CONTRACTING ON AVERAGE RANDOM IFS WITH REPELLING FIXED POINT.

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Abstract. We consider random iterated function systems which consist of strictly increasing and (not necessarily strictly) convex functions on a compact interval or on a half line. We assume that the system is contracting on average in a sense which is wide enough to permit the existence of a common fixpoint at which some functions of the system are expanding and perhaps none of them are contracting (see Figure 1). We prove that the Hausdorff dimension of any of the possibly uncountably many invariant measures is smaller than or equal to the accumulated entropy divided by the Liapunov exponent.

1. Introduction

Our aim in this paper is to prove that the Hausdorff dimension is less than or equal to the entropy/Liapunov exponent for all the (usually uncountably many) measures which are invariant w.r.t. a *random iterated function system* RIFS $(\mathcal{F}, \mathbf{p})$ on I, where $I = [0, b]$ is a compact interval or a half line ($b = \infty$), and $\mathcal{F} = (f_1, \dots, f_m)$ is a set of m strictly increasing, convex $\mathcal{C}^2$ mappings defined on I whose derivatives are bounded away from zero. Further, $\mathbf{p} = (p_1, \dots, p_m)$ is a probability vector which gives us the probability with which we apply f_i. Let μ be the $\{p_1, \dots, p_m\}^{\mathbb{N}}$ Bernoulli measure on the symbolic space $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$.

We assume that $(\mathcal{F}, \mathbf{p})$ is of similar flavor to the one on Figure 1 (see 2.1-2.4 for the precise definition). Furthermore, we make the following assumption.

Date: November 28, 2005.

2000 *Mathematics Subject Classification.* Primary 42A85 Secondary 11R06, 26A46, 26A30, 26A78, 28A80

Key words and phrases. Hausdorff dimension, Contracting on average

The work was started when the second author visited LAMFA, University of Picardie, France. The research of Simon was supported by OTKA Foundation #T42496 and the research of Tóth was supported by OTKA TS40719 and T42496.

Principal assumption: In the rest of the paper we always assume that $(\mathcal{F}, \mathbf{p})$ is contracting on average in the sense that the Liapunov exponent

$$
\begin{aligned}
(1.1)\quad \chi : &= \int\limits_{\Sigma} \log f'_{i_1}(\Pi(\sigma \mathbf{i})) d\mu(\mathbf{i}) \\
&= \int \mathbb{E}(\log f'_i(x)) d\nu(x) = \int \sum_{i=1}^{m} p_i \log f'_i(x) d\nu(x) < 0,
\end{aligned}
$$

where

$$\nu := \Pi_* \mu,$$

is the push down measure of μ and $\Pi : \{1, \dots, m\}^{\mathbb{N}} \to I$ defined by

$$\Pi(i_1, i_2, \dots) = \lim_{n \to \infty} f_{i_1} \circ \dots \circ f_{i_n}(0)$$

is the natural projection. Notice that $\{f_{i_1} \circ \dots \circ f_{i_n}(0)\}_{n=1}^{\infty}$ is an increasing sequence. Therefore $\Pi(i_1, i_2, \dots)$ always exists. If $b = \infty$ then it can happen that $\Pi(\mathbf{i}) = \infty$. In this case in (1.1) we mean $f'_k(\Pi(\mathbf{i})) := \lim\limits_{x \to \infty} f'_k(x)$ (which exists since f'_k is increasing). However, (1.1) implies that for μ almost all $\mathbf{i} = (i_1, i_2, \dots)$ we have $\Pi(\mathbf{i}) < \infty$ even if $b = \infty$. (See Fact 1.)

It was assumed in related results previously known (see [9]) that

$$\mathbb{E}(\sup\nolimits_{\mathrm{x}} \log f'_i(x)) < 0 \tag{1.2}$$

which is a much stronger assumption (which definitely does not apply in a case where we have a common repelling fixpoint).

We introduce the above notion of contracting on average which is different from the one most commonly used (see [2] and below) in the literature for the following reasons:

(1) RIFS with common repelling fix points are not contracting on average in the most commonly used sense,
(2) We cannot possibly have the best upper bound on dimensions of the form entropy/Liapunov exponent if the exponent is defined like in [2] or in [9] (see 1.3) and if not all the maps are linear,
(3) The Liapunov exponent for RIFS contracting on average in the most commonly used sense (see (1.3)) is not invariant under coordinate change with a $\mathcal{C}^1$ map whose derivative is not separated from zero (e.g. on the half line). On the other hand, the entropy and the dimension are invariant.

The RIFS $\{\mathcal{F}, \mathbf{p}\}$ in the literature is most commonly called contracting on average (see [2]) if all the maps in $\mathcal{F}$ are Lipschitzian and for μ-almost all $\mathbf{i} \in \Sigma$

$$\chi_L := \lim_{n\to\infty} \frac{1}{n} \log \|f_{i_1,\dots,i_n}\| < 0, \tag{1.3}$$

where we write $f_{i_1,\dots,i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$ throughout the paper, and $\|f\|$ denotes the Lipschitz constant of a function f. In this case there is a unique invariant probability measure (see [2]). The authors in [9] proved that the Hausdorff dimension of the invariant measure is less than or equal to $\frac{H_\mu}{-\chi_L}$ for RIFS satisfying (1.3), where H_μ is the accumulation entropy of μ relative to $\mathcal{F}$ (see the definition in Section 2).

However, χ_L is not invariant under a coordinate change whose derivative is not separated from zero, which is a significant drawback. It can happen that the RIFS $(\mathcal{F}, \mathbf{p})$ is contracting on average in the sense (1.3) but after a coordinate change by a smooth map φ whose derivative is not bounded away from zero (if the domain of $\mathcal{F}$ is not compact), the resulting new RIFS $\mathcal{G} := \{g_1, \dots, g_m\}$, $g_i(u) := \varphi \circ f_i \circ \varphi^{-1}(u), i = 1, \dots, m$ (with the same probability vector) is not contracting on average in the sense of (1.3). For example, special attention was paid in the literature to the RIFS defined on $[0, \infty)$

$$f_1(x) = \lambda^{-1}x, \quad f_2(x) = x + 1; \qquad p_0 = p_1 = \frac{1}{2}.$$

See Section 6 for a short account about the importance of this system. This system clearly satisfies (1.3). However, as is detailed in Section 6, after the coordinate change $\varphi : [0, \infty] \to [0, 1]$ defined by

$$\varphi(x) = \frac{x}{1 + x}, \quad \varphi(\infty) = 1$$

we obtain the RIFS $\mathcal{G} := \{g_1, g_2\}$ defined on $[0, 1]$

$$g_1(u) := \frac{u}{u + \lambda(1 - u)}, \quad g_2 := (2 - u)^{-1} \quad (0 \le u \le 1)$$

also with probability $(\frac{1}{2}, \frac{1}{2})$ (see Figure 1), which does not satisfy (1.3). See Section 6 for details.

Our result is also related to the paper [10], where the authors investigated the Hausdorff dimension of invariant measures for parabolic (so, not contracting) IFS's with overlaps. At some places during the proofs we use ideas similar to those in [10]. Myjak and Szarek [8] also investigated related problems about the Hausdorff dimension of invariant measures for non-contractive IFS. D. Steinsaltz [11] considered RIFS which are contracting on average in another sense, which is more general than (1.3).

2. Notation and main result

Let $I = [0, b] \subset \overline{\mathbb{R}^+}$ be an interval with $b \in \mathbb{R}^+ \cup \{\infty\}$ and $\mathcal{F} := \{f_1, \ldots, f_m\}$ $(m \geq 2)$ be a system of $\mathcal{C}^2$ maps on $[0, b]$ if $b < \infty$ and on $[0, \infty)$ if $b = \infty$ having the properties

$$f_i : I \to I, \quad 0 = \min_i \mathrm{Fix}(f_i), \tag{2.1}$$

$$\inf_i \inf_{x \in I} f_i'(x) > 0, \tag{2.2}$$

$$\forall x < b, \ f_m(x) > x. \tag{2.3}$$

(2.4) Every map f_i is convex but different from the identity map.

The assumption (2.3) simply states that there is a map which is above diagonal on $[0, b)$ and without loss of generality we may assume that this is f_m. In particular, $f_m(0) > 0$. We remark that the maps are not assumed to be strictly convex. Even, if $b = \infty$ we do not assume that our maps are Lipschitz continuous.

Let $\mu = \{p_1, p_2, \ldots, p_m\}^{\mathbb{N}}$ be the product measure on the symbolic space $\Sigma = \{1, 2, \cdots, m\}^{\mathbb{N}}$ of the probability $(p_1, \cdots, p_m)$ (i.e. $\sum_{i=1}^{m} p_i = 1$, $p_i > 0$). For any finite sequence $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, we write

$$p_{i_1,\ldots,i_n} = p_{i_1} p_{i_2} \cdots p_{i_n}, \qquad f_{i_1,\ldots,i_n} = f_{i_1} \circ \cdots \circ f_{i_n}.$$

As we shall see, the Liapunov exponent χ is invariant under coordinate change. Moreover, we shall see that χ has another equivalent expression (see Lemma 5):

$$\chi_{\mathcal{F}} := \lim_{n \to \infty} \frac{1}{n} \log f'_{i_1,\ldots,i_n}(0) \qquad \mu \text{ a.s.} \tag{2.5}$$

The Perron-Frobenius operator of $(\mathcal{F}, \mathbf{p})$ is defined as follows:

$$L(\phi) = \sum_{i=1}^{m} p_i \phi \circ f_i$$

where ϕ is continuous function with compact support. The adjoint operator L^* of L acts on Radon measures on I.

Definition 1. *We say that a probability measure ν_0 is invariant if $L^* \nu_0 = \nu_0$.*

Let ν_0 be an invariant measure. For every Borel set $A \subset [0, b]$ and for every n we have

$$\nu_0(A) = \sum_{i_1,\dots,i_n} p_{i_1,\dots,i_n} \nu_0 \left(f^{-1}_{i_1,\dots,i_n}(A) \right). \tag{2.6}$$

On the probability space (Σ, μ), define inductively a stochastic process $\{X_n\}_{n \geq 0}$ with state space I as follows

$$X_0(\mathbf{i}) = x, \quad X_n(\mathbf{i}) = f_{i_n}(X_{n-1}(\mathbf{i})) \quad n \geq 1).$$

It is clear that $\{X_n\}$ is a Markov chain with transition probability $P(y, B) = L1_B(y)$, i.e.

$$P(y, B) = \sum_{i=1}^{m} p_i 1_B(f_i(y)) \qquad (y \in I, B \in \mathcal{B}(I)).$$

The invariant measures defined above are the invariant measures of the Markov chain with transition probability $P(y, B)$. For an RIFS which is contracting on average, the push down measure ν is an invariant measure. This is an immediate consequence of the following Fact.

Fact 1. *It follows from (1.1) that* $\Pi(\mathbf{i})$ *is finite (even if* $b = \infty$*) for almost all* $\mathbf{i} \in \Sigma$.

Proof. Let $H = \{\mathbf{i} : \Pi(\mathbf{i}) = \infty\}$. To get contradiction we assume that $\mu(H) > 0$. Then by ergodicity $\mu(H) = 1$. In this case $\chi = \sum_{k=1}^{m} p_k \log d_k$, where $d_k := \lim_{x \to \infty} f'_k(x) < \infty$ follows from (1.1). Using Birkhoff Ergodic Theorem we obtain that for μ almost all $\mathbf{i}$

$$\frac{1}{n} \sum_{\ell=1}^{n} \log d_{i_\ell} \longrightarrow \chi < 0. \tag{2.7}$$

Fix a $\chi < \chi_1 < 0$. Then for μ almost all $\mathbf{i}$ there exists N such that for all $n \geq N$ we have

$$\forall x, \quad f'_{i_1,\dots,i_n}(x) < \exp \sum_{\ell=1}^{n} \log d_{i_\ell} < e^{n\chi_1}. \tag{2.8}$$

Using this and the fact that for every q we have

$$|f_{i_1,\dots,i_{N+q}}(0) - f_{i_1,\dots,i_N}(0)| \leq \sum_{k=0}^{q-1} \left| f_{i_1,\dots,i_{N+k}}(f_{i_{N+K+1}}(0)) - f_{i_1,\dots,i_{N+k}}(0) \right|.$$

From Lagrange's Theorem we get that for μ almost all $\mathbf{i}$ the sequence $\{f_{i_1,\dots,i_n}(0)\}$ is bounded. $\square$

Furthermore, if b is a common fixed point of the system $\mathcal{F}$, the Dirac measure δ_b is also an invariant measure (it may happen that $\nu = \delta_b$) and then we shall prove that all invariant measures are convex combinations of the push down measure ν and the Dirac measure δ_b. However, if b is not a common fixed point of the system, the push down measure will be proved to be the unique invariant measure. Also we will prove that if the push down measure ν is different from δ_b, then it is non-atomic.

For $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, consider the cylinder

$$[i_1, \ldots, i_n] := \{\mathbf{j} \in \Sigma : j_k = i_k,\ k = 1, 2, \cdots, n\}.$$

If $f_{j_1,\ldots,j_n} = f_{i_1,\ldots,i_n}$, we write $(j_1, \ldots, j_n) \sim (i_1, \ldots, i_n)$. Let D_n be the set of equivalence classes for this equivalence relation. Following [7, 9] we introduce the n-th *accumulation* of μ (relative to the system $\mathcal{F}$) by

$$\mu_n([i_1, \ldots, i_n]) := \sum_{(j_1,\ldots,j_n)\sim(i_1,\ldots,i_n)} \mu([j_1, \ldots, j_n]).$$

Then we define the accumulated entropy (relative to the system $\mathcal{F}$) of μ by

$$H_\mu := \lim_{n\to\infty} -\frac{1}{n} \sum_{[y]\in D_n} \mu_n([y]) \log \mu_n([y]).$$

Obviously, $H_\mu \leq h_\mu$ where h_μ is the usual entropy of μ (μ being considered as a shift-invariant measure on Σ). For an $\mathbf{i} \in \Sigma$ we define

$$\mathcal{E}_n(\mathbf{i}) := \{\mathbf{j} \in \Sigma : (j_1, \ldots, j_n) \sim (i_1, \ldots, i_n)\}.$$

The following lemma was proved in [9, Lemma 2.2].

Proposition 1. (Existence of local entropy) *For μ a.e. $\mathbf{i} \in \Sigma$*

$$\lim_{n\to\infty} \frac{\log \mu(\mathcal{E}_n(\mathbf{i}))}{n} = -H_\mu. \tag{2.9}$$

Our main result is the following

Theorem 1. *Let $(\mathcal{F}, \mathbf{p})$ be a contracting on average RIFS (satisfying (1.1)). Suppose the assumptions (2.1)–(2.4) hold. Then for every invariant measure ν_0 we have*

$$dim_H(\nu_0) \leq \frac{H_\mu}{-\chi}.$$

3. Different forms of the Liapunov exponent

Proposition 2. (Existence of $\chi_{\mathcal{F}}$) *The limit (2.5) defining $\chi_{\mathcal{F}}$ exists almost surely.*

Proof. The existence is ensured by the Kingman Ergodic Theorem (see [6, p.38]). Let $F_n(\mathbf{i}) = \log f'_{i_1,\dots,i_n}(0)$. We have only to show that F_n is a superadditive process since the derivative of f_i, $i = 1,\dots,m$ are uniformly bounded away from zero. That is

$$F_{n+k}(\mathbf{i}) \geq F_n(\mathbf{i}) + F_k(\sigma^n \mathbf{i}) \qquad (\forall n, k \geq 1).$$

Since $f_{i_1,\dots,i_{n+k}} = f_{i_1,\dots i_n}(f_{i_{n+1},\dots,i_{n+k}})$, we can write

$$F_{n+k}(\mathbf{i}) = \log f'_{i_1,\dots i_n}(f_{i_{n+1},\dots,i_{n+k}}(0)) + \log f'_{i_{n+1},\dots,i_{n+k}}(0).$$

Notice that $f_{i_{n+1},\dots,i_{n+k}}(0) \geq 0$ and that $f'_{i_1,\dots,i_n}$ is increasing. Then the first term is greater than or equal to $F_n(\mathbf{i})$. The second term is exactly $F_k(\sigma^n \mathbf{i})$. Thus the superadditivity is verified. □

It follows from the definition of χ (see 1.1) that

Lemma 1. *For μ-a.e. $\mathbf{i} \in \Sigma$:*

$$\chi = \lim_{n\to\infty} \frac{1}{n} \log f'_{i_1,\dots,i_n}(\Pi(\sigma^n \mathbf{i})). \tag{3.1}$$

Proof. From the definition of Π we get $f_{i_{k+1},\dots,i_n}(\Pi(\sigma^n \mathbf{i})) = \Pi\left(\sigma^k \mathbf{i}\right)$ for $0 \leq k < n$. By the Birkhoff Ergodic Theorem, we obtain that for μ a.e. $\mathbf{i}$:

$$\begin{aligned}
\chi &= \int \log f'_{i_1}(\Pi(\sigma \mathbf{i}))\, d\mu(\mathbf{i}) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log f'_{i_k}\left(\Pi(\sigma^k \mathbf{i})\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log f'_{i_k}\left(f_{i_{k+1},\dots,i_n}(\Pi(\sigma^n \mathbf{i}))\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log f'_{i_1,\dots,i_n}(\Pi(\sigma^n \mathbf{i})).
\end{aligned}$$

So, the last limit exists for μ-a.e. $\mathbf{i}$. □

Using that $f'_{i_1,\dots,i_n}$ is a monotone increasing function it follows from Lemma 1 that

Corollary 1. $\chi_{\mathcal{F}} \leq \chi$.

Since it was our principal assumption that $\chi < 0$, thus we obtain that

$$\chi_{\mathcal{F}} < 0. \tag{3.2}$$

In the dimension theory of contracting conformal IFS the so called "Bounded distortion lemma" has a very important role. We can hope here only to prove a weaker result:

Definition 2. *We say that the* weak distortion property *holds on a closed interval interval $J \subset [0,b)$ if*

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{x\in J} f'_{i_1,\dots,i_n}(x) = \lim_{n\to\infty} \frac{1}{n} \log \min_{x\in J} f'_{i_1,\dots,i_n}(x). \tag{3.3}$$

In the next two steps we are going to prove that the weak distortion property holds on every proper closed subinterval of the form $J = [0,t] \subsetneq [0,b]$. For a $k \in \mathbb{N}$ let

$$J_k := [0, f_{m_1,\dots,m_k}(0)], \tag{3.4}$$

where we choose $(m_1,\dots,m_k)$ such that for all $(i_1,\dots,i_k)$ we have

$$f_{m_1,\dots,m_k}(0) \geq f_{i_1,\dots,i_k}(0). \tag{3.5}$$

Proposition 3. (Weak distortion) *The weak distortion property holds on J_k for every $k \in \mathbb{N}$.*

Proof. Since $\chi_{\mathcal{F}} < 0$, we can choose $0 < \varepsilon < -\chi_{\mathcal{F}}$. Fix $N = N(\varepsilon)$ such that $\mu(\Omega_\varepsilon) > 1-\varepsilon$ where

$$\Omega_\varepsilon := \left\{ \mathbf{i} \in \Sigma : \forall n \geq N, f'_{i_1,\dots,i_n}(0) < e^{n(\chi_{\mathcal{F}}+\varepsilon)} \right\}.$$

Let $\Omega_\varepsilon^c := \Sigma \setminus \Omega_\varepsilon$. Define

$$\mathcal{R}_1 := \{ i_1 \in \{1,\dots,m\} : [i_1, m_1,\dots,m_k] \subset \Omega_\varepsilon^c \}$$

and

$$\mathcal{R}_n := \{ (i_1,\dots,i_n) : (i_1,\dots,i_j) \notin \mathcal{R}_j, \forall 1 \leq j < n; [i_1,\dots,i_n,m_1,\dots,m_k] \subset \Omega_\varepsilon^c \}.$$

Let

$$R_n := \bigcup_{(i_1,\dots,i_n)\in\mathcal{R}_n} [i_1,\dots,i_n], \qquad W_n := \bigcup_{(i_1,\dots,i_n)\in\mathcal{R}_n} [i_1,\dots,i_n,m_1,\dots,m_k]$$

The rest of the proof of the lemma is organized in 5 claims.

Claim 1. *If $(i_1,\dots,i_\ell) \in \mathcal{R}_\ell$ and $(j_1,\dots,j_n) \in \mathcal{R}_n$ with $\ell \neq n$, then*

$$[i_1,\dots,i_\ell] \cap [j_1,\dots,j_n] = \emptyset. \tag{3.6}$$

In fact, we may assume that $\ell < n$. Since $(j_1,\dots,j_n) \in \mathcal{R}_n$, by definition $(j_1,\dots,j_\ell) \notin \mathcal{R}_\ell$. Thus $(i_1,\dots,i_\ell) \neq (j_1,\dots,j_\ell)$. This implies the claim immediately.

Claim 2. *The set $\bigcup_{n=1}^{\infty} W_n$ is equal to the disjoint union*

$$\bigcup_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in\mathcal{R}_n} [i_1,\dots,i_n,m_1,\dots,m_k] \subset \Omega_\varepsilon^c.$$

In fact, by the definition of $\mathcal{R}_n$, we have $[i_1, \ldots, i_n, m_1, \ldots, m_k] \subset \Omega_\varepsilon^c$. The disjointness is ensured by the previous claim.

Claim 3. $\mu\left(\bigcup_{n=1}^{\infty} R_n\right) \leq \frac{\varepsilon}{p_{m_1,\ldots,m_k}}$.

Notice that $\mu([i_1, \ldots, i_n]) = \frac{1}{p_{m_1,\ldots,m_k}} \mu([i_1, \ldots, i_n, m_1, \ldots, m_k])$. Summing up this equality for all $(i_1, \ldots, i_n) \in \mathcal{R}_n$, we get $\mu\left(\bigcup_n R_n\right) = \frac{1}{p_{m_1,\ldots,m_k}} \mu\left(\bigcup_n W_n\right)$. Thus Claim 3 follows from Claim 2 and the fact that $\mu\left(\Omega_\varepsilon^c\right) < \varepsilon$.

Claim 4. *For all* $\mathbf{i} \notin \bigcup_{n=1}^{\infty} R_n$ *and* $n \geq N(\epsilon)$ *we have*

$$f'_{i_1,\ldots,i_n}(f_{m_1,\ldots,m_k}(0)) < \frac{1}{f'_{m_1,\ldots,m_k}(0)} e^{(n+k)(\chi_{\mathcal{F}}+\varepsilon)}. \tag{3.7}$$

Fix $n \geq N$ and $\mathbf{i} \notin \bigcup_{\ell=1}^{\infty} R_\ell$. Since $(i_1, \ldots, i_\ell) \notin \mathcal{R}_\ell$ for all $\ell \leq n$, there exists $\mathbf{j} \in [i_1, \ldots, i_n, m_1, \ldots, m_k] \cap \Omega_\varepsilon$. On one hand, by the definition of Ω_ε, the fact on $\mathbf{j}$ implies

$$f'_{j_1,\ldots,j_{n+1},\ldots,j_{n+k}}(0) < e^{(n+k)(\chi_{\mathcal{F}}+\varepsilon)}.$$

On the other hand, since $(j_1, \ldots, j_{n+1}, \ldots, j_{n+k}) = (i_1, \ldots, i_n, m_1, \ldots, m_k)$, we have $f'_{i_1,\ldots,i_n,m_1,\ldots,m_k}(0) = f'_{j_1,\ldots,j_{n+1},\ldots,j_{n+k}}(0)$. Thus the above inequality is just the claim.

Claim 5. *For all* $\mathbf{i} \notin \bigcup_n R_n$ *and* $n \geq N$ *we have*

$$\max_{x \in J_k} f'_{i_1,\ldots,i_n}(x) \leq \frac{1}{f'_{m_1,\ldots,m_k}(0)} e^{(n+k)(\chi_{\mathcal{F}}+\varepsilon)}. \tag{3.8}$$

By the chain rule, $f'_{i_1,\ldots,i_n}(x)$ is a product of increasing positive functions. So $f'_{i_1,\ldots,i_n}(x)$ itself is an increasing function. By using Claim 4 and the definition of J_k, we obtain the Claim 5.

Now we can finish the proof of Proposition 3. It follows from Claims 3 and 5 that for μ a.e. $\mathbf{i} \in \Sigma$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \max_{x \in J_k} f'_{i_1,\ldots,i_n}(x) \leq \chi_{\mathcal{F}}; \tag{3.9}$$

On the other hand, using again the fact that $f'_{i_1,\ldots,i_n}(x)$ is a monotone increasing function we see that

$$\lim_{n\to\infty} \frac{1}{n} \log \min_{x \in J_k} f'_{i_1,\ldots,i_n}(x) = \lim_{n\to\infty} \frac{1}{n} \log f'_{i_1,\ldots,i_n}(0) = \chi_{\mathcal{F}}.$$

This, together with (3.9), completes the proof of Proposition 3. □

Lemma 2. *For any $0 < t < b$ the weak distortion property (3.10) holds on the interval $[0,t]$ as well. That is*

$$\lim_{n\to\infty}\frac{1}{n}\log\min_{x\in[0,t]} f'_{i_1,\dots,i_n}(x)=\lim_{n\to\infty}\frac{1}{n}\log\max_{x\in[0,t]} f'_{i_1,\dots,i_n}(x)=\chi_{\mathcal{F}} \tag{3.10}$$

Proof. It is a consequence of Proposition 3 and the following fact: J_k tends to $[0,b]$ as $k\to\infty$. This is so because $f_m(z) > z$ for $z < b$ and $f_m(b) = b$. Then the iterates $f_{m,\cdots,m}(0)$ tend to b. □

Proposition 4. *If the RIFS is contracting on average and $b=\infty$ then the push down measure ν is supported on $[0,\infty)$. That is $\nu(\{\infty\}) = 0$.*

Proof. What we have to show is $\Pi(\mathbf{i}) < \infty$ μ-a.e. Consider the interval $J := [0,u]$ with $u = \max\limits_{1\le i\le m} f_i(0)$. Take an $\varepsilon > 0$ such that $\chi_{\mathcal{F}} + \epsilon < 0$. By the definition of $\chi_{\mathcal{F}}$ and by Lemma 2, there exists an $N \in \mathbb{N}$ and a set $\Omega\subset\Sigma$ such that $\mu(\Omega) > 1-\varepsilon$ and for all $\mathbf{i}\in\Omega$, $\forall n\ge N$, and $\forall y\in J$

$$f'_{i_1,\dots,i_n}(y) < e^{n(\chi_{\mathcal{F}}+\varepsilon)}. \tag{3.11}$$

Let $L := \max\limits_{i_1,\dots,i_N}\{f_{i_1,\dots,i_N}(0)\}$. Then for any $k > N$, by the Lagrange theorem we have

$$\begin{aligned} f_{i_1,\dots,i_k}(0) &\le L+\sum_{\ell=N}^{k-1}|f_{i_1,\dots,i_{\ell+1}}(0)-f_{i_1,\dots,i_\ell}(0)| \\ &\le L+u\sum_{\ell=N}^{k-1}\max_{y\in J}|f'_{i_1,\dots,i_\ell}(y)| \\ &\le L+u\sum_{\ell=N}^{k-1}e^{\ell(\chi_{\mathcal{F}}+\varepsilon)}\le L+u\frac{e^{N(\chi_{\mathcal{F}}+\varepsilon)}}{1-e^{\chi_{\mathcal{F}}+\varepsilon}}:=K \end{aligned} \tag{3.12}$$

Therefore, $\Pi(\mathbf{i}) < K$ holds for all $\mathbf{i}\in\Omega$. It follows that $\Pi(\mathbf{i}) < \infty$ for μ-almost all $\mathbf{i}$. □

Proposition 5 (Integral representation of $\chi_{\mathcal{F}}$)**.**

$$\chi = \chi_{\mathcal{F}}.$$

Proof. There are two cases to settle:
Case I: for $b<\infty$, $\nu(\{b\}) = 1$ holds.

Let $\varepsilon > 0$ be arbitrary fixed. Using Corollary 1 it is enough to prove that

$$\chi_{\mathcal{F}} > \chi-\varepsilon. \tag{3.13}$$

Let $L := \max_i \max_x \left|\left(\log f_i'\right)'(x)\right|$. This is finite since $f \in \mathcal{C}^2$ and $f_i'(x)$ is separated from zero. Put $M := \max_i |\log f_i'(b)|$. We write $c := \min_i \log f_i'(0)$. We often use in this proof the fact that both $f'_{i_1,\dots,i_n}$ and $f_{i_1,\dots,i_n}$ are monotone increasing functions. Using that the system is contracting on average, it follows from f_i' increasing, $i = 1, \dots, m$ that $c < 0$. Let $\delta > 0$ be chosen such that

$$\delta\left(1 + L + M - 2c\right) < \varepsilon$$

Using that $\lim_{n\to\infty} f_{i_1,\dots,i_n}(0) = b$ for almost all $\mathbf{i} \in \Sigma$, we can choose an N such that for the set

$$\Omega_\delta := \bigcup_{f_{i_1,\dots,i_N}(0) > b-\delta} [i_1, \dots, i_N]$$

we have

$$\mu(\Omega_\delta) > 1 - \frac{\delta}{2}. \tag{3.14}$$

Since from Birkhoff Ergodic Theorem

$$\frac{1}{n}\#\left\{0 \le \ell < n : \sigma^\ell \mathbf{i} \in \Omega_\delta\right\} \longrightarrow \mu(\Omega_\delta),$$

by (3.14) we can choose a K_1 such that the measure of the set

$$H_\delta = \left\{\mathbf{i} \in \Sigma : \forall n > K_1 \text{ we have } \frac{1}{n}\#\left\{0 \le \ell < n : \sigma^\ell \mathbf{i} \in \Omega_\delta\right\} > 1 - \delta\right\}$$

is

$$\mu(H_\delta) > 1 - \delta. \tag{3.15}$$

Finally, it follows from Birkhoff Ergodic Theorem that for μ almost all $\mathbf{i} \in \Sigma$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \log f'_{i_\ell}(b) = \chi = \sum_{j=1}^{m} p_j \log f_j'(b). \tag{3.16}$$

Therefore we can choose K_2, such that the measure of the set

$$Z_\delta := \left\{\mathbf{i} \in \Sigma : \forall n \ge K_2,\ \left|\frac{1}{n}\sum_{\ell=1}^{n} \log f'_{i_\ell}(b) - \chi\right| < \delta\right\}$$

satisfies

$$\mu(Z_\delta) > 1 - \delta. \tag{3.17}$$

We write

$$K := \max\left\{K_1, K_2\right\}.$$

Let R be any natural number satisfying:

$$R > K + N \text{ and } \frac{N+1}{R} < \delta,$$

and let

$$\mathbf{i} \in H_\delta \bigcap Z_\delta.$$

Now we prove that

$$\frac{1}{R} \log f'_{i_1,\dots,i_R}(0) > (1-\delta)\chi - \varepsilon, \tag{3.18}$$

which completes the proof of (3.13) since δ can be arbitrarily small. In the following formula we partition the integers between 1 and $R-N-1$ into two sets: The "good" ones are $G_R := \left\{1 \le \ell \le R-N-1 : \sigma^\ell \mathbf{i} \in \Omega_\delta\right\}$. While the "bad" ones are $W_R := \left\{1 \le \ell \le R-N-1 : \sigma^\ell \mathbf{i} \notin \Omega_\delta\right\}$. We have

$$\begin{aligned}
\frac{1}{R}\log f'_{i_1,\dots,i_R}(0) &= \frac{1}{R}\sum_{\ell=1}^{R-N-1} \log f'_{i_\ell}\left(f_{i_{\ell+1},\dots,i_R}(0)\right) + \frac{1}{R}\sum_{\ell=R-N}^{R} \log f'_{i_\ell}\left(f_{i_{\ell+1},\dots,i_R}(0)\right) \\
&\ge \frac{1}{R}\sum_{\ell=1}^{R-N-1} \log f'_{i_\ell}\left(f_{i_{\ell+1},\dots,i_R}(0)\right) + \delta c \\
&\ge \frac{1}{R}\sum_{\ell\in W_R} c + \frac{1}{R}\sum_{\ell \in G_r}\left(\log f'_{i_l}(b) - \delta L\right) + \delta c \\
&\ge \frac{1}{R}\sum_{\ell=1}^{R-N-1} \log f'_{i_l}(b) - \delta M + 2\delta c - \delta L \\
&\ge (1-\delta)\left(\sum_{j=1}^{m} p_j \log f'_j(b) - \delta\right) - \delta M + 2\delta c - \delta L \\
&\ge \chi - \varepsilon.
\end{aligned}$$

Where we used Lagrange's theorem for the function $x \to \log f'_{i_l}(x)$ in the fourth step and we used (3.16) in the fifth step.

Case II: $\nu(\{b\}) < 1$.Then there exist k and $\{n_p\}_{p=1}^{\infty}$ such that for all p, $\Pi(\sigma^{n_p}(\mathbf{i})) \in J_k$. We immediately get the statement of our Lemma from Lemma 1 and Proposition 3, where J_k is as it was defined in (3.4). □

So we have proved that $\chi_{\mathcal{F}} = \chi$. Now we prove that our Liapunov exponent is invariant under coordinate change. Given two sets of maps $\mathcal{F} = \{f_1,\dots,f_m\}$, $\mathcal{G} = \{g_1,\dots,g_m\}$ satisfying the conditions (2.1)–(2.4) and a probability vector $\mathbf{p} = \{p_1,\dots,p_m\}$. We write $b_{\mathcal{F}}$ and $b_{\mathcal{G}}$ for b appearing in the conditions (2.1)–(2.4). We consider two systems $(\mathcal{F},\mathbf{p})$ and $(\mathcal{G},\mathbf{p})$ and write their Liapunov exponents by $\chi_{\mathcal{G}}$ and $\chi_{\mathcal{F}}$.

Proposition 6. *If $\mathcal{F}$ is conjugate to $\mathcal{G}$ in the sense that there exist some $\varphi : [0, b_{\mathcal{F}}] \to [0, b_{\mathcal{G}}]$ strictly increasing bijection which is $\mathcal{C}^1$ on $[0, b_{\mathcal{F}})$ such that*

$$g_i(u) = \varphi \circ f_i \circ \varphi^{-1}(u) \tag{3.19}$$

holds for all $u \in [0, b_{\mathcal{G}}]$, then $\chi_{\mathcal{F}} = \chi_{\mathcal{G}}$.

We remark that for the most commonly used notion of Liapunov exponent χ_L of a RIFS satisfying (1.3), the same does not hold as our motivating example shows. We also remark that $\mathcal{F}$ is conjugate to $\mathcal{G}$ doesn't mean that $\mathcal{G}$ is conjugate to $\mathcal{F}$.

Proof. First we observe that for all $x \in [0, b)$ we have $\varphi'(x) \neq 0$ (although $\lim_{x \to b_{\mathcal{F}}} \varphi'(x) = 0$ is possible). Then

$$g'_{i_1,\dots,i_n}(0) = \varphi'(f_{i_1,\dots,i_n}(\varphi^{-1}(0))) \cdot f'_{i_1,\dots,i_n}(\varphi^{-1}(0)) \cdot (\varphi^{-1})'(0). \tag{3.20}$$

However, using the facts $\varphi^{-1}(0) = 0$ and $f_{i_1,\dots,i_n}(0) < \Pi(\mathbf{i}) < b_{\mathcal{F}}$, we obtain that

$$0 < \varphi'(0) < \varphi'(f_{i_1,\dots,i_n}(\varphi^{-1}(0))) < \varphi'(\Pi(\mathbf{i})) < \infty$$

since almost surely $\Pi(\mathbf{i}) < b_{\mathcal{F}}$. Thus

$$\chi_{\mathcal{G}} = \lim_{n \to \infty} \frac{1}{n} \log g'_{i_1,\dots,i_n}(0) = \lim_{n \to \infty} \frac{1}{n} \log f'_{i_1,\dots,i_n}(0) = \chi_{\mathcal{F}}.$$

□

4. Essential uniqueness of invariant measures

Although in general there are infinitely many invariant measures, we are going to prove that every invariant measure is a convex combination of the push down measure ν and some atomic measure.

Proposition 7. *Let ν_0 be an invariant probability measure such that $\nu_0(\{b\}) = 0$. Then ν_0 is a non-atomic measure, that is $\nu_0(\{x\}) = 0$ for all x.*

Proof. We set $h(x) := \nu_0(\{x\})$ and $H := \{x : h(x) > 0\}$. We have only to show that $H = \emptyset$. Suppose $H \neq \emptyset$. We will deduce a contradiction. Let $z = \max_{x \in H} h(x)$. Since $\sum_{x \in H} h(x) \le 1$, the set $H_0 := \{x : h(x) = z\}$ is finite and nonempty. It follows from (2.6) that for every $x \in H_0$ and for all $1 \le i \le m$, $f_i^{-1}(x)$ exists and belongs to H_0. Put $y := \max H_0$. In particular, $f_m^{-1}(y) \in H_0$. However, this is impossible since $f_m^{-1}(y) > y$. □

In the case of contracting IFS we know that the invariant measure must be unique. For the systems considered in the present paper, it may occur that δ_b is invariant (when b is a common fixed point) and there is still the push down measure ν supported on $[0, b)$, which is invariant. Then there are at least two essentially different invariant measures $\delta_{\{b\}}$ and ν (it is possible that $\nu = \delta_b$). Therefore the convex combinations of δ_b and ν are invariant measures. Below we point out that there are no more invariant measures.

Proposition 8 (Essential uniqueness)**.** *Let ν_0 be an invariant probability measure such that $\nu_0(\{b\}) = 0$. Then $\nu_0 = \nu$. (We remind that ν is the push down measure of μ.)*

Proof. First, we are going to prove

$$\nu_0([0,t)) \le \nu([0,t)) \qquad (0 < t < b). \tag{4.1}$$

Fix a $0 < t < b$. Let $A_n := \{\mathbf{i} \in \Sigma : f_{i_1,\dots i_n}(0) > t\}$. Then $\{A_n\}$ is an increasing sequence so,

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup A_n\right) = \nu((t,b]). \tag{4.2}$$

Using that $f^{-1}_{i_1,\dots,i_n}(t,b] = [0,b]$ for an $\mathbf{i} \in A_n$, we get

$$\begin{aligned} \nu_0((t,b]) &= \sum_{i_1,\dots,i_n} p_{i_1,\dots,i_n} \nu_0(f^{-1}_{i_1,\dots,i_n}(t,b]) \\ &\ge \sum_{(i_1,\dots,i_n)\in A_n} p_{i_1,\dots,i_n} \underbrace{\nu_0(f^{-1}_{i_1,\dots,i_n}(t,b])}_{1} = \mu(A_n). \end{aligned} \tag{4.3}$$

So,

$$\nu_0((t,b]) \ge \nu((t,b]). \tag{4.4}$$

From this and Proposition 7 we get

$$\nu_0([0,t)) \le \nu([0,t)). \tag{4.5}$$

We can apply Proposition 7 for the measure ν too. Namely, it follows from (4.4) that $1 = \nu_0\left(\cup_{t<b}[0,t]\right) \le \nu\left(\cup_{t<b}[0,t]\right) = \nu\left([0,b)\right)$. So we get

$$\nu(\{b\}) = 0 \tag{4.6}$$

which results that we can apply Proposition 7.

Finally we show the reverse inequality

$$\nu([0,t)) \le \nu_0([0,t)) \qquad (0 < t < b). \tag{4.7}$$

Let $\delta > 0$ be arbitrary positive number. Take a $\xi \in (0, b)$ such that $\nu_0([\xi, b]) < \delta$. Notice that if $f_{i_1,\cdots,i_n}(\xi) \le t$, then we have $f^{-1}_{i_1,\cdots,i_n}([t, b]) \subset [\xi, b]$ by the monotonicity, thus $\nu_0(f^{-1}_{i_1,\cdots,i_n}([t, b])) < \delta$. So

$$\begin{aligned}
\nu_0([t,b]) &= \sum_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n} \cdot \nu_0\left(f^{-1}_{i_1,\ldots,i_n}([t,b])\right) \\
&= \sum_{f_{i_1,\ldots,i_n}(\xi)>t} + \sum_{f_{i_1,\ldots,i_n}(\xi)\le t} \\
&\le \sum_{f_{i_1,\ldots,i_n}(\xi)>t} p_{i_1,\ldots,i_n} + \delta \\
&= \mu\{\mathbf{i} \mid f_{i_1,\cdots,i_n}(\xi) > t\} + \delta
\end{aligned} \tag{4.8}$$

holds for all n. On the other hand, ν is also an invariant measure. So, we can apply (2.6) for ν to get

$$\nu([0,t)) = \sum_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n} \nu\left(f^{-1}_{i_1,\ldots,i_n}([0,t))\right). \tag{4.9}$$

Using that $f_{i_1,\ldots,i_n}(0) \ge t$ implies $f^{-1}_{i_1,\ldots,i_n}([0,t)) = \emptyset$ we obtain

$$\nu([0,t)) \le \sum_{f_{i_1,\ldots,i_n}(0)<t} p_{i_1,\ldots,i_n} = \mu\{\mathbf{i} \mid f_{i_1,\ldots,i_n}(0) < t\}. \tag{4.10}$$

also holds for all n. If we knew that for any $\varepsilon > 0$ there exists n such that

$$\mu\{\mathbf{i} \mid f_{i_1,\cdots,i_n}(0) \le t \text{ and } f_{i_1,\cdots,i_n}(\xi) > t\} < \varepsilon, \tag{4.11}$$

then choosing this n in the inequalities (4.8) and (4.10) we would conclude that

$$\nu([0,t)) + \nu_0([t,b]) \le 1 + \varepsilon + \delta.$$

Here we have used the identity $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$. So,

$$\nu([0,t)) \le 1 - \nu_0([t,b]) + \varepsilon + \delta = \nu_0([0,t)) + \varepsilon + \delta,$$

which would complete the proof of (4.7) since $\varepsilon, \delta > 0$ were arbitrary.

To complete the proof we have only to show that for any $\varepsilon > 0$ there exists n such that (4.11) holds.

Since the push down measure ν is non-atomic, we can find $\eta > 0$ such that $\mu(H_0) < \frac{\varepsilon}{5}$ where $H_0 := \{\mathbf{i} \mid \Pi(\mathbf{i}) \in (t - \eta, t + \eta)\}$. Since $\Pi(\mathbf{i}) = \lim_n f_{i_1,\ldots,i_n}(0)$ for all $\mathbf{i}$ and $\Pi(\mathbf{i}) = \lim_n f_{i_1,\ldots,i_n}(\xi)$ for μ-almost all $\mathbf{i}$. By the Egorov theorem we can find an integer N and a small set $H_1 \subset \Sigma$ with $\mu(H_1) < \frac{\varepsilon}{5}$ such that

$$\mid f_{i_1,\cdots,i_n}(0) - \Pi(\mathbf{i}) \mid < \frac{\eta}{2}, \quad \mid f_{i_1,\ldots,i_n}(\xi) - \Pi(\mathbf{i}) \mid < \frac{\eta}{2}$$

for $n \geq N$ and $\mathbf{i} \notin H_1$. So $\mathbf{i} \notin H_0 \cup H_1$ then for any $n > \max\{N_1, N_2\}$ (4.11) holds. On the other hand $\mu(H_0 \cup H_1) < 2\varepsilon/5$. This completes our proof. $\square$

5. Proof of Theorem 1

As we already discussed, without loss of generality we may assume that

$$\nu_0 = \nu, \quad \nu((0,b)) = 1.$$

Choose a $k \in \mathbb{N}$ such that $\nu(J_k) > 0$, where J_k was defined as in Proposition 3. Since $\nu((0,b)) > 0$ such a k exists. We denote by G the (good) set of those $\mathbf{i} \in \Sigma$ for which the following three conditions are satisfied:

$$\lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in J_k} f'_{i_1,\dots,i_n}(x) = \chi_{\mathcal{F}}, \qquad \lim_{n\to\infty} \frac{\log \mu(\mathcal{E}_n(\mathbf{i}))}{n} = -H_\mu, \tag{5.1}$$

furthermore, there exists an infinite sequence $\{n_p\}_{p=1}^{\infty}$ of distinct natural numbers such that $\Pi(\sigma^{n_p}\mathbf{i}) \in J_k$. Then

$$\mu(G) = 1.$$

For the rest of the proof we fix an arbitrary $\varepsilon < -\chi$. It follows from the definition of G (see (5.1)) that for every $\mathbf{i} \in G$ we can choose $N = N(\mathbf{i}, \varepsilon)$ such that for every $n \geq N$ the following hold:

$$\left| \frac{1}{n} \log \sup_{x\in J_k} f'_{i_1,\dots,i_n}(x) - \chi_{\mathcal{F}} \right| < \varepsilon, \tag{5.2}$$

$$\left| \frac{\log \mu(\mathcal{E}_n(\mathbf{i}))}{n} - (-H_\mu) \right| < \varepsilon. \tag{5.3}$$

Lemma 3. *Let $\mathbf{i} \in G$ and $\varepsilon > 0$ be arbitrary fixed. We choose N as above. Put $r_n := 2|J_k| \cdot e^{n(\chi_{\mathcal{F}}+\varepsilon)}$. Then for every $n \geq N$ we have*

$$f_{i_1,\dots,i_{n_p}}(J_k) \subset \left[\Pi(\mathbf{i}) - r_{n_p}, \Pi(\mathbf{i}) + r_{n_p}\right], \tag{5.4}$$

Proof. Choose $p \in \mathbb{N}$ such that $n_p \geq N$ and let $x \in J_k$. Using that $\Pi(\sigma^{n_p}\mathbf{i}) \in J_k$ by definition, we obtain that

$$\begin{aligned} |\Pi(\mathbf{i}) - f_{i_1,\dots,i_n}(x)| &= \left| f_{i_1,\dots,i_{n_p}}(\Pi(\sigma^{n_p}\mathbf{i})) - f_{i_1,\cdots,i_{n_p}}(x) \right| \\ &\leq |J_k| \max_{y\in J_k} f'_{i_1,\dots,i_{n_p}}(y) \leq |J_k|\, e^{n_p(\chi_{\mathcal{F}}+\varepsilon)}. \end{aligned} \tag{5.5}$$

$\square$

Now we can prove our Theorem.

Proof of Theorem 1. We use the well known theorem that for a probability Radon measure m on $\mathbb{R}^d$ we have

$$\dim_H(m) = \text{ess sup}_x \liminf_{r\to 0} \frac{\log m(B(x,r))}{\log r}, \tag{5.6}$$

where $B(x,r)$ is the ball centered at x with radius r (see [3]). Let us write

$$I(\mathbf{i}, n) := [\Pi(\mathbf{i}) - r_n, \Pi(\mathbf{i}) + r_n].$$

By Lemma 3, with $n = n_p$ we have

$$\nu(I(\mathbf{i}, n)) = \mu \circ \Pi^{-1}(I(\mathbf{i}, n)) \geq \mu \circ \Pi^{-1}(f_{i_1,\cdots,i_n}(J_k)).$$

Observe that

$$\Pi^{-1}(f_{i_1,\dots,i_n}(J_k)) \supset \bigcup_{(j_1,\dots,j_n)\sim(i_1,\dots,i_n)} \left(j_1, \dots, j_n, \Pi^{-1}(J_k)\right),$$

where $(j_1, \dots, j_n, \Pi^{-1}(J_k))$ is the subset in Σ defined by

$$\{\tau \in \Sigma : \tau_k = j_k \text{ for } 1 \leq k \leq n \text{ and } \Pi(\sigma^n(\tau)) \in J_k\}.$$

We obtain from (5.4) that

$$\nu(I(\mathbf{i}, n)) \geq \sum_{[j_1,\dots,j_n]\in\mathcal{E}_n(\mathbf{i})} p_{j_1\cdots j_n} \cdot \mu(\Pi^{-1}(J_k)). \tag{5.7}$$

Because k was chosen in such a way that $\mu(\Pi^{-1}(J_k)) > 0$, we get

$$\liminf_{n\to\infty} \frac{\log \nu(I(\mathbf{i}, n))}{\log e^{n(\chi_{\mathcal{F}}+\varepsilon)}} \leq \frac{1}{\chi_{\mathcal{F}} + \epsilon} \lim_{n\to\infty} \frac{1}{n} \log \sum_{[j_1,\dots,j_n]\in\mathcal{E}_n(\mathbf{i})} p_{j_1,\dots,j_n} \leq \frac{H_\mu}{-(\chi_{\mathcal{F}} + \varepsilon)}.$$

Then we get the desired conclusion by using (5.6). □

6. The origin of our motivating example

In the study of Oppenheim expansion of Laurent series over a finite field, we meet the following random series ([4]). Let $\lambda \geq 2$ be an integer and $q \geq 2$ be another integer. Let ν be the distribution measure of the following random series

$$\sum_{n=0}^{\infty} \epsilon_n \lambda^{-n} \tag{6.1}$$

where $\{\epsilon_n\}$ is an i.i.d. sequence of $\mathbb{N}$-valued random variables whose common law is geometric, i.e.

$$p_k := P(\epsilon_1 = k) = \frac{q-1}{q^k} \qquad (k = 1, 2, \cdots).$$

The measure ν can be viewed as the invariant measure of the infinite IFS defined by $f_k(x) = \frac{x+k}{\lambda}$ ($k = 1, 2, \cdots$) and with probability

$(p_1, p_2, \cdots)$. By considering the Fourier transform of the measure ν, we see that ν is singular with respect to the Lebesgue measure. In fact,

$$\widehat{\nu}(t) = e^{it/(\lambda-1)} \prod_{n=0}^{\infty} \frac{q-1}{q - e^{i\lambda^{-n}t}}. \tag{6.2}$$

It turns out that $|\widehat{\nu}(2\pi\lambda^n)| = |\widehat{\nu}(2\pi)| > 0$ ([4]). A natural question arises: what is the Hausdorff dimension of ν ?

Let us consider the following IFS:

$$f_0(x) = \frac{x}{\lambda}, \qquad f_1(x) = x + 1$$

with probability $p_0 = (q-1)/q$ and $p_1 = 1/q$. Let σ be the invariant measure of this IFS. Its Fourier transform satisfies

$$\widehat{\sigma}(t) = \frac{q-1}{q}\widehat{\sigma}(\lambda^{-1}t) + \frac{1}{q}e^{it}\widehat{\sigma}(t).$$

So,

$$\widehat{\sigma}(t) = \frac{q-1}{q-e^{it}}\widehat{\sigma}(\lambda^{-1}t).$$

Repeated application of this yields:

$$\widehat{\sigma}(t) = \frac{q-1}{q-e^{it}} \cdot \frac{q-1}{q-e^{i\lambda^{-1}t}} \cdots \frac{q-1}{q-e^{i\lambda^{-n}t}} \cdot \widehat{\sigma}(\lambda^{-(n+1)}t) \tag{6.3}$$

Using (6.2) we obtain that $\widehat{\nu}(t) = e^{it/(\lambda-1)}\widehat{\sigma}(t)$. This implies that μ is a translation of σ by $\frac{1}{\lambda-1}$. Therefore both measures ν and σ have the same dimension. The question of dimension for μ is the same for σ which is an invariant measure of a finite system.

Since both f_0 and f_1 are linear, we have

$$f'_{i_1,\cdots,i_n} = \lambda_{i_1} \cdots \lambda_{i_n}$$

where $\lambda_0 = \frac{1}{\lambda}$ and $\lambda_1 = 1$. It follows that

$$\chi_{\mathcal{F}} = -\frac{q-1}{q}\log\lambda.$$

Thus we get

Theorem 2. *We have*

$$\dim_{\mathrm{H}} \nu = \dim_{\mathrm{H}} \sigma \leq \frac{q}{(q-1)\log\lambda}\left(\frac{q-1}{q}\log\frac{q}{q-1} + \frac{1}{q}\log q\right). \tag{6.4}$$

The upper bounded is effective if it is smaller than 1. This is the case if $\lambda \geq \frac{q^q}{q-1}$. Notice that the preceding arguments hold even for real numbers $\lambda > 1$ and $q > 1$. In [9, Example 3.2] the authors discussed the special case of $p_0 = p_1 = \frac{1}{2}$. It has been proved in [9, Theorem 2.2] that for $\lambda > 4$ the dimension of the measure ν :

$$\dim_H(\nu) \leq \frac{2\log 2}{\log \lambda} < 1. \tag{6.5}$$

Let us make the following coordinate change $\varphi[0,\infty) \to [0,1)$:

$$\varphi(x) := \frac{x}{x+1}, \text{ where }, \quad \varphi(\infty) := 1.$$

We define

$$g_i(x) := \begin{cases} (\varphi \circ f_i \circ \varphi^{-1})(u), & \text{if } 0 \leq u \leq 1, \\ 1, & \text{if } u = 1. \end{cases}$$

Then

$$g_0(u) = \frac{u}{u + \lambda(1-u)}, \quad g_1(u) = \frac{1}{2-u}$$

and

$$g_0'(u) = \frac{\lambda}{(u+\lambda(1-u))^2}, \quad g_1'(u) = \frac{1}{(2-u)^2}.$$

The appropriate measure on $[0,1]$:

$$\eta := \nu \circ \varphi^{-1} = \mu \circ \Pi_{\mathcal{G}},$$

where $\mathcal{G} = \{g_0, g_1\}$ and

$$\Pi_{\mathcal{G}} := \lim_{n\to\infty} g_{i_1,\dots,i_n}(0) = \lim_{n\to\infty} \varphi \circ f_{i_1,\dots,i_n}(\varphi^{-1}(0)) = \lim_{n\to\infty} \varphi \circ f_{i_1,\dots,i_n}(0) = \varphi(\Pi_{\mathcal{F}}(\mathbf{i})).$$

Therefore we know that

$$\dim_H \eta = \dim_H \nu \leq \frac{H_\mu}{-\chi}.$$

However, one can easily see that the logarithmic growth rate of the Lipschitz constant is:

$$\frac{1}{n}\log \|g_{i_1,\dots i_n}\| \longrightarrow \frac{1}{2}\log\lambda > 0$$

thus [9, Theorem 2.2] does not apply to the system $\{g_0, g_1\}$.

7. Application: another example

Let $f_1, f_2 : [0,1] \to [0,1]$ (see Figure 2) be defined by

$$f_1(x) = \frac{1}{\sqrt{2}} \cdot x + 1 - \frac{1}{\sqrt{2}},$$

$$f_2(x) = \begin{cases} \beta x & \text{if } x \in [0, \frac{1}{2}] \\ h(x) & \text{if } x \in [\frac{1}{2}, (2-\beta)^{-1}] \\ 2x - 1 & \text{if } x \in ((2-\beta)^{-1}, 1] \end{cases}$$

where $\beta < 1$ is small and $h(x)$ is defined to make f_2 a strictly increasing, convex C^2 map.

Proposition 9. *Consider the IFS defined above $\mathcal{F} = \{f_1, f_2\}$ with the probability $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$. Let $\rho = \frac{\sqrt{5}-1}{2} = 0,618...$ be the golden mean. For the push down measure ν we have*

$$\dim_H \nu \leq \frac{\log 4}{\log \sqrt{2} - \rho \log \beta - (1-\rho) \log 2}.$$

This estimate is effective, i.e. $\dim_H \nu < 1$ *whenever* $0 < \beta < 0,121135...$.

As Figure 2 shows, this IFS clearly satisfies all of our assumptions made in Section 2. Our aim is to estimate $\chi_{\mathcal{F}}$ from above by using the integral representation of $\chi_{\mathcal{F}}$. In order to do so, we are led to give a lower bound on $\nu([0, \frac{1}{2}])$ since f_2 is strongly contracting on this interval. In fact, we will prove that $\nu([0, \frac{1}{2}]) \geq \rho > 0.618$. Then choosing $\beta > 0$ small enough we can prove that $\dim_{\mathrm{H}} \nu < 1$.

To estimate $\nu([0, \frac{1}{2}])$, we introduce the following new IFS. Let $\mathcal{G} := \{g_1, g_2\}$ with probability vector $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$, where $g_1, g_2 : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ are respectively defined as follows

$$g_1(x) = \frac{1}{\sqrt{2}} \cdot x + 1 - \frac{1}{\sqrt{2}}$$

$$g_2(x) = \begin{cases} \frac{1}{2} & \text{if} \quad x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - 1 & \text{if} \quad x \in (\frac{3}{4}, 1] \end{cases}$$

The reason for introducing the new system is as follows (see the next two lemmas). Put

$$x_{-2} := 0 \text{ and } x_k := f_1^{k+2}(0) \text{ for } k = -1, 0, 1, 2, \dots$$

It is nothing but the orbit of 0 under f_1, ordered by $\{-2, -1, 0, 1, 2, \cdots\}$ so that $\{x_n\}_{n \geq 0}$ is the orbit of $\frac{1}{2}$ under g_1 (notice that $x_0 = \frac{1}{2}$). Notice that $g_1 = f_1|_{[\frac{1}{2}, 1]}$ and $g_2 \geq f_2|_{[\frac{1}{2}, 1]}$.

Lemma 4. *For any $n \geq 1$ and $i_1, \dots, i_n$, we have*

$$g_{i_1, \dots, i_n}(x_0) \in \{x_k\}_{k=0}^{\infty}.$$

Proof. We prove the lemma by induction on n. The case $n = 1$ is true because $g_1(x_0) = x_1$ and $g_2(x_0) = \frac{1}{2} = x_0$. Assume that the inclusion is true for n. Let $g_{i_1, \dots, i_n}(x_0) = x_k = g_1^k(x_0)$ for some $k \geq 0$. Since $g_1(x_k) = x_{k+1}$, we have only to check the three facts:

$$g_2(x_0) = x_0, \quad g_2(x_1) = x_0, \quad g_2(x_k) = x_{k-2} \text{ for } k \geq 2.$$

The first fact is seen. The second one is because of $x_1 = 1 - \frac{\sqrt{2}}{4} < \frac{3}{4}$. Notice that $x_2 = \frac{3}{4}$ and that x_k is increasing. So the third fact is equivalent to $2g_1^k(x_0) - 1 = g_1^{k-2}(x_0)$ which is easy to check. □

Lemma 5. *For every $n \geq 1$ and $i_1, \dots, i_n$, we have the equivalence*

$$g_{i_1,\dots,i_n}(x_0) = x_0 \text{ or } x_1 \Longleftrightarrow f_{i_1,\dots,i_n}(0) < \frac{1}{2}. \tag{7.1}$$

The proof of this lemma is left to the reader. It follows from the geometry of the graph of these two functions.

Now we are led to consider the random walk $X_{k+1} = g_{i_{k+1}}(X_k)$ starting from x_0 with states $\{x_k\}_{k\geq 0}$. Or equivalently we can consider a random walk on the non negative integers starting from 0. The random walker stays at 0 or moves to his neighbor 1 with equal probability $\frac{1}{2}$. When the random walker is at 1 he jumps to his neighbors 0 or 2 also with equal probability $\frac{1}{2}$. If the walker is at any $n > 1$ then he jumps to $n-2$ or to $n+1$ with equal probability $\frac{1}{2}$. The probability transition matrix is given below:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \cdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The reason that we are interested in this random walk is as follows. Let $q_k^{(n)} = P(X_n = k)$ be the probability that the random walker is at k after n steps. By the construction of the random walk we have (see [5, p. 393])

$$q_k^{(n)} = \sum_{g_{i_1,\dots,i_n}(\frac{1}{2}) = x_k} p_{i_1,\dots,i_n}. \tag{7.2}$$

We compute the stationary measure $\mathbf{q} = (q_k)_{k\geq 0}$ for this random walk, where $q_k = \lim_n q_k^{(n)}$. It is the probability vector such that $\mathbf{q}P = \mathbf{q}$. In other words, the solution of the following system:

$$\begin{cases} \sum_{k\geq 0} q_k = 1, \quad q_k \geq 0 \ (\forall k \geq 0) \\ \qquad q_k = \frac{1}{2}\left(q_{k-1} + q_{k+2}\right) \\ \qquad q_0 = \frac{1}{2}\left(q_0 + q_1 + q_2\right) \end{cases}$$

The solution is $q(k) = \rho^{k+2}$ for $k \geq 0$. So,

$$q(0) + q(1) = \rho > 0.618 \tag{7.3}$$

Also we know by [5, p. 393] and by (7.2) that

$$q(k) = \lim_{n\to\infty} q_k^{(n)} = \lim_{n\to\infty} \sum_{g_{i_1,\dots,i_n}(\frac{1}{2})=x_k} p_{i_1,\dots,i_n}. \tag{7.4}$$

Now we are ready to estimate χ. Fix an $1 \leq i \leq m$ and write $\phi(x) = \log f_i'(x)$. Notice that ϕ is bounded and $\Pi(\mathbf{i}) = \lim_n f_{i_1,\dots,i_n}(0)$. We have

$$\begin{aligned} \int_0^1 \phi(x)d\nu(x) &= \int (\phi(\Pi(\mathbf{i}))d\mu(\mathbf{i}) = \int \phi\left(\lim_{n\to\infty} f_{i_1,\dots,i_n}(0)\right) d\mu(\mathbf{i}) \\ =& \lim_{n\to\infty} \int \phi\left(f_{i_1,\dots,i_n}(0)\right) d\mu(\mathbf{i}) = \lim_{n\to\infty} \sum_{i_1,\dots,i_n} p_{i_1,\dots,i_n} \cdot \phi\left(f_{i_1,\dots,i_n}(0)\right) \\ \leq& \phi\left(\frac{1}{2}\right) \lim_{n\to\infty} \sum_{f_{i_1,\dots,i_n}(0)<\frac{1}{2}} p_{i_1,\dots,i_n} + \phi(1) \lim_{n\to\infty} \sum_{f_{i_1,\dots,i_n}(0)\geq\frac{1}{2}} p_{i_1,\dots,i_n}. \end{aligned}$$

By Lemma 5, the first limit in the last expression is equal to

$$\lim_{n\to\infty} \sum_{\ell=0}^{1} \sum_{g_{i_1,\dots,i_n}(\frac{1}{2})=x_\ell} p_{i_1,\dots,i_n} = q_0 + q_1 = \rho$$

and then the second limit is equal to $1-\rho$. Here we used (7.4) and (7.3). Thus we get

$$\int_0^1 \phi(x)d\nu(x) \leq \rho\phi\left(\frac{1}{2}\right) + (1-\rho)\phi(1).$$

Notice that $f_1'(1/2) = f_1'(1) = 1/\sqrt{2}$, $f_2'(1/2) = \beta$ and $f_2'(1) = 2$. We can obtain

$$\sum_{i=1}^{m} p_i \int \log f_i'(x) d\nu(x) \le \frac{\rho}{2} \log\left(\frac{\beta}{\sqrt{2}}\right) + \frac{1-\rho}{2} \log\left(\frac{2}{\sqrt{2}}\right)$$

$$= \frac{1}{2}(\rho \log \beta + (1-\rho)\log 2 - \log\sqrt{2}).$$

So by Theorem 1 we get

$$\dim_H \nu \le \frac{2\log 2}{\log\sqrt{2} - \rho\log\beta - (1-\rho)\log 2}.$$

This completes the proof of Proposition 9.

Acknowledgement

We would like to say thanks to the referees for the many valuable suggestions.

References

[1] M. Barnsley, J. H. Elton, *A new class of Markov Processes for image encoding*, Annals of Applied Probability, **20**, 14-32, 1988.

[2] R. P. Diaconis and D. Freedman, *Iterated random functions*, SIAM Review, **41**, (1999), 45-76.

[3] A. H. Fan, *Sur les dimensions de mesures*, Studia Math, **111**, (1994), 1-17.

[4] A. H. Fan and J. Wu, *Metric properties and exceptional sets of the Oppenheim expansions over the field of Laurent series*, Constr. Approx. **20** (2004), 465-495.

[5] W. Feller, An introduction of probability theory and its applications, Vol. 1, *John Wiley & Sons*, New York, 1968.

[6] U.Krengel, Ergodic Theorems, *Walter de Gruyter*, 1985.

[7] S. Lalley, *Random series in powers of algebraic numbers: Hausdorff dimension of the limit distributuin*, J. London Math. Soc. , **57**, (1998), 629-654.

[8] J. Myjak, T. Szarek, *On Hausdorff dimension of invariant measures arising from non-contractive iterated function systems*, Annali di Mathematica **181**, (2002), 223–237.

[9] M. Nicol, N. Sidorov, D. Broomhead, *On the fine structure of stationary measures in systems which contract on average*, J. Theoret Probab. **15** no.3, (2002), 715-730.

[10] K. Simon, B. Solomyak, M. Urbanski, *Invariant meausures for parabolic IFS with overlaps and random continued fractions*, Trans. Amer. Math. Soc. **353**, (2001), 5145–5164.

[11] D. Steinsaltz, *Locally contractive iterated function systems*, Annals of Probability, **27**, (1999), 1952-1979.

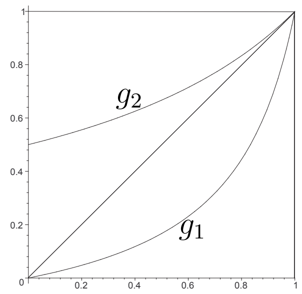

FIGURE 1. Repelling common fixpoint at $x = 1$.

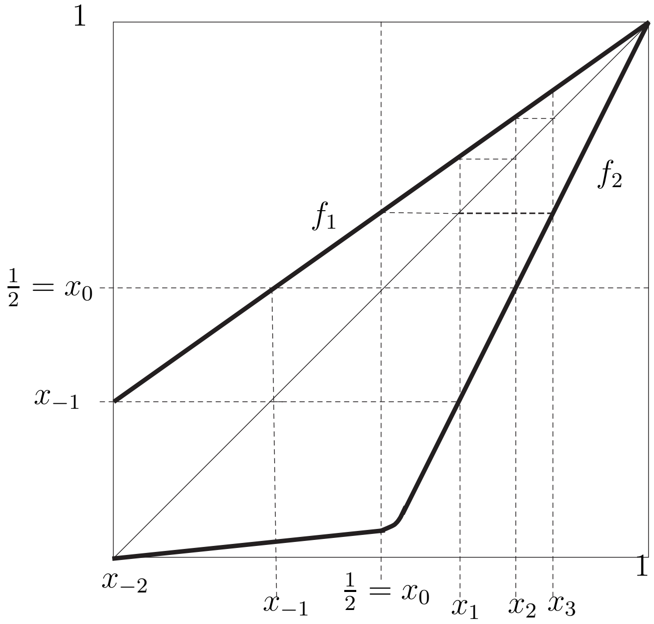

FIGURE 2. Repelling common fixed point at $x = 1$

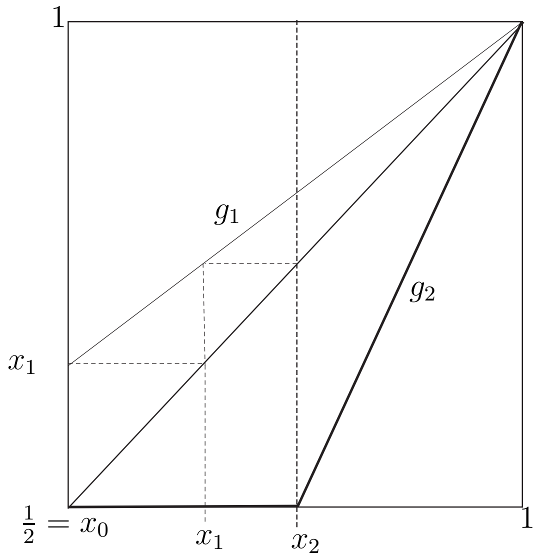

FIGURE 3. Maps g_1, g_2

Figure Captions:
The file: motivation.tex, first.tex, second.tex are attached.
Figure: motivation.eps caption:
Repelling common fixpoint at $x = 1$.

Figure: first3.eps caption:
Repelling common fixed point at $x = 1$

Figure: second.eps caption:
Maps g_1, g_2

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