Multifractal analysis for self-affine IFS

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3 September, 2007

The problem

Given $(X, \mathcal{B}, \mu, T)$, where μ is ergodic and given a Hölder continuous real function f on X.We know that: For a μ typical $x \in X$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f d\mu.$$

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For a given β we would like to find out how big is the set

$$\left\{ x : \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) \to \beta \right\}.$$

The same for IFS

Given an IFS $\{S_i\}_{i=1}^m$ on $\mathbb{R}^d$. On the symbolic space $\Sigma := \{1, \ldots, m\}^{\mathbb{N}}$ we define the natural projection $\Pi : \Sigma \to \Lambda$ (Λ is the attractor):

$$\Pi(i_1, i_2, \ldots) := \lim_{n\to\infty} \underbrace{S_{i_1} \circ \cdots \circ S_{i_n}}_{S_{i_1\ldots i_n}}(\mathbf{0}).$$

given a Hölder continuous $f : \Sigma \to \mathbb{R}$.The problem previously stated for dynamical systems has the following form for IFS:

$\dim_{\mathrm{H}}(K_\beta) = ?$

$$K_\beta := \Pi\left\{\mathbf{i} \in \Sigma : \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) = \beta\right\}$$

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An Example:

Let $S_1(x) = x/3, S_2(x) := x/3 + 2/3$.
$\Sigma := \{1,2\}^{\mathbb{N}}$, Λ is the triadic Cantor set and Π is the natural coding of the elements of Λ by $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$. Further, let $I_n(x)$ be the n-th level cylinder interval that contains $x \in \Lambda$. We define the Bernoulli measure for a given $0 < p < 1$ on C by $\mu(I_n(x)) := p_{i_1} \cdots p_{i_n}$, where $p_1 := p, p_2 := 1 - p$. Let $f(\mathbf{i}) := \log p_{i_1}$.

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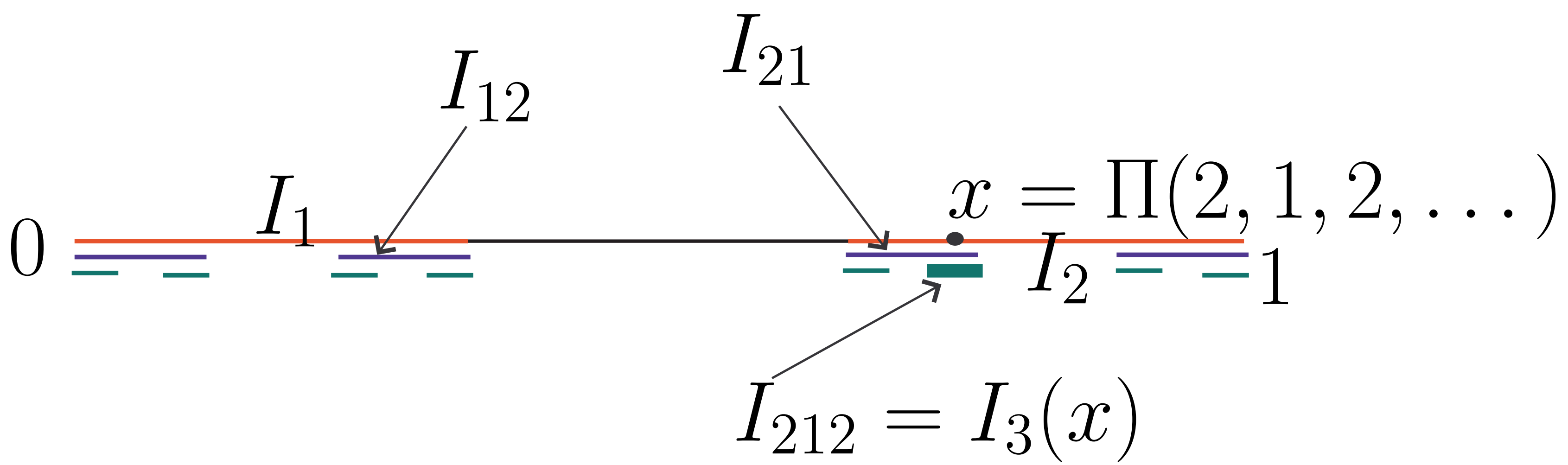
$$\frac{1}{n} \log \mu(I_n(x)) \to -h_\mu = \sum_{i=1}^{2} p_i \log p_i = \int f \, d\mu.$$

In picture:



$\mu(I_3(x)) = p_1 \cdot p_2 \cdot p_1$.
Since we defined $f(\mathbf{i}) = \log p_{i_1}$ so

$$\frac{1}{n} \cdot \log \mu(I_n(x)) = \frac{1}{n} \cdot \sum_{k=0}^{n-1} f(\sigma^k(\mathbf{i})).$$

Irregular points

For a $\beta \neq h_\mu$ we want to know

$$K_\beta = \dim_{\mathrm{H}} \left\{ x \in C : \frac{1}{n} \log \mu(I_n(x)) \to \beta \right\} =?$$

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Then

$$K_\beta = \dim_{\mathrm{H}} \Pi \left\{ \underbrace{\mathbf{i} \in \Sigma : \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) \to \beta}_{\Delta(\beta)} \right\}.$$

Dimension of measures

Let $\eta \in \mathcal{E}_\sigma(\Sigma)$. We also denote by η the push down measure $\Pi_* \eta$ on Λ.

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$$\dim_{\mathrm{H}}(\text{ arbitrary } \eta - \text{full measure set}) \geq \frac{h_\eta}{\log 3}.$$

Ergodic Theorem

For $\eta \in \mathcal{E}_\sigma(\Sigma)$ with $\int f d\eta = \beta$ by Birkhoff ergodic Theorem we have

$$\eta(\Delta(\beta)) = 1,$$

where I remind that $\Delta(\beta)$ was defined:

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$\Delta(\beta) = \left\{ \mathbf{i} \in \Sigma : \frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) \to \beta \right\}$. Thus

$$K_\beta = \dim_{\mathrm{H}} \Pi(\Delta(\beta)) \geq \sup_{\substack{\eta \in \mathcal{E}_\sigma(\Sigma) \\ \int f d\eta = \beta}} \underbrace{\frac{h_\eta}{\log 3}}_{\dim_{\mathrm{H}}(\eta)}.$$

Large deviations

We may assume that $\beta < h_\mu (= h_\mu(\sigma)$.

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We may assume that $\beta < h_\mu (= h_\mu(\sigma)$. To get upper bound we estimate $\ell_{n,\varepsilon} :=$

$$\#\left\{ (i_1, \ldots, i_n) \in \{1,2\}^n : \underbrace{\frac{1}{n}\sum_{k=1}^{n} \log p_{i_k}}_{\log \mu(I_n)} < \beta + \varepsilon \right\}$$

for fixed small $\varepsilon > 0$.

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$$\lim_{n\to\infty} \frac{1}{n} \log \ell_{n,\varepsilon} = h_\nu, \tag{1}$$

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where $\nu = \left(\frac{p^{t_0}}{p^{t_0}+(1-p)^{t_0}}, \frac{(1-p)^{t_0}}{p^{t_0}+(1-p)^{t_0}}\right)^{\mathbb{N}}$, where t_0 is the zero of the derivative of

$t \longrightarrow (\beta+\varepsilon)\cdot t - \log\left(p^t + (1-p)^t\right).$

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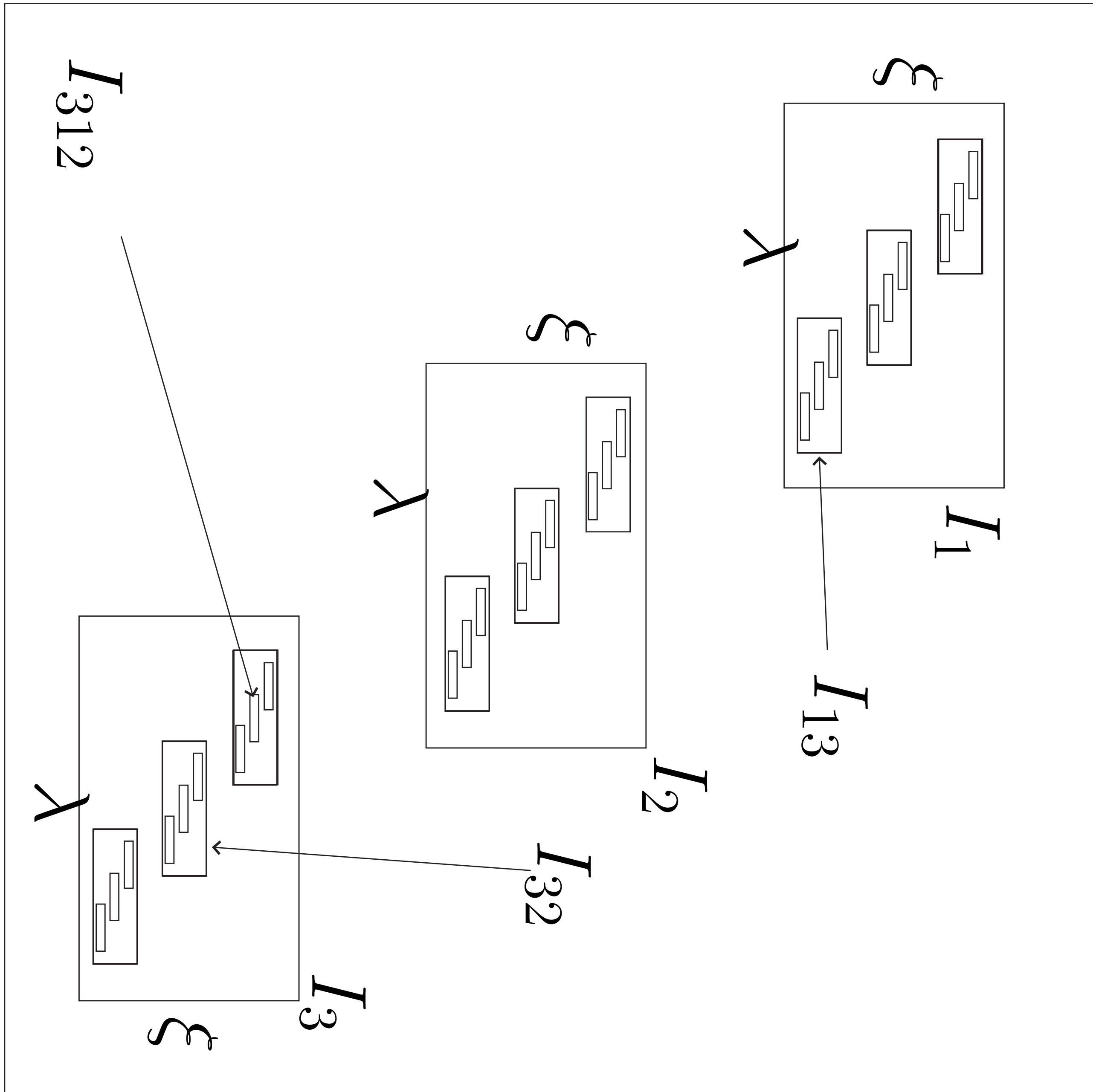
$$h_\nu = \sup_{\substack{\eta \in \mathcal{M}_\sigma(\Sigma) \\ \int f d\eta = \beta+\varepsilon}} h_\eta.$$

Self affine case

Given an $m \geq 2$ and a contracting self-affine IFS on $\mathbb{R}^2$ of the form

$$\mathscr{S} := \left\{ S_i(\mathbf{x}) = \begin{bmatrix} \lambda_i & 0 \\ 0 & \xi_i \end{bmatrix} \cdot \mathbf{x} + \mathbf{t}_i \right\}_{i=1}^{m},$$

where $0 < |\lambda_i|, |\xi_i| < \frac{1}{2}$.
We need to require that all the matrices are diagonal only for the upper bound. On the other hand we require that the norm of the matrices is smaller than $1/2$ only for the lower bound.



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Lyapunov dimension

Let $\mu \in \mathcal{M}_\sigma(\Sigma)$. The Lyapunov exponents of μ:

$$\lambda(\mu) = \int\limits_{\omega\in\Sigma} \log \lambda_{\omega_1} \mathsf{d}\mu(\omega) \text{ and } \xi(\mu) = \int\limits_{\omega\in\Sigma} \log \xi_{\omega_1} \mathsf{d}\mu(\omega),$$

where ω_1 is the first component of $\omega = (\omega_1, \omega_2, \dots)$.

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where ω_1 is the first component of $\omega = (\omega_1, \omega_2, \ldots)$. Now we define the Lyapunov dimension $D(\mu)$ of μ. If $\lambda(\mu) \geq \xi(\mu)$ then

(2) $$D(\mu) := \min\left\{-\frac{h(\mu)}{\lambda(\mu)}, 1 - \frac{h(\mu)+\lambda(\mu)}{\xi(\mu)}\right\}.$$

self-affine sets

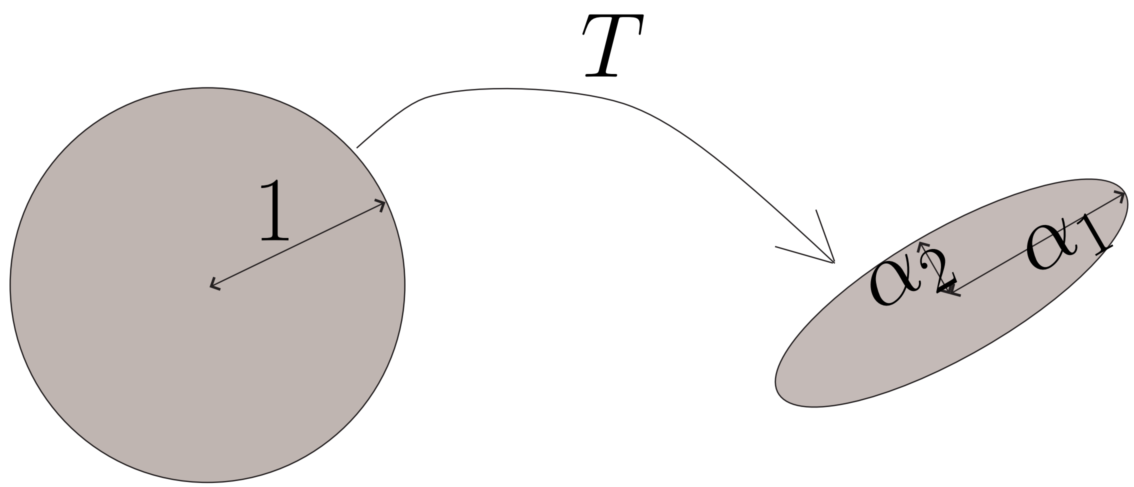
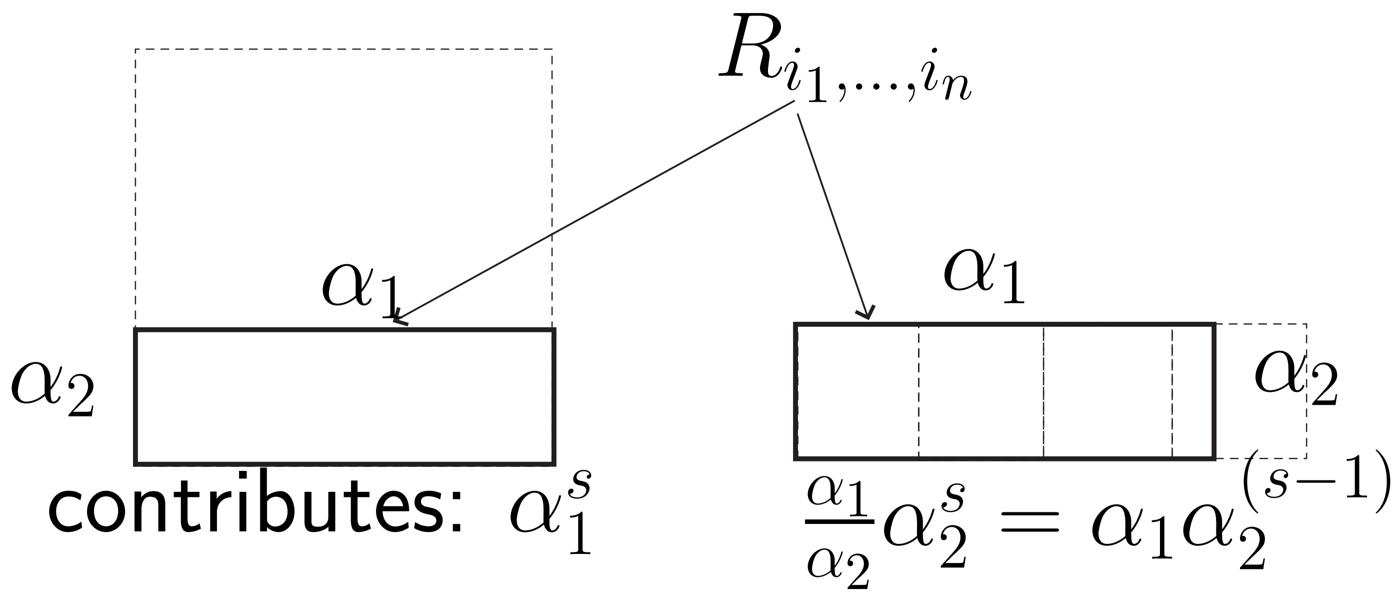


Figure 1: The singular values α_1, α_2 of the linear transformation T

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup \underbrace{S_{i_0 \ldots i_{n-1}}([0,1]^2)}_{R_{i_0 \ldots i_{n-1}}},$$

We should opt between covering the cylinder $R_{i_0 \ldots i_{n-1}} = S_{i_0 \ldots i_{n-1}}([0,1]^2)$ in one of the two following ways:

Two coverings of Λ



where $\alpha_1 \geq \alpha_2$ are the singular values of $S_{i_1\ldots i_n}$

Ergodic averages

We would like to study the ergodic averages of f

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k \mathbf{i})$$

from the following point of view:

Ergodic averages

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from the following point of view: For a suitable β we want to know how big is the Hausdorff dimension of the Π projection the set of $\mathbf{i} \in \Sigma$ for which the ergodic averages of f converge to β.

A far-fetched ambition

It would be **desirable** to find the Hausdorff dimension of

$$K_\beta := \Pi \left\{ \underbrace{\mathbf{i} \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) = \beta}_{\Delta(\beta)} \right\}$$

as a function of β. That is the function:

(3) $$\beta \to \dim_{\mathrm{H}}(K_\beta).$$

A realistic goal instead

In our IFS $\left\{ S_i(\mathbf{x}) = \begin{bmatrix} \lambda_i & 0 \\ 0 & \xi_i \end{bmatrix} \cdot \mathbf{x} + \mathbf{t}_i \right\}_{i=1}^{m}$ we consider the diagonal matrices as fixed and the m translation vectors $\mathfrak{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_m) \in \mathbb{R}^{2 \cdot m}$ as parameters.

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$$\mathscr{S}^{\mathfrak{t}} := \left\{ S_i^{\mathfrak{t}}(\mathbf{x}) = \begin{bmatrix} \lambda_i & 0 \\ 0 & \xi_i \end{bmatrix} \cdot \mathbf{x} + \mathbf{t}_i \right\}_{i=1}^{m},$$

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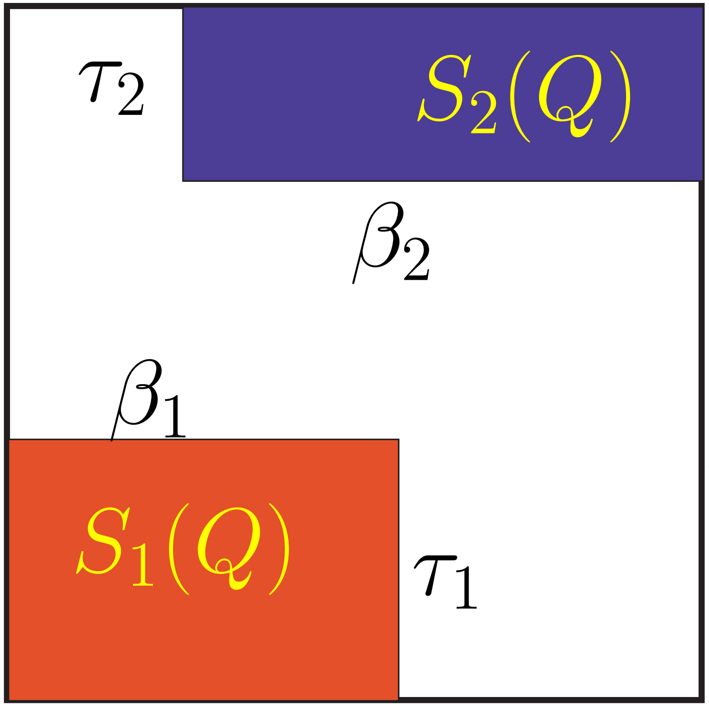
and we denote accordingly

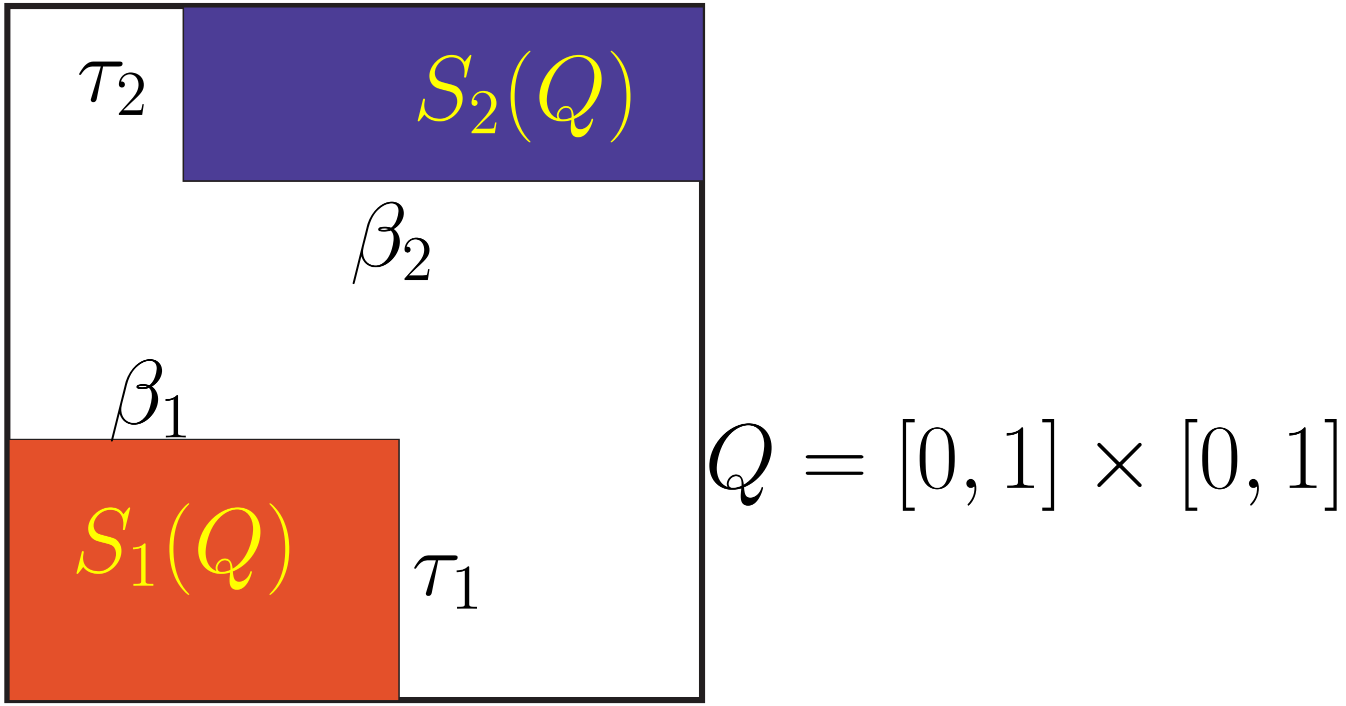
$$\Lambda^{\mathfrak{t}}, \Pi^{\mathfrak{t}}, K_\beta^{\mathfrak{t}}.$$

Neunhauserer



$Q = [0, 1] \times [0, 1]$

Neunhauserer



Let P be the set of $(\beta_1, \beta_2, \tau_1, \tau_2)$ for which:
$\tau_1 + \tau_2 < 1$, and $\beta_1 + \beta_2 > 1$.

Neunhauserer

Further let d be the solution of the equation

$$\beta_1 \cdot \tau_1^d + \beta_2 \cdot \tau_2^d = 1.$$

Then

- $\dim_{\mathrm{B}} \Lambda = 1 + d$
- If further we assume that $\beta_1, \beta_2 < 0.649$ then for Leb. a.a. $(\beta_1, \beta_2, \tau_1, \tau_2)$ we have

$$\dim_{\mathrm{H}} \Lambda = 1 + d.$$

- If $\beta_1 = \beta_2$ is reciprocal of a PV number and $\tau_1 = \tau_2 < \frac{1}{2}$ then $\dim_{\mathrm{H}} \Lambda < 1 + d$.

The main result

Theorem. 1 (Jordan, S.) *Assume that* $\forall i$, $0 < \lambda_i, \xi_i < \frac{1}{2}$. *Then for almost all* $\mathfrak{t} \in \mathbf{R}^{2\cdot m}$ *for any*

$$\beta \in \left(\underbrace{\min_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \int f d\mu \right\}}_{\beta_{\min}}, \underbrace{\max_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \int f d\mu \right\}}_{\beta_{\max}} \right),$$

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$$\dim_{\mathrm{H}} K_\beta^{\mathfrak{t}} = \min \left\{ \sup_{\substack{\eta \in \mathcal{E}_\sigma(\Sigma) \\ \int f \, d\eta = \beta}} D(\eta), 2 \right\}.$$

The lower estimate

$\forall \mu \in \mathcal{E}_\sigma(\Sigma)$ with $\int f d\mu = \beta$ we have $\mu(\Delta(\beta)) = 1$.
Thus $\forall \mathfrak{t}$: $\dim_{\mathrm{H}}(K_\beta^{\mathfrak{t}}) \geq \dim_{\mathrm{H}}(\mu \circ (\Pi^{\mathfrak{t}})^{-1})$. It follows from a theorem of Jordan, Pollicott, Simon that:
For $\mathcal{L}eb_{2 \cdot m}$ almost all $\mathfrak{t}$ we have

$$\boxed{\dim_{\mathrm{H}}(\mu \circ (\Pi^{\mathfrak{t}})^{-1}) = \min \{D(\mu), d\}.}$$

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An example

$m = 4$ and $(p_1, \ldots, p_4) = (1/2, 1/4, 1/8, 1/8)$ and $\lambda = 0.4, \xi = 1/6$.

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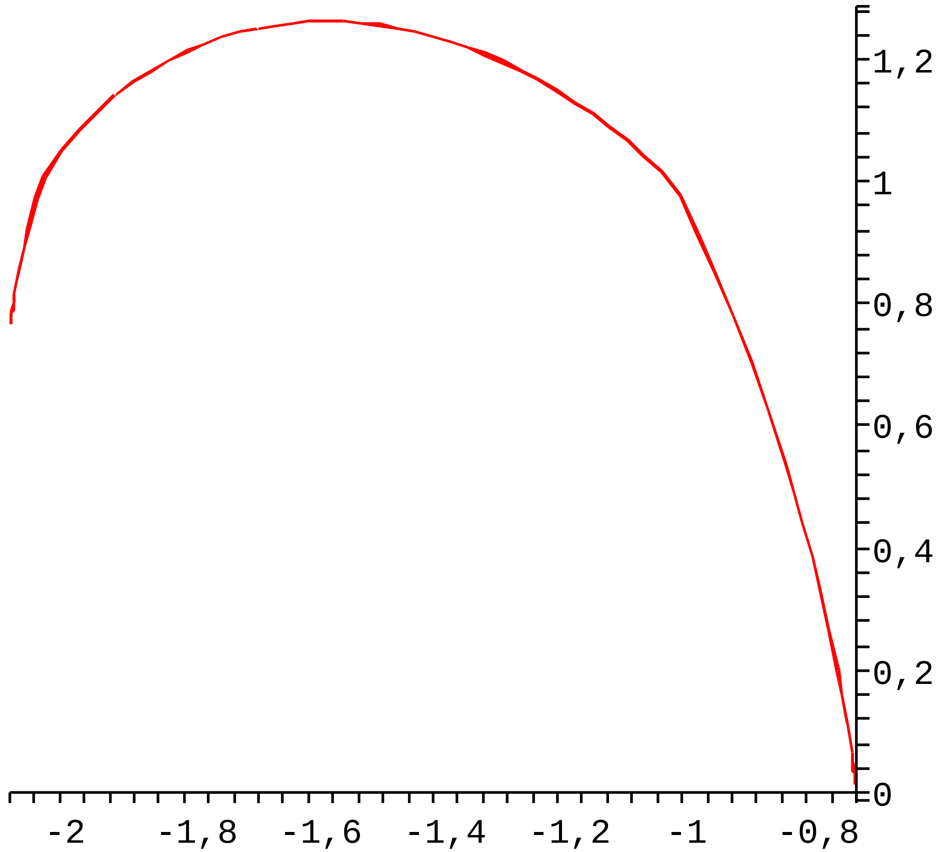

Falconer, Solomyak Thm

Let A_i be $d \times d$ matrices with $\|A_i\| < 1$ and $\mathbf{t}_i \in \mathbb{R}^d$ for $i = 1, \ldots, m$. Consider the self affine IFS:

$$\{f_i\}_{i=1}^m = \{A_i \cdot x + t_i\} . \tag{4}$$

Theorem (Falconer, Solomyak) If $\|A_i\| < \frac{1}{2}$ then for Lebesgue almost all $(t_1, \ldots, t_m) \in \mathbb{R}^{d \cdot m}$ the Hausdorff dimension of the attractor is equal to the singularity dimension.

Theorem (Käenmäki) If $\|A_i\| < \frac{1}{2}$ then the singularity dimension is equal to the dimension of an ergodic measure for almost all translations.

Jordan, Pollicott, S.

We consider a one parameter family of self affine IFS $\{f_i^u\}_{i=1}^m$, on $\mathbb{R}^d$, where the parameter $u \in U$ a compact set which is endowed with a Borel measure $\mathcal{M}$. We introduce a so called **self affine transversality condition**. **Assuming this** we prove that for $\mathcal{M}$-a.a. parameter $u \in U$:

- the dimension of an ergodic measure μ is $D(\mu)$.
- If $D(\mu) > d$ then μ is absolute continuous.
- **The dimension of the attractor** is equal to the singularity dimension. Further, these are equal to the dimension of the Käenmäki measure.

The function $Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)$

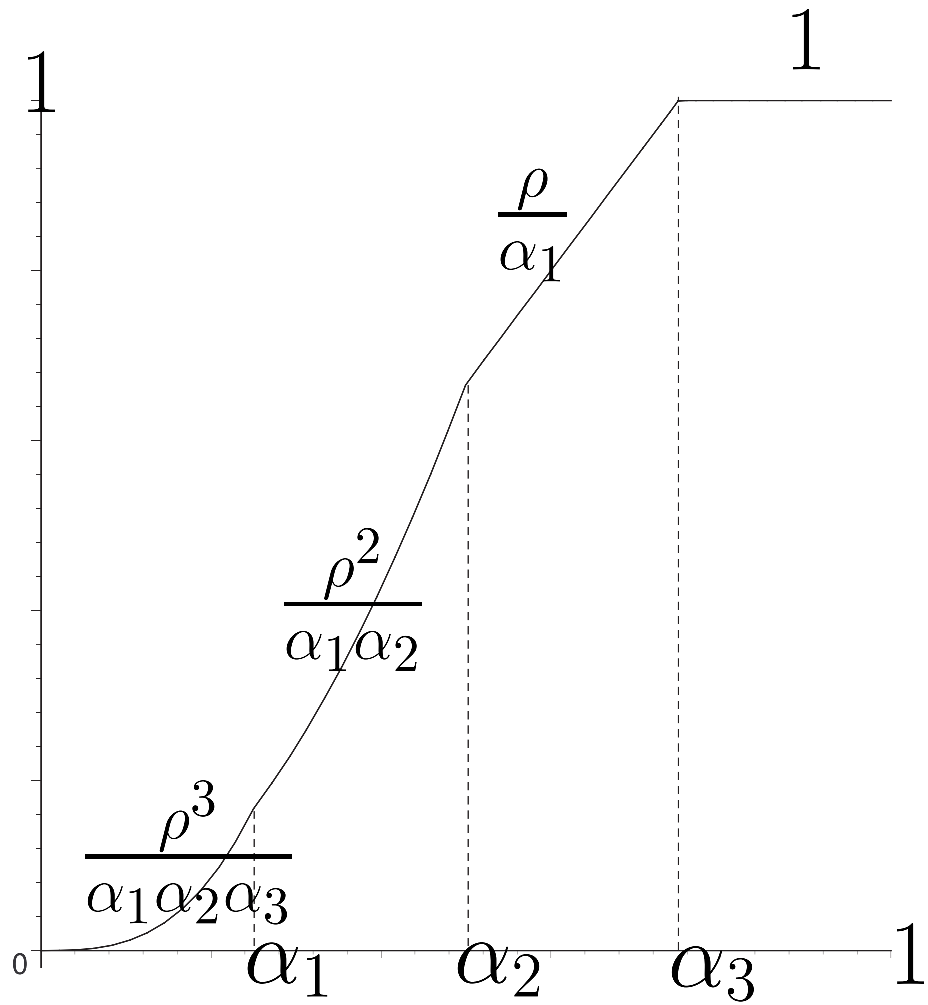

Figure 3: The function $\rho \to Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)$, $\alpha_k := \alpha_k(\mathbf{i} \wedge \mathbf{j})$

Transversality condition

Self-affine transversality condition: There is a constant $C > 0$ (independent of $\mathbf{i}, \mathbf{j}$) such that for all $\mathbf{i}, \mathbf{j} \in \Sigma$, $\mathbf{i} \neq \mathbf{j}$ we have

(5)

$$\mathcal{M}\left\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| < \rho\right\} < C \cdot Z_{\mathbf{i}\wedge\mathbf{j}}(\rho),$$

where U was the parameters space which was supposed to be a compact metric space and $\mathcal{M}$ was a finite Borel measure on U. Further, by definition $\alpha_k(\mathbf{i} \wedge \mathbf{j})$ was the k-th biggest singular value of $A_{\mathbf{i}\wedge\mathbf{j}}$.

Random perturbation

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_0 \dots i_{n-1}} f_{i_0 \dots i_{n-1}}(B).$$

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Let $\mathcal{T}$ be the m-adic tree, $\mathbf{i}_n = (i_0, \dots, i_{n-1}) \in \mathcal{T}$

$$f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}} := (f_{i_0} + y_{i_0}) \circ (f_{i_1} + y_{i_0 i_1}) \circ \cdots \circ (f_{i_{n-1}} + y_{i_0 \dots i_{n-1}}),$$

$$\mathbf{y}_{\mathbf{i}_n} := \left(y_{i_0}, y_{i_0 i_1}, \dots, y_{i_0 \dots i_{n-1}}\right) \in \underbrace{D \times \cdots \times D}_{n}$$

$$\Lambda^{\mathbf{y}} := \bigcap_{n=0}^{\infty} \bigcup_{\mathbf{i}_n} f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}}(B),$$

$\mathcal{E}_\sigma(\Sigma)$ or $\mathcal{M}_\sigma(\Sigma)$

Let Ψ_i^s is the logarithm of the price of an s cover. Assume that $\Psi_1^s > \Psi_2^s$. Let $l(q,s) := qf - q\beta + \Psi_1^s$ Let $\mu_{q,s}$ be the equilibrium measure for $l(q,s)$. We fix β. Then for every s $\exists$ a $q = q(s)$: $\int (qf - q\beta) d\mu_{q,s} = 0$. Let

(6) $$s = \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), \int f d\mu = \beta \right\}$$

Then $P(q(s)f - q(s)\beta + \Psi_1^s) = 0$ and the supremum is achieved by the equilibrium state for $q(s)f - q(s)\beta + \Psi_1^s$