every $\alpha < \frac{|\log\beta|}{|\chi(\eta)|}$,

$$\int_{\mathbb{R}} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 (1+|\xi|)^{\alpha-1}\, d\xi < \infty, \tag{7.1}$$

for η_∞ a.e. $\mathbf{y}$. Fix $\alpha < \frac{|\log\beta|}{|\chi(\eta)|}$ for the rest of the proof. By the the Law of Large Numbers and Egorov's Theorem, for any $\varepsilon > 0$ we can find $F_\varepsilon \subset \mathbb{R}^{\mathbb{N}}$ such that $\eta_\infty(F_\varepsilon) > 1 - \varepsilon$ and $(y_{1\ldots n})^{1/n} \to e^{\chi(\eta)}$ uniformly on F_ε. It suffices to verify (7.1) for η_∞ a.e. $\mathbf{y} \in F_\varepsilon > 0$, for an arbitrary $\varepsilon > 0$. Fix $\varepsilon > 0$ for the rest of the proof. The result will follow by Fubini's Theorem if we can show that

$$\int_{F_\varepsilon} \int_{\mathbb{R}} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 (1+|\xi|)^{\alpha-1}\, d\xi\, d\eta_\infty(\mathbf{y}) < \infty.$$

Recall that $\nu_{\mathbf{y}}$ is the conditional distribution of X in (2.9) given $\mathbf{y}$, with $\lambda = 1$, which can be viewed as a sum of independent discrete random variables. Thus, $\nu_{\mathbf{y}}$ is the infinite convolution product

$$\nu_{\mathbf{y}} = \ast\prod_{n=1}^{\infty} \left(\sum_{j=1}^m p_j \delta_{d_j y_{1\ldots(n-1)}} \right),$$

where δ is the Dirac's delta. Its Fourier transform is

$$\widehat{\nu}_{\mathbf{y}}(\xi)=\prod_{n=1}^{\infty}\sum_{j=1}^{m}p_je^{id_jy_{1\ldots(n-1)}\xi}=:\prod_{n=1}^{\infty}\psi_n(\mathbf{y},\xi).$$

Now the argument essentially follows the proof of [7, Théorème II]. We have

$$\begin{aligned}|\psi_n(\mathbf{y},\xi)|^2 &= \sum_{j,k=1}^{m}p_jp_ke^{i(d_j-d_k)y_{1\ldots(n-1)}\xi}\\ &= \sum_{j=1}^{m}p_j^2+\sum_{j\neq k}p_jp_ke^{i(d_j-d_k)y_{1\ldots(n-1)}\xi}.\end{aligned}$$

Clearly,

$$|\widehat{\nu}_{\mathbf{y}}(\xi)|^2\leq\prod_{n=1}^{\ell+1}|\psi_n(\mathbf{y},\xi)|^2=:f_{\xi,\ell+1}.$$

Denote by F_ε^n the projection of F_ε onto $\mathbb{R}^n$ (the first n coordinates). Then, since $f_{\xi,\ell+1}$ depends only on $y_1,\ldots,y_\ell$, we obtain

$$\begin{aligned}\int_{F_\varepsilon}f_{\xi,\ell+1}\,d\eta_\infty(\mathbf{y}) &= \int_{F_\varepsilon^\ell}f_{\xi,\ell+1}\,d\eta(y_1)\ldots d\eta(y_\ell)\\ &\leq \int_{F_\varepsilon^{\ell-1}}f_{\xi,\ell}\,d\eta(y_1)\ldots d\eta(y_{\ell-1})\int_{\mathbb{R}}|\psi_{\ell+1}(\mathbf{y},\xi)|^2\,d\eta(y_\ell).\end{aligned}\tag{7.2}$$

Recall that $\beta=\sum_{j=1}^m p_j^2$, so

$$\begin{aligned}\int_{\mathbb{R}}|\psi_{\ell+1}(\mathbf{y},\xi)|^2\,d\eta(y_\ell) &= \beta+\sum_{j\neq k}p_jp_k\int_{\mathbb{R}}e^{i(d_j-d_k)y_{1\ldots(n-1)}\xi}\,d\eta(y_\ell)\\ &= \beta+\sum_{j\neq k}p_jp_k\widehat{\eta}((d_j-d_k)y_{1\ldots(n-1)}\xi).\end{aligned}$$

Integration by parts (see e.g. [8, p. 25]) shows that the Fourier transform of a compactly supported function of bounded variation is bounded above by $c|t|^{-1}$. Since $|d_j-d_k|\geq b>0$ for $j\neq k$, we obtain that for some $C>0$,

$$\int_{\mathbb{R}}|\psi_{\ell+1}(\mathbf{y},\xi)|^2\,d\eta(y_\ell)\leq\beta\left(1+\frac{C}{y_{1\ldots(\ell-1)}|\xi|}\right)\tag{7.3}$$

Recall that $\alpha<|\log\beta|/|\chi(\eta)|$; choose $\rho<e^{\chi(\eta)}$ such that $\alpha<|\log\beta|/|\chi(\eta)|$. Since $y_{1\ldots n}^{1/n}$ converges to $e^{\chi(\eta)}$ uniformly on F_ε, we can find $N\in\mathbb{N}$ such that

$$y_{1\ldots n}\geq(n+1)^2\rho^{n+1}\quad\forall\,n\geq N,\ \forall\,\mathbf{y}\in F_\varepsilon.$$

It follows from (7.2) and (7.3) that for $\ell \geq N+1$,

$$\begin{aligned}\int_{F_\varepsilon} f_{\xi,\ell+1}\, d\eta_\infty(\mathbf{y}) &\leq \int_{F_\varepsilon} f_{\xi,\ell}\, d\eta_\infty(\mathbf{y}) \cdot \beta \left(1 + \frac{C}{\ell^2 \rho^\ell |\xi|}\right) \\ &\leq \int_{F_\varepsilon} f_{\xi,\ell}\, d\eta_\infty(\mathbf{y}) \cdot \beta(1 + C\ell^{-2}), \end{aligned} \tag{7.4}$$

provided that $|\xi| \geq \rho^{-\ell}$. Clearly this condition holds for $\ell' < \ell$ if it holds for ℓ, so we can iterate (7.4) to obtain, assuming $|\xi| \geq \rho^{-\ell}$:

$$\int_{F_\varepsilon} f_{\xi,\ell+1}\, d\eta_\infty(\mathbf{y}) \leq \int_{F_\varepsilon} f_{\xi,N}\, d\eta_\infty(\mathbf{y}) \cdot \beta^{\ell+1-N} \prod_{k=N}^{\ell} (1 + Ck^{-2}) \leq C'\beta^\ell,$$

where $C' > 0$ depends on N but not on ξ. We conclude that

$$\int_{F_\varepsilon} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2\, d\eta_\infty(\mathbf{y}) \leq C''|\xi|^{\log\beta/|\log\rho|}.$$

But $\log\beta/|\log\rho| < -\alpha$, hence

$$\int_{\mathbb{R}} \left(\int_{F_\varepsilon} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2\, d\eta_\infty(\mathbf{y})\right) (1 + |\xi|)^{\alpha-1}\, d\xi < \infty,$$

and the proof is complete. □

7.2. **Proof of Proposition 2.2.** Consider the probability space $\Sigma \times \mathbb{R}^{\mathbb{N}}$ with the measure $\mathbb{P} := \mu \times \eta_\infty$. Under the assumptions of the proposition, $Z_n := \log p_{i_n} - \log\lambda_{i_n} - \log y_n$ are i.i.d. non-constant random variables with mean zero. Let

$$B_n := \left\{(\mathbf{i}, \mathbf{y}) :\ \sum_{j=1}^{n} Z_j > \sqrt{n}\right\} = \left\{(\mathbf{i}, \mathbf{y}) :\ \frac{p_{i_1\ldots i_n}}{\lambda_{i_1\ldots i_n} y_{1\ldots n}} > e^{\sqrt{n}}\right\}.$$

By the Law of Iterated Logarithm, we have

$$\mathbb{P}\left(\limsup B_n\right) = 1. \tag{7.5}$$

For $(\mathbf{i}, \mathbf{y}) \in \limsup B_n$ and $k \in \mathbb{N}$ let

$$\tau_k = \tau_k(\mathbf{i}, \mathbf{y}) = \min\{n \geq k :\ (\mathbf{i}, \mathbf{y}) \in B_n\},$$

which is well-defined and finite. Let

$$A_n = \{(\mathbf{i}, \mathbf{y}) :\ |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i})| \leq n\};$$

note that A_n is independent of $i_1, \ldots, i_n, y_1, \ldots, y_n$. Since the probability $\mathbb{P}$ is σ-invariant and $\Pi_{\mathbf{y}}(\mathbf{i})$ is finite a.s., we have $\mathbb{P}(A_n) = \mathbb{P}\{(\mathbf{i}, \mathbf{y}) : \ \Pi_{\mathbf{y}}(\mathbf{i}) \le n\} \uparrow 1$. Now,

$$\begin{aligned}\mathbb{P}\left[\bigcup_{n\ge k}(A_n \cap B_n)\right] &\ge \mathbb{P}\left(\bigcup_{n\ge k}[A_n \cap \{\tau_k = n\}]\right)\\ &= \sum_{n=k}^{\infty} \mathbb{P}\{\tau_k = n\} \cdot \mathbb{P}(A_n) \ge \mathbb{P}(A_k) \to 1,\end{aligned}$$

as $k \to \infty$. In the last displayed line we used that A_n and $\{\tau_k = n\}$ are independent events and that $\sum_{n=k}^{\infty} \mathbb{P}\{\tau_k = n\} = 1$ by (7.5). It follows that

$$\mathbb{P}(\limsup(A_n \cap B_n)) = 1.$$

By Fubini, there exists $\Omega \subset \mathbb{R}^{\mathbb{N}}$ such that $\eta_\infty(\Omega) = 1$ and

$$\Sigma_{\mathbf{y}} := \{\mathbf{i} \in \Sigma : \ (\mathbf{i}, \mathbf{y}) \in \limsup(A_n \cap B_n)\}$$

has $\mu(\Sigma_{\mathbf{y}}) = 1$ for every $\mathbf{y} \in \Omega$. We claim that $\mathcal{L}(\Pi_{\mathbf{y}}(\Sigma_{\mathbf{y}})) = 0$ for every $\mathbf{y} \in \Omega$, which will imply that $\nu_{\mathbf{y}} = \mu \circ \Pi_{\mathbf{y}}^{-1} \perp \mathcal{L}$. Fix $\mathbf{y} \in \Omega$ and $k \in \mathbb{N}$. Observe that $\Sigma_{\mathbf{y}} \subset \bigcup_{n\ge k}\{\mathbf{i} : (\mathbf{i}, \mathbf{y}) \in A_n \cap B_n\}$. For any $(\mathbf{i}, \mathbf{y}) \in A_n \cap B_n$ and $(\mathbf{j}, \mathbf{y}) \in [\mathbf{i}, n] \cap A_n$ we have

$$\begin{aligned}|\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| &= \lambda_{i_1 \ldots i_n} y_{1 \ldots n} |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i}) - \Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{j})|\\ &\le 2n\lambda_{i_1 \ldots i_n} y_{1 \ldots n}\\ &\le 2ne^{-\sqrt{n}} p_{i_1 \ldots i_n}.\end{aligned}$$

Here we used first that $(\mathbf{i}, \mathbf{y}), (\mathbf{j}, \mathbf{y}) \in A_n$, and then that $(\mathbf{i}, \mathbf{y}) \in B_n$. Summing over all cylinders of length n (using that $\sum_{i_1 \ldots i_n} p_{i_1 \ldots i_n} = 1$), and then summing over n we obtain

$$\mathcal{L}(\Pi_{\mathbf{y}}(\Sigma_{\mathbf{y}})) \le \sum_{n\ge k} 2ne^{-\sqrt{n}} \to 0, \quad \text{as } k \to \infty.$$

□

7.3. **Proof of Corollaries 2.4-2.6.** By the Variational Principle (see e.g. [19]), $h_{\text{top}}(\Gamma) = \sup_\mu h(\mu)$, where the supremum is over ergodic σ-invariant measures supported on Γ. Thus, Theorem 2.1(a) implies Corollary 2.4(a), and Theorem 2.1(b) implies the lower estimate for $\dim_H(S_\Gamma(\mathbf{y}))$ in Corollary 2.4(b).

In Corollary 2.5, we have $\Gamma = \Sigma$, for which the measure of maximal entropy is $(\frac{1}{m}, \dots, \frac{1}{m})^{\mathbb{N}}$. Part (a) of Corollary 2.5 then follows from Theorem 2.3(b) by the Variational Principle, and Corollary 2.5(b) is a special case of Corollary 2.4(b).

It remains to verify the upper estimate for $\dim_H(S_{\mathbf{y}})$ in Corollary 2.4(b). By the Law of Large Numbers and Egorov's Theorem, we can find $F_\varepsilon \subset \mathbb{R}^{\mathbb{N}}$ such that $\eta_\infty(F_\varepsilon) > 1 - \varepsilon$ and $y_{1\dots n}^{1/n} \to e^{\chi(\eta)}$ uniformly for $\mathbf{y} \in F_\varepsilon$. Fix an arbitrary $\alpha > h_{\rm top}(\Gamma)/|\chi(\eta)|$. It suffices to show that for every $\varepsilon > 0$ we have $\overline{\dim}_{\rm box}(S_{\mathbf{y}}) < \alpha$, for a.e. $\mathbf{y} \in F_\varepsilon$. Here $\overline{\dim}_{\rm box}$ denotes the upper box-counting dimension. Fix $\varepsilon > 0$ for the rest of the proof.

Let $\delta > 0$ be such that $\chi(\eta) + \delta < 0$ and $\alpha > h_{\rm top}(\Gamma)/(|\chi(\eta)| - \delta)$. We can find $N \in \mathbb{N}$ such that

$$y_{1\dots n} \le \exp(n(\chi(\eta) + \delta)), \quad \forall\, n \ge N, \ \forall\, \mathbf{y} \in F_\varepsilon.$$

Then for $\mathbf{i}, \mathbf{j} \in \Sigma$ such that $|\mathbf{i} \wedge \mathbf{j}| \ge n \ge N$, we have

$$\begin{aligned} |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| &= \left| \sum_{\ell=n}^{\infty} y_{1\dots\ell}(d_{i_{\ell+1}} - d_{j_{\ell+1}}) \right| \\ &\le 2d_{\max} \sum_{\ell=n}^{\infty} e^{\ell(\chi(\eta)+\delta)} = \frac{2d_{\max} e^{n(\chi(\eta)+\delta)}}{1 - e^{\chi(\eta)+\delta}}, \end{aligned}$$

where $d_{\max} := \max_{i \le m} |d_i|$. It follows that the diameter of the set $\Pi_{\mathbf{y}}([\omega])$ for ω of length $n \ge N$ and $\mathbf{y} \in F_\varepsilon$ is bounded above by const $\cdot\, e^{n(\chi(\eta)+\delta)}$. Denote by $\#\mathcal{W}_n(\Gamma)$ the number of cylinders $[\omega]$ of length n such that $[\omega] \cap \Gamma \ne \emptyset$. We obtain a cover of $S_{\mathbf{y}}$ by $\#\mathcal{W}_n(\Gamma)$ intervals of length const $\cdot\, e^{n(\chi(\eta)+\delta)}$, hence

$$\overline{\dim}_{\rm box}(S_{\mathbf{y}}) \le \limsup_{n\to\infty} \frac{\log(\#\mathcal{W}_n(\Gamma))}{n(-\chi(\eta) - \delta)} = \frac{h_{\rm top}(\Gamma)}{|\chi(\eta)| - \delta} < \alpha.$$

Here we used the definition of the upper box-counting dimension and the definition of topological entropy. The proof is complete. □

Proof of Corollary 2.6. This essentially follows the proof of Proposition 2.2. Let $\mu = (\frac{1}{m}, \dots \frac{1}{m})$. Then $Z_n = -\log m - \log \lambda - \log y_n$ are i.i.d. non-constant random variables with mean zero on $\mathbb{R}^{\mathbb{N}}$. Define $B'_n = \left\{ \mathbf{y} : \sum_{j=1}^n Z_j > \sqrt{n} \right\}$. Then $\eta_\infty(\limsup B'_n) = 1$ by the Law of Iterated Logarithm. We can define $\tau_k = \tau_k(\mathbf{y})$ similarly to the proof of Proposition 2.2. Then let $A'_n = \{\mathbf{y} : \ |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i})| \le n \ \forall\, \mathbf{i} \in \Sigma\}$. We have $\eta_\infty(A'_n) \uparrow 1$ and $\eta_\infty(\limsup(A'_n \cap B'_n)) = 1$ repeating the argument in the proof of Proposition 2.2. Now we can take $\Sigma_{\mathbf{y}} = \Sigma$ and conclude as in the proof of Proposition 2.2, obtaining that $\mathcal{L}(S_{\mathbf{y}}) = \mathcal{L}(\Pi_{\mathbf{y}}(\Sigma)) = 0$ for all $\mathbf{y} \in \limsup(A'_n \cap B'_n)$. □

8. Open questions

Question 1. Is the condition $\log m > 2|\chi(\eta)|$ in Corollary 2.5(a) for the random set $S_{\mathbf{y}}$ to contain an interval almost surely, sharp? Perhaps, $\log m > |\chi(\eta)|$ is already sufficient? This would mean that as soon the random set $S_{\mathbf{y}}$ has positive Lebesgue measure, it has non-empty interior (almost surely). This is interesting, in particular, for the example considered by Arratia, see Corollary 3.2.

Question 2. The results on continuous density and intervals in random sets (see Theorem 2.3 and Corollary 2.5(a)) are obtained only in the case when μ is a product measure. Extend this to the general case of ergodic μ.

Question 3. In Theorem 7.3 we prove a lower bound for the a.s. value of the Sobolev dimension $\dim_s(\nu_{\mathbf{y}})$. Is this actually an equality? If $|\log\beta| < |\chi(\eta)|$, then the matching upper bound can be obtained from the fact that the Sobolev dimension equals the correlation dimension when it is less than one, but what about the case $|\log\beta| \ge |\chi(\eta)|$?

Acknowledgment. We are grateful to Richard Arratia and Jim Pitman for useful discussions.

References

[1] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, 1996.

[2] R. Arratia, On the central role of scale invariant Poisson processes on $(0,\infty)$, *Microsurveys in discrete probability (Princeton, NJ, 1997)*, 21–41, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **41**, Amer. Math. Soc., Providence, RI, 1998.

[3] P. Diaconis and D. A. Freedman, Iterated random functions, *SIAM Review* **41**, No. 1, 45–76.

[4] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

[5] K. J. Falconer, *Techniques in Fractal Geometry*, Wiley, 1997.

[6] A. H. Fan, K. Simon, H. Tóth, Contracting on average random IFS with repelling fix point, *J Stat. Phys.*, to appear.

[7] J.-P. Kahane, R. Salem, Sur la convolution d'une infinité de distributions de Bernoulli, *Colloq. Math.* **6** (1958), 193–202.

[8] Y. Katznelson, *Harmonic Analysis*, Dover, New York, 1976.

[9] Y. Kifer, Fractal dimensions and random transformations, *Trans. Amer. Math. Soc.* **348** (1996), 2003–2038.

[10] Y. Kifer, Random f-expansions, *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, 385–407, Proc. Sympos. Pure Math., **69**, Amer. Math. Soc., Providence, RI, 2001.

[11] K.-S. Lau, S.-M. Ngai and H. Rao, Iterated function systems with overlaps and self-similar measures, *J. London Math. Soc. (2)* **63** (2001), no. 1, 99–116.

[12] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[13] J. Neunhäuserer, Properties of some overlapping self-similar and some self-affine measures, *Acta Math. Hungar.* **92** (2001), 143–161.

[14] S.-M. Ngai and Y. Wang, Self-similar measures associated with IFS with non-uniform contraction ratios, *Asian J. Math.* **9** (2005), no. 2, 227–244.

[15] M. Nicol, N. Sidorov, D. Broomhead, On the fine structure of stationary measures in systems which contract on average. *J. Theoret. Probab.* **15** (2002), no.3, 715-730.

[16] Y. Peres and W. Schlag, Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions, *Duke Math. J.* **102** (2000), no. 2, 193–251.

[17] Y. Peres and B. Solomyak, Self-similar measures and intersections of Cantor sets, *Trans. Amer. Math. Soc.* **350**, no. 10 (1998), 4065–4087.

[18] K. Simon, B. Solomyak, and M. Urbański, Invariant measures for parabolic IFS with overlaps and random continued fractions, *Trans. Amer. Math. Soc.* **353** (2001), 5145–5164.

[19] P. Walters, *An introduction to Ergodic Theory*, Springer, 1982.

Yuval Peres, Department of Statistics, University of California, Berkeley
E-mail address: peres@stat.Berkeley.edu

Károly Simon, Institute of Mathematics, Technical University of Budapest, H-1529 B.O.box 91, Hungary
E-mail address: simonk@math.bme.hu

Boris Solomyak, Box 354350, Department of Mathematics, University of Washington, Seattle WA 98195
E-mail address: solomyak@math.washington.edu