First homework assignment. Due at 12:15 on 22 September 2016.

Homework 1. We roll two dices. X is the result of one of them and Z the sum of the results. Find $\mathbb{E}[X|Z]$.

Homework 2. Let X be a r.v.. Assume that Y another r.v. for which $\mathbb{P}(Y=0 \text{ or } Y=1)=1$. Prove that $Y \in \sigma(X)$ iff there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ Borel Measurable function such that $Y=\varphi(X)$.

Homework 3. Let X and Y be random variables on the same probability space. Prove that X and Y are independent iff for every $\varphi : \mathbb{R} \to \mathbb{R}$ bounded measurable functions we have

$$\mathbb{E}[\varphi(Y)|X] = \mathbb{E}[\varphi(Y)].$$

Homework 4. Assume that X, Y are jointly continuous r.v. with joint distribution function $f(x)$. Prove that

$$\mathbb{E}[Y|X] = \mathbb{E}[Y|\sigma(X)] = \frac{\int_{\mathbb{R}} y f(X,y)dy}{\int_{\mathbb{R}} f(X,y)dy}.$$

Homework 5. Let $Y \in \sigma(\mathcal{G})$. Prove that

$$\mathbb{E}[X|\mathcal{G}] \geq Y \Longleftrightarrow \forall A \in \mathcal{G}\ \mathbb{E}[X \cdot \mathbb{1}_A] \geq \mathbb{E}[Y \cdot \mathbb{1}_A].$$

Homework 6. Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$\mathbb{E}[X|Y] = Y \text{ and } \mathbb{E}[Y|X] = X$$

Show that $\mathbb{P}(X=Y)=1$.

Homework 7. Prove the general version of Bayes's formula: Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let $G \in \mathcal{G}$. Show that

$$\mathbb{P}(G|A) = \frac{\int_G \mathbb{P}(A|\mathcal{G})\, d\mathbb{P}}{\int_\Omega \mathbb{P}(A|\mathcal{G})} d\mathbb{P} \tag{1}$$

Homework 8. Prove the conditional variance formula

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[\mathrm{X}|\mathrm{Y}]), \tag{2}$$

where $\mathrm{Var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$.

Homework 9. Let $X_1, X_2, \ldots$ iid r.v. and N is a non-negative integer valued r.v. that is independent of X_i, $i \geq 1$. Prove that

$$\mathrm{Var}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}[N]\,\mathrm{Var}(X) + (\mathbb{E}[X])^2 \mathrm{Var}(N). \tag{3}$$

Second homework assignment. Due at 12:15 on 29 September 2016.

Homework 10. Let X_t be a Poisson(1). That is a Poisson process with rate $\lambda = 1$. (See Durrett's book p. 139 if you forgot the definition.) Find: $\mathbb{E}[X_1|X_2]$ and $\mathbb{E}[X_2|X_1]$.

Homework 11. Construct a martingale which is NOT a Markov chain.

Homework 12. For every $i = 1, \ldots, m$ let $\left\{M_n^{(i)}\right\}_{n=1}^{\infty}$ be a sequence of martingales w.r.t. $\{X_n\}_{n=1}^{\infty}$. Show that

$$M_n := \max_{1 \leq i \leq n} M_n^{(i)}$$

is a submartingal w.r.t. $\{X_n\}$.

Homework 13. Let $\xi_1, \xi_2, \dots$ standard normal variables. (Recall that in this case the moment generating function $M(\theta) = \mathbb{E}\left[e^{\theta \xi_i}\right] = e^{\theta^2/2}$.) Let $a, b \in \mathbb{R}$ and

$$S_n := \sum_{k=1}^{n} \xi_k \text{ and } X_n := e^{aS_n - bn}$$

Prove that

(a) $X_n \to 0$ a.s. iff $b > 0$

(b) $X_n \to 0$ in L^r iff $r < \frac{2b}{a^2}$.

Homework 14. Let $S_n := X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ are iid with $X_1 \sim \text{Exp}(1)$. Verify that

$$M_n := \frac{n!}{(1 + S_n)^{n+1}} \cdot e^{S_n}$$

is a martingale w.r.t. the natural filtration $\mathcal{F}_n$.

Third homework assignment. Due at 12:15 on 6 October 2016.

Homework 15. Prove that the following two definitions of λ-system $\mathcal{L}$ are equivalent:

Definition 1. (a) $\Omega \in \mathcal{L}$.

(b) If $A, B \in \mathcal{L}$ and $A \subset B$ then $B \setminus A \in \mathcal{L}$

(c) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$ (that is $A_n \subset A_{n+1}$ and $A = \cup_{n=1}^{\infty} A_n$) then $A \in \mathcal{L}$.

Definition 2. (i) $\Omega \in \mathcal{L}$.

(ii) If $A \in \mathcal{L}$ then $A^c \in \mathcal{L}$.

(iii) If $A_i \cap A_j = \emptyset$, $A_i \in \mathcal{L}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{L}$.

Homework 16. There are n white and n black balls in an urn. We pull out all of them one-by-one without replacement. Whenever we pull:

- a black ball we have to pay 1\$,
- a white ball we receive 1\$.

Let $X_0 := 0$ and X_i be the amount we gained or lost after the i-th ball was pulled. We define

$$Y_i := \frac{X_i}{2n - i}, \text{ for } 1 \le i \le 2n - 1, \quad \text{and } Z_i := \frac{X_i^2 - (2n - i)}{(2n - i)(2n - i - 1)} \text{ for } 1 \le i \le 2n - 2.$$

(a) Prove that $Y = (Y_i)$ and $Z = (Z_i)$ are martingales.

(b) Find $\text{Var}(X_i) =?$

Homework 17. Let X, Y be two independent $\text{Exp}(\lambda)$ r.v. and $Z := X + Y$. Show that for any non-negative measurable h we have $\mathbb{E}[h(X)|Z] = \frac{1}{Z} \int_0^Z h(t)dt$.

Homework 18. Let $X_1, X_2, \dots$ be iid r.v. with $\mathbb{P}(X_n = -n^2) = \frac{1}{n^2}$ and $\mathbb{P}\left(X_n = \frac{n^2}{n^2-1}\right) = 1 - \frac{1}{n^2}$. Let $S_n := X_1 + \cdots + X_n$. Show that

(a) $\lim\limits_{n \to \infty} S_n / n = \infty$.

(b) $\{S_n\}$ is a martingale which converges to ∞ a.s..

Homework 19. (This was withdrawn on this week and it is reassigned next week.) A player's winnings per unit stake on game n are ξ_n, where $\{\xi\}_{n=1}^{\infty}$ are i.i.d. r.v.

$$\mathbb{P}(\xi_n = 1) = p \text{ and } \mathbb{P}(\xi_n = -1) = q := 1 - p,$$

where surprisingly, $p \in (1/2, 1)$, that is $p > q$. That is with probability $q < 1/2$ the player losses her stake and with probability p she gets back twice of her stake. Let C_n be the player's stake on game n. We assume that C_n is previsible, that is $C_{n+1} \in \mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$ for all n. Further, we assume that $0 \leq C_n \leq Y_{n-1}$, where Y_{n-1} is the fortune of the player at time n. We call $\alpha := p \log p + q \log q + \log 2$ the entropy. Prove that

(a) $\log Z_n - n\alpha$ is a supermartingale. This means that the rate of winnings $\mathbb{E}[\log Y_n - \log Y_0] \leq n\alpha$

(b) There exists a strategy for which $\log Z_n - n\alpha$ is a martingale.

Fourth homework assignment. Due at 12:15 on 13 October 2016.

Homework 20. (Reassigned from last week) A player's winnings per unit stake on game n are ξ_n, where $\{\xi\}_{n=1}^{\infty}$ are i.i.d. r.v.

$$\mathbb{P}(\xi_n = 1) = p \text{ and } \mathbb{P}(\xi_n = -1) = q := 1 - p,$$

where surprisingly, $p \in (1/2, 1)$, that is $p > q$. That is with probability $q < 1/2$ the player losses her stake and with probability p she gets back twice of her stake. Let C_n be the player's stake on game n. We assume that C_n is previsible, that is $C_{n+1} \in \mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$ for all n. Further, we assume that $0 \leq C_n \leq Y_{n-1}$, where Y_{n-1} is the fortune of the player at time n. We call $\alpha := p \log p + q \log q + \log 2$ the entropy. Prove that

(a) $\log Y_n - n\alpha$ is a supermartingale. This means that the rate of winnings $\mathbb{E}[\log Y_n - \log Y_0] \leq n\alpha$

(b) There exists a strategy for which $\log Y_n - n\alpha$ is a martingale.

Homework 21. Let $\mathbf{X}, \mathbf{Z}$ be $\mathbb{R}^d$-valued r.v. defined on the $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\mathbb{E}\left[e^{i\mathbf{t}\cdot\mathbf{X}+i\mathbf{s}\cdot\mathbf{Z}}\right] = \mathbb{E}\left[e^{i\mathbf{t}\mathbf{X}}\right] \cdot \mathbb{E}\left[e^{i\mathbf{s}\mathbf{Z}}\right], \quad \forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^n$. Prove that $\mathbf{X}, \mathbf{Z}$ are independent.

Homework 22. Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in $\mathbb{R}^n$. Let $\mathbf{X}_1 := (X_1, \dots, X_p)$ and $\mathbf{X}_2 := (X_{p+1}, \dots, X_n)$. Let Σ, Σ_1 and Σ_2 the covariance matrix of X, X_1 and X_2 respectively. Prove that $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent if and only if

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.$$

Homework 23. Let $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ and let B be a non-singular matrix. Find the distribution of $\mathbf{X} = B \cdot \mathbf{Y}$.

Homework 24. Let $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in $\mathbb{R}^2$, $\mathbf{Y} = (Y_1, Y_2)$ and let $a_1, a_2 \in \mathbb{R}$. Find the distribution of $a_1 Y_1 + a_2 Y_2$.

Fifth homework assignment. Due at 12:15 on 20 October 2016.

Homework 25. Construct a random vector (X, Y) such that both X and Y are one-dimensional normal distributions but (X, Y) is NOT a bivariate normal distribution.

Homework 26. Let $\mathbf{X} = (X_1, \dots, X_d)$ be standard multivariate normal distribution. Let Σ be an $n \times n$ positive semi-definite, symmetric matrix. and let $\boldsymbol{\mu} \in \mathbb{R}^d$. Prove that there exists an affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$, such that $T(\mathbf{X}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

Homework 27. Let X be the height of the father and Y be the height of the son in sample of father-son pairs. Then (X, Y) is bivariate normal. Assume that

$$\mathbb{E}[X] = 68 \text{ (in inches)}, \ \mathbb{E}[Y] = 69, \ \sigma_X = \sigma_Y = 2, \ \rho = 0.5,$$

where ρ is the correlation of (X, Y). Find the conditional distribution of Y given $X = 80$ (6 feet 8 inches). (That is find the parameters in $Y \sim \mathcal{N}(\mu_Y, \sigma^2(Y))$.)

Homework 28 (Extension of part (iii) of Doob's optional stopping Theorem)**.** Let X be a supermartingale. Let T be a stopping time with $\mathbb{E}[T] < \infty$.(Like in part (iii) of Doob's optional stopping Theorem.) Assume that there is a C such that

$$\mathbb{E}[|X_k - X_{k-1}||\mathcal{F}_{k-1}](\omega) \leq C, \quad \forall k > 0 \text{ and for a.e. } \omega.$$

Prove that $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.

Homework 29. Let X_n be a discrete time birth-death process with probabilities with transition probabilities

$$p(i, i+1) = p_i, \ p(i, i-1) = 1 - p_i =: q_i, p_0 = 1$$

We define the following function:

$$g : \mathbb{N} \to \mathbb{R}^+, \quad g(k) := 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \frac{q_i}{p_i}$$

,

(a) Prove that $Z_n := g(X_n)$ is a martingale for the natural filtration.

(b) Let $0 < i < n$ be fixed. Find the probability that a process started from i gets to n earlier than to 0.

Sixth homework assignment. Due at 12:15 on 3 November 2016.

Homework 30. Let $\{\varepsilon_n\}$ be an iid sequence of real numbers satisfying $\mathbb{P}(\varepsilon_n = \pm 1) = \frac{1}{2}$. Show that $\sum\limits_n \varepsilon_n a_n$ converges alsmost surely iff $\sum\limits_{k=1}^{\infty} a_n^2 < \infty$.

Homework 31. Let $X = (X_n)$ be an L^2 random walk that is a martingale. Let σ^2 be the variance of the k-th increment $Z_k := X_k - X_{k-1}$ for all k. Prove that the quadratic variance is $A_n = n\sigma^2$.

Homework 32. Prove the assertion of Remark 5.6 from File C.

Homework 33. Let $M = (M_n)$ be a martingale with $M_0 = 0$ and $|M_k - M_{k-1}| < C$ for a $C \in \mathbb{R}$. Let $T \geq 0$ be a stopping time and we assume that $\mathbb{E}[T] \leq \infty$. Let

$U_n := \sum\limits_{k=1}^{n} (M_k - M_{k-1})^2 \cdot \mathbb{1}_{T \geq k}, \quad V_n := 2 \sum\limits_{1 \leq i < j \leq n} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbb{1}_{T \geq j}$.

$U_\infty := \sum\limits_{k=1}^{\infty} (M_k - M_{k-1})^2 \cdot \mathbb{1}_{T \geq k}, \quad V_\infty := 2 \sum\limits_{1 \leq i < j} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbb{1}_{T \geq j}$. Prove that

(a) $M_{T \wedge n}^2 = U_n + V_n$ and $M_T^2 = U_\infty + V_\infty$.

(b) Further, if $\mathbb{E}[T^2] < \infty$ then $\lim\limits_{n \to \infty} U_n = U_\infty$ a.s. and $\mathbb{E}[U_\infty] < \infty$ and $\mathbb{E}[V_n] = \mathbb{E}[V_\infty] = 0$.

(c) Conclude that $\lim\limits_{n \to \infty} \mathbb{E}[M_{T \wedge n}^2] = \mathbb{E}[M_T^2]$.

Homework 34 (Wald equalities)**.** Let $Y_1, Y_2, \ldots$ be iid r.v. with $Y_i \in L^1$. Let $S_n := Y_1 + \cdots + Y_n$ and we write $\mu := \mathbb{E}[Y_i]$. Given a stopping time $T \geq 1$ satisfying: $\mathbb{E}[T] < \infty$. Prove that

(a)

$$\mathbb{E}[S_T] = \mu \cdot \mathbb{E}[T]. \tag{4}$$

(b) Further, assume that Y_i are bounded ($\exists C_i \in \mathbb{R}$ s.t. $|Y_i| < C_i$) and $\mathbb{E}[T^2] < \infty$. We write $\sigma^2 := \mathrm{Var}(Y_i)$. Then

$$\mathbb{E}\left[(S_T - \mu T)^2\right] = \sigma^2 \cdot \mathbb{E}[T]. \tag{5}$$

Hint: Introduce an appropriate martingale and apply the result of the previous exercise.

Sevens homework assignment. Due at 12:15 on 10 November 2016.

Homework 35 (Branching Processes)**.** You might want to recall what you have learned about Branssching Processes. (See File C of the course "Stochastic Processes".) A Branching Process $Z = (Z_n)_{n=0}^{\infty}$ is defined recursively by a given family of $\mathbb{Z}^+$ valued iid rv. $\left\{X_k^{(n)}\right\}_{k,n=1}^{\infty}$ as follows:

$$Z_0 := 1, \quad Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}, \quad n \geq 0.$$

Let $\mu = \mathbb{E}\left[X_k^{(n)}\right]$ and $\mathcal{F}_n = \sigma(Z_0, Z_1, \dots Z_n)$. We write $f(s)$ for the generating function. that is

$$f(s) = \sum_{\ell=0}^{\infty} \underbrace{\mathbb{P}\left(X_k^{(n)} = \ell\right)}_{p_\ell} \cdot s^\ell \text{ for any } k, n.$$

Further, let

$$\{extinction\} := \{Z_n \to 0\} = \{\exists n, Z_n = 0\} \quad \{explosion\} = \{Z_n \to \infty\}.$$

let $q := \mathbb{P}([extinction]) :=$. Recall that we learned that q is the smaller (if there are two) fixed point of $f(s)$. That is q is the smallest solution of $f(q) = q$. Prove that

(a) $\mathbb{E}[Z_n] = \mu^n$. Hint: Use induction.

(b) For every $s \geq 0$ we have $\mathbb{E}\left[s^{Z_{n+1}} | \mathcal{F}_n\right] = f(s)^{Z_n}$. Explain why it is true that q^{Z_n} is a martingale and $\lim_{n\to\infty} Z_n = Z_\infty$ exists a.s.

(c) Let $T := \min\{n : Z_n = 0\}$. ($T = \infty$ if $Z_n > 0$ always.)

(d) Prove that $q = \mathbb{E}\left[q^{Z_T}\right] = \mathbb{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] + \mathbb{E}\left[q^{Z_T} \cdot \mathbb{1}_{T<\infty}\right]$.

(e) Prove that $\mathbb{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] = 0$.

(f) Conclude that if $T(\omega) = \infty$ then $Z_\infty = \infty$.

(g) Prove that

$$\mathbb{P}(\text{extinction}) + \mathbb{P}(\text{explosion}) = 1. \tag{6}$$

Homework 36 (Branching Processes cont.)**.** Here we assume that

$$\mu = \mathbb{E}\left[X_k^{(n)}\right] < \infty \text{ and } 0 < \sigma^2 := \mathrm{Var}(X_k^{(n)}) < \infty.$$

Prove that

(a) $M_n = Z_n / \mu^n$ is a martingale for the natural filtration $\mathcal{F}_n$

(b) $\mathbb{E}\left[Z_{n+1}^2 | \mathcal{F}_n\right] = \mu^2 Z_n^2 + \sigma^2 Z_n$. Conclude that

$$M \text{ is bounded in } L^2 \iff \mu > 1.$$

(c) If $\mu > 1$ then $M_\infty := \lim_{n\to\infty} M_n$ exists (in L^2 and a.s.) and

$$\mathrm{Var}(M_\infty) = \frac{\sigma^2}{\mu(\mu-1)}.$$

Homework 37 (Branching Processes cont.)**.** Assume that $q = 1$ (q was defined in the one but last exercise as the probability of extinction). Prove that $M_n = Z_n/\mu^n$ is NOT a UI martingale.

Eights homework assignment. Due at 12:15 on 24 November 2016.

Homework 38. Let $X_1, X_2, \dots$ be iid rv. with **continuous** distribution distribution function. Let E_i be the event that a record occurs at time n. That is $E_1 := \Omega$ and $E_n := \{X_n > X_m,\ \forall m < n\}$. Prove that $\{E_i\}_{i=1}^{\infty}$ independent and $\mathbb{P}(E_i) = \frac{1}{i}$.

Homework 39 (**Continuation**)**.** Let $E_1, E_2, \dots$ be independent with $\mathbb{P}(E_i) = 1/i$. Let $Y_i := \mathbb{1}_{E_i}$ and $N_n := Y_1 + \cdots + Y_n$. (In the special case of the previous homework, N_n is the number of records until time n.) Prove that

(a) $\sum_{k=1}^{\infty} \frac{Y_k - 1/k}{\log k}$ converges almost surely.

(b) Using Krocker's Lemma conclude that $\lim_{n\to\infty} \frac{N_n}{\log n} = 1$ a.s..

(c) Apply this to the situation of the previous exercise to get an estimate on the number of records until time n.

Homework 40. Let $\mathcal{C}$ be a class of rv on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that the following assertions (1) and (2) are equivalent:

1. $\mathcal{C}$ is UI.
2. Both of the following two conditions hold:
   - **(a)** $\mathcal{C}$ is L^1-bounded. That is $A := \sup\{\mathbb{E}[|X|] : X \in \mathcal{C}\} < \infty$ AND
   - **(b)** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
   $$F \in \mathcal{F} \text{ and } \mathbb{P}(F) < \delta \Longrightarrow \mathbb{E}[|X|; F] < \varepsilon.$$

Homework 41. Let $\mathcal{C}$ and $\mathcal{D}$ be UI classes of rv.. Prove that $\mathcal{C} + \mathcal{D} := \{X + Y : X \in \mathcal{C} \text{ and } Y \in \mathcal{D}\}$ is also UI. Hint: use the previous exercise.

Homework 42. Let $\mathcal{C}$ be a UI family of rv.. Let us define
$\mathcal{D} := \{Y : \exists X \in \mathcal{C},\ \exists \mathcal{G} \text{ sub-}\sigma\text{-algebra of } \mathcal{F} \text{ s.t. } Y = \mathbb{E}[X|\mathcal{G}]\}$. Prove that $\mathcal{D}$ is also UI.

Ninth homework assignment. Due at 12:15 on 1 December 2016.

Homework 43. Let $X_1, X_2, \dots$ be iid. rv. with $\mathbb{E}[X^+] = \infty$ and $\mathbb{E}[X^-] < \infty$. (Recall $X = X^+ - X^-$ and $X^+, X^- \geq 0$.) Use SLLN to prove that $S_n/n \to \infty$ a.s., where $S_n := X_1 + \cdots + X_n$. Hint: For an $M > 0$ let $X_i^M := X_i \wedge M$ and $S_n^M := X_n^M + \cdots + X_n^M$. Explain why $\lim_{n\to\infty} S_n^M/n \to \mathbb{E}\left[X_i^M\right]$ and $\liminf_{n\to\infty} S_n/n \geq \lim_{n\to\infty} S_n^M/n$.

Homework 44. Let $X_1, X_2, \dots$ be iid rv with $\mathbb{E}[|X_i|] < \infty$. Prove that $\mathbb{E}[X_1|S_n] = S_n/n$. (This is trivial intuitively from symmetry, but prove it with formulas.)

Homework 45. Let $\mathcal{C}$ be a class of random variables of $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following condition implies that $\mathcal{C}$ is UI:
If $\exists p > 1$ and $A \in \mathbb{R}$ such that $\mathbb{E}[|X|^p] < A$ for all $X \in \mathcal{C}$. (L^p bounded for some $p > 1$.)

Homework 46. Let $\mathcal{C}$ be a class of random variables of $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following condition implies that $\mathcal{C}$ is UI:

$\exists Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, s.t. $\forall X \in \mathcal{C}$ we have $|X(\omega)| \leq Y(\omega)$. ($\mathcal{C}$ is dominated by an integrable (non-negative) r.v..)

Homework 47. (Infinite Monkey Theorem) Monkey typing random on a typewriter for infinitely time will type the complete works of Shakespeare eventually.

Tenth homework assignment. Due at 12:15 on 8 December 2016.

Homework 48 (Azuma-Hoeffding Inequality)**.**

(a) Assume that Y is a r.v. which takes values from $[-c, c]$ and $\mathbb{E}[Y] = 0$ holds. Prove that for all $\theta \in \mathbb{R}$ we have

$$\mathbb{E}\left[e^{\theta Y}\right] \leq \cosh(\theta c) \leq \exp\left(\frac{1}{2}\theta^2 c^2\right).$$

Hint: Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then by the convexity of f we have

$$f(y) \leq \frac{c-y}{2c} f(-c) + \frac{c+y}{2c} f(c).$$

(b) Let M be a martingale with $M_0 = 0$ such that or a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ we have

$$|M_n - M_{n-1}| \leq c_n, \qquad \forall n.$$

Then the following inequality holds for all $x > 0$:

$$\mathbb{P}\left(\sup_{k \leq n} M_k \geq x\right) \leq \exp\left(-\frac{1}{2}x^2 / \sum_{k=1}^{n} c_k^2\right).$$

This exercise is from the Williams book.

Hint: Use submartingale inequlaity as in the proof of LIL. Then present M_n (in the exponent) like a telescopic sum, then use the orthogonality of martingale increments. Use part (a), then find the minimum in θ of the expression in the exponent.

Homework 49 (Exercise for Markov chain CLT)**.** Consider the following Markov chain $X = \{X\}_{n=0}^{\infty}$: The state space is $\mathbb{Z}$. The transition probabilities are as follows: $p(0,1) = p(0,-1) = \frac{1}{2}$. For an arbitrary $x \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ we have

$$p(x, x+1) = p(x, 0) = \frac{1}{2}, \text{ and } p(-x, -x-1) = p(-x, 0) = \frac{1}{2}.$$

(a) Find the stationary measure π for X.

Definitions

- Define the operator $P : L^1(\mathbb{Z}, \pi) \to L^1(\mathbb{Z}, \pi)$ by $(Pg)(i) := \sum_{j \in \mathbb{Z}} p(i,j) g(j)$ and let I be the identity on $L^1(\mathbb{Z}, \pi)$. Basically P is an infinite matrix and g is an infinite column vector and the action of the operator P on g is the product of this infinite matrix P with the infinite column vector g like $(P \cdot g)(i)$, $i \in \mathbb{Z}$.
- Let $f : \mathbb{Z} \to \mathbb{R}$ be an arbitrary function satisfying the following conditions:

  $$\forall x \in \mathbb{Z},\ f(x) = -f(-x), \text{ and } \exists a < \sqrt{2} \text{ s.t. } f(x) < a^{|x|} \text{ for all } x \text{ large enough.}$$

  For example: polynomials of the form $f(x) = \sum_{i=1}^{n} b_{2i-1} x^{2i-1}$

(b) Construct an $U \in L^2(\pi)$ such that $((I - P) \cdot U)(i) = f(i)$.

(c) From now on we always assume that $f(x) = x^{-3}$. Determine

$$\sigma^2 := \mathbb{E}_\pi\left[(U(X_1) - U(X_0) + f(X_0))^2\right].$$

(d) Prove that $\mathbb{P}(-3\sigma\sqrt{n} \leq f(X_1) + \cdots + f(X_n) \leq 3\sigma\sqrt{n}) \geq 0.99$ for sufficiently large n.