Markov Processes and Martingales

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C File

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Hitting time for SRW

Let X be a r.v. with

$$\mathbb{P}(X=1)=\mathbb{P}(X=-1)=\frac{1}{2}.$$

Let $X_1, X_2, \dots$ be iid, $X_i \stackrel{d}{=} X$. Let $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$ for $n \geq 1$. We define

(1) $$T := \min\{n : S_n = 1\}$$

Our goal on the next slides is to find the probability generating function $\alpha \mapsto \mathbb{E}\left[\alpha^T\right]$ of T.
We write $\mathcal{F}_n := \sigma(X_1,\dots,X_n) = \sigma(S_0, S_1, \dots, S_n)$. Then $S = (S_n)$ is adapted to $\{\mathcal{F}_n\}$. Clearly,

Hitting time for SRW cont.

$$\mathbb{E}\left[e^{\theta X}\right] = \frac{1}{2}\left(e^{\theta}+e^{-\theta}\right).$$

So,

(2) $$\mathbb{E}\left[(\operatorname{sech}\theta \cdot e^{\theta X_n}) = 1,\right]$$

where $\operatorname{sech}\theta = 1/\cosh\theta = 2/(e^{\theta}+e^{-\theta})$. (Read: hyperbolic secant). So, $M^{\theta} = (M_n^{\theta})$,

(3) $$M_n^{\theta} = (\operatorname{sech}\theta)^n e^{\theta S_n}.$$

is a martingale by A File Example 1.8. Using that T is a stopping time:

Hitting time for SSRW cont.

(4) $$\mathbb{E}\left[M_{T\wedge n}^{\theta}\right] = \mathbb{E}\left[(\operatorname{sech}\theta)^{T\wedge n} e^{\theta S_{T\wedge n}}\right] = 1 \quad \forall n.$$

Assume that $\theta > 0$. Then

- $\exp(\theta \cdot S_{T\wedge n}) \leq e^{\theta}$. So, $M_{T\wedge n}^{\theta} \leq 1$ (this follows from(3) since $\operatorname{sech}\theta \leq 1$).
- $\lim\limits_{n\to\infty} M_{T\wedge n}^{\theta} = M_T^{\theta}$, where $M_T^{\theta} = 0$ if $T = \infty$.

Using (4) and the Dominated Conv. Thm.

(5) $$\mathbb{E}\left[M_T^{\theta}\right] = 1 = \mathbb{E}\Big[\underbrace{(\operatorname{sech}\theta)^T e^{\theta}}_{0 \text{ if } T=\infty}\Big],$$

Hitting time for SSRW cont.

Note that $e^{\theta \cdot S_T} = e^{\theta}$ by the definition of T. So we obtained:

(6) $$\mathbb{E}\left[(\operatorname{sech}\theta)^T\right] = e^{-\theta}, \quad \theta > 0.$$

Observe that

(7) $$\lim_{\theta\to 0^+}(\operatorname{sech}\theta)^T = \begin{cases} 1, & \text{if } T < \infty; \\ 0, & \text{if } T = \infty. \end{cases}$$

Using Dominated Conv. Thm., by (6), (7)

$$1 = \lim_{\theta\to 0^+}\mathbb{E}\left[(\operatorname{sech}\theta)^T\right] = \mathbb{E}\left[\mathbb{1}_{\{T<\infty\}}\right] = \mathbb{P}(T<\infty).$$

Hitting time for SSRW cont.

In (6) let $\alpha := \operatorname{sech}\theta$. Then using (6) we get

(8) $$\sum_{n=1}^{\infty}\alpha^n \cdot \mathbb{P}(T=n) = \mathbb{E}\left[\alpha^T\right]$$
$$= e^{-\theta} = \alpha^{-1}\left[1-\sqrt{1-\alpha^2}\right].$$

Hence,

(9) $$\mathbb{P}(T=2m-1) = (-1)^{m+1}\cdot\binom{\frac{1}{2}}{m}.$$

Explanation of (9)

Recall that for an $|x|<1$ we have

$$(1+x)^{\beta} = \sum_{n=0}^{\infty}\binom{\beta}{n}x^n = \sum_{n=0}^{\infty}\frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!}x^n.$$

Using this with $\beta = 1/2$ and $x = -\alpha^2$ we get

(10) $$\alpha^{-1}\left(1-\sqrt{1-\alpha^2}\right) = \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-1)^{n+1}\alpha^{2n-1}.$$

Putting together (8) and (10) yields (9). ■ The heart of the matter was (8). Now we give an alternative proof for (8).

An easier proof for (8)

Fact 1.1

Consider a (simple symmetric) random walker who starts from -1. Let

- *T_1 be the first time she reaches 0,*
- *T_2 be the time she needs to get to 1 AFTER she reached 0.*

(So, she needs time $T_1 + T_2$ to get to 1 from -1.) Then

(a) $T_1 \stackrel{d}{=} T_2 \stackrel{d}{=} T$ *(T was defined in (1)).*

(b) *T and T_1 are independent .*

An easier proof for (8) cont.

This is immediate from the Strong Markov Property of SSRW. (We learned about the Strong Markov Property on slide # **??** of File A of the course Stochastic Processes.

An easier proof for (8) cont.

Let $\alpha \in (0,1]$. we define $f(\alpha) := \mathbb{E}\left[\alpha^T\right]$. (T was defined in (1)). Then

$$f(\alpha) = \frac{1}{2}\mathbb{E}\left[\alpha^T | X_1 = 1\right] + \frac{1}{2}\mathbb{E}\left[\alpha^T | X_1 = -1\right]$$

$$= \frac{1}{2}\alpha + \frac{1}{2}\mathbb{E}\left[\alpha^{1+T_1+T_2}\right] = \frac{1}{2}\alpha + \alpha f(\alpha)^2.$$

This implies that $f(\alpha) = \alpha^{-1}\left[1 - \sqrt{1-\alpha^2}\right]$, that is (8) holds.

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Superharmonic functions

First recall from File A: Definition 2.2, Theorem 2.6 and Theorem 6.6 (Optional stopping thm). Let S be a countable set and let $P = (p(i,j))_{i,j\in S}$ be stochastic matrix. Let μ be any measure on S. Let $Z = (Z_n)_{n=0}^{\infty}$ be the Markov chain corresponding to the transition probability matrix P and initial distribution μ. That is

(11)
$$\mathbb{P}_\mu\left(Z_0 = i_0, Z_1 = i_1 \dots, Z_n = i_n\right) = \mu_{i_0} \cdot \prod_{k=1}^{n-1} p(i_k, i_{k+1}).$$

Let $\mathcal{F}_n := \sigma(Z_0, \dots, Z_n)$. Then

(12) $$\mathbb{P}_\mu\left(Z_{n+1} = j | \mathcal{F}_n\right) = p(Z_n, j).$$

Superharmonic functions cont.

Recall that for a function $h : S \to \mathbb{R}^+$ we defined the function $Ph : S \to \mathbb{R}^+$ by

$$(Ph)(i) = \sum_{j\in S} p(i,j)h(j), \quad i \in S.$$

Assume that h is P-superharmonic . That is

(13) $$(Ph)(i) \leq h(i), \quad \forall i \in S.$$

It follows from File A Theorem 2.6 that $h(Z_n)$ is a supermartingal.

Superharmonic functions cont.

Namely,

$$\mathbb{E}_\mu\left[h(Z_{n+1})|\mathcal{F}_n\right] = \sum_{p(Z_n,j)} h(j) = (Ph)(Z_n) \leq h(Z_n).$$

Lemma 2.1

The Markov chain $Z = (Z_n)$ is irreducible and recurrent iff every non-negative superharmonic function is constant.

We prove only the $\Longrightarrow$ implication.

Superharmonic functions cont.

Proof

Assume that the chain $Z = (Z_n)$ is irreducible and recurrent. Then $f_{ij} := \mathbb{P}_i\left(T_j < \infty\right) = 1, \quad \forall i,j \in S$, where $T_j := \min\left\{n \geq 1 : Z_n = j\right\}$. Let h be a superharmonic function. Consider the process: $h(Z_n)$. By Theorem 2.6, $h(Z_n)$ is a supermartingale. Then by File A, Theorem 6.8, (a corollary of the Optional Stopping Theorem) we have

$$h(j) = \mathbb{E}_i\left[h(Z_{T_j})\right] \leq \mathbb{E}_i\left[h(Z_0)\right] = h(i).$$

That is h is constant.

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Uncrossings

Given a process $X = (X_n)$ and numbers $a < b$. We define the upcrossings of $[a, b]$ by X_n by time $N \in \mathbb{N}$ as follows: Let $S_0 := 0$, $a < b$ and let us define the following stopping times:

$$R_k := \min \{n \geq S_{k-1} : X_n > b\}$$
$$S_k := \min \{n \geq R_k : X_n < a\} .$$

The upcrossings of $[a, b]$ by time N is

$$U_N[a, b](\omega) := \max \{k : R_k(\omega) \leq N\} .$$

Now we construct a game which corresponds to this process as we did on slide # 24 in File A.

Upcrossing cont.

Imagine that somebody plays games at times $k = 1, 2, \ldots$. Let $X_k - X_{k-1}$ be the net winnings per unit stake in game n.
C_n is the player's stake at time n (specified on the next slide) which is decided based upon the history of the game up to time $n-1$. The winning on game n is $C_n(X_n - X_{n-1})$. The total winning after n game is

(14) $$Y_n := \sum_{1 \leq k \leq n} C_k(X_k - X_{k-1}) =: (C \bullet X)_n .$$

By definition: $Y_0(\omega) \equiv 0$. $(C \bullet X)_0 = 0$ and $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. So, we assumed above that $C_n \in \mathcal{F}_{n-1}$. Clearly, $Y_n - Y_{n-1} = C_n(X_n - X_{n-1})$.

Uncrossing cont., the definition of C_n

Starting from time $n = 0$, we define C_n the stake at time n as follows:

$$C_1 := \mathbb{1}_{\{X_0 < a\}}$$
$$C_n = \mathbb{1}_{\{C_{n-1}=1\} \cap \{X_{n-1} \leq b\}} + \mathbb{1}_{\{C_{n-1}=0\} \cap \{X_{n-1} < a\}} .$$

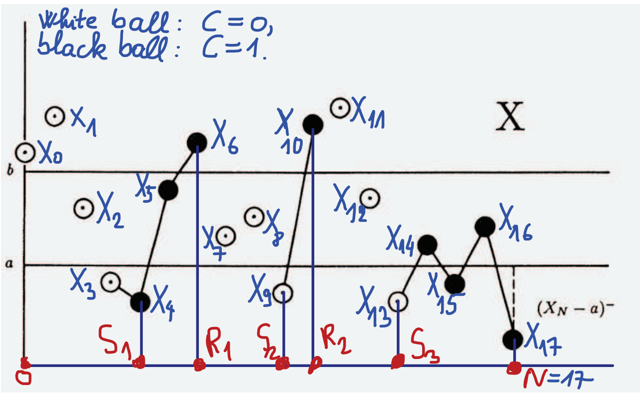
On the next Figure we color $(n, X_n) \in \mathbb{R}^2$

- white if $C_n = 0$,
- black if $C_n = 1$.

In summary:

The first black ball appears where we go below a for the first time. The only way to get a black ball is:

- either after a white ball which is below a or
- after a black ball which is below b.



Uncrossing cont.

Observe that

Remark 3.1

(a) All the increase of Y is due to a run of black balls which does not end at N.
(b) All the decrease of Y is due to a run of black balls which ends at N.

Uncrossing cont.

Remark cont.

(c) every run of black balls which ends earlier than N follows a white ball which is below a (let say X_k) and ends with a black ball above b (let say X_ℓ). Then there is exactly one uprcossing between k and ℓ which corresponds to this run of black balls.
(d) The maximum decrease in Y caused by the last run of black balls is $(X_N - a)^-$, where

$$(X_N - a)^- := \max \{0, -(X_N - a)\} .$$

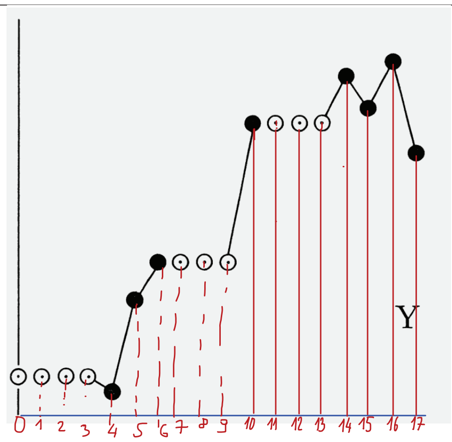


Figure: This and the previous Figure is from Williams' book.

Uncrossing cont.

It follows from Remark 3.1 that

$$(15) \quad Y_N(\omega) \geq (b-a)U_N[a,b](\omega) - [X_N(\omega - a)]^- .$$

Lemma 3.2 (Doob's Upcrossing Lema)

Let $X = (X_n)$ be a supermartingale. Then

$$(16) \quad (b-a)\mathbb{E}[U_N[a,b]] \leq \mathbb{E}\left[(X_N - a)^-\right].$$

Uncrossing cont.

Proof.

Using that C is previsible, bounded and non-negative we may apply Theorem 4.2 of File A to obtain that $Y = C \bullet X$ is also a supermartingale. Then

$$\mathbb{E}[Y_N] \leq \mathbb{E}[Y_0] = 0.$$

This and (15) together imply that (16) holds. □

Uncrossing cont.

Corollary 3.3

Let $X = (X_n)$ be an L^1-bounded supermartingale:

$$(17) \quad \sup_n \mathbb{E}[(|X_n|)] < \infty.$$

For an $a < b$ let $U_\infty[a,b] := \lim_{N\to\infty} U_N[a,b]$ Then

$$(18) \quad (b-a)\mathbb{E}[U_\infty[a,b]] \leq |a| + \sup_n \mathbb{E}[|X_n|].$$

Hence,

$$(19) \quad \mathbb{P}(U_\infty[a,b] = \infty) = 0.$$

Uncrossing cont.

Proof.

By (16), for every N we have

$$(20) \quad (b-a)\mathbb{E}[U_N[a,b]] \leq |a| + \mathbb{E}[|X_N|] \leq |a| + \sup_n \mathbb{E}[|X_n|].$$

Let $N \uparrow \infty$ and use Monotone Convergency Theorem. □

Definition 3.4

For definiteness for a general $X = (X_n)$ we define $X_\infty(\omega) := \limsup_{n\to\infty} X_n(\omega)$.

Doob's Forward Convergence Theorem

Theorem 3.5 (Doob's Forward Convergence Theorem)

Let $X = (X_n)$ be an L^1-bounded ((17) holds) supermartingale. Then

$$X_\infty = \lim_{n\to\infty} X_n, \text{ and } X_\infty < \infty \text{ a.s.}$$

Let Λ be the set of $\omega \in \Omega$ for which $X_n(\omega)$ does NOT converge to a limit in $[-\infty, \infty]$. So there is no limit even is we alow that the limit can be infinite. So, for an $\omega \in \Lambda^c$ the limit $\lim_{n\to\infty} X_N(\omega) \in [-\infty, \infty]$ exist.

Forward Convergence Theorem cont.

Proof.

$$\Lambda = \bigcup_{\substack{a,b\in\mathbb{Q},\\ a<b}} \underbrace{\left\{\omega : \liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)\right\}}_{\Lambda_{a,b}} \subset \bigcup_{\substack{a,b\in\mathbb{Q},\\ a<b}} \underbrace{\{\omega : U_\infty[a,b](\omega) = \infty\}}_{\widetilde{\Lambda}_{a,b}},$$

By (19) we have $\mathbb{P}\left(\widetilde{\Lambda}_{a,b}\right) = 0$, hence $\mathbb{P}(\Lambda) = 0$. So, for almost all ω the following limit exists:

$$X_\infty = \lim_{n\to\infty} X_n(\omega) \in [-\infty, \infty].$$

Forward Convergence Theorem cont.

proof cont.

Now we use Fatau Lemma:

$$\mathbb{E}[|X_\infty|] = \mathbb{E}\left[\liminf_{n\to\infty} |X_n|\right] \leq \liminf_{n\to\infty} \mathbb{E}[|X_n|] \leq \sup_n \mathbb{E}[|X_n|] < \infty,$$

by assumption. This shows that $\mathbb{P}(X_\infty < \infty) = 1$. So the limit exists and less than ∞ almost everywhere. ■

Be careful. It can happen that the limit does not exists in L^1.

Forward Convergence Theorem cont.

Corollary 3.6

*Assume that $X = (X_n)$ is a non-negative supermartingale. Then the limit $X_\infty := \lim_{n\to\infty} X_n$ exists almost surely. (This was Theorem **??** in File E on the course "Stochastic Processes".)*

Proof.

X is L^1-bounded. In deed

$$\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leq \mathbb{E}[X_0].$$

Then we apply Theorem 3.5. □

Pythagorean formula for L^2 martingales

Below we always assume that $M = (M_n)$ is a martingale in L^2:

(21) $$M_n \in L^2 \quad \forall n.$$

Then the Pythagorean formula holds:

(22) $$\mathbb{E}\left[M_n^2\right] = \mathbb{E}\left[M_0^2\right] + \sum_{k=1}^{n} \mathbb{E}\left[(M_k - M_{k-1})^2\right].$$

This follows from the orthogonality (in L^2) of the increments (see Theorem **??** in File E).

Pythagorean formula for L^2 martingales cont.

Namely,

(23) $$M_n = M_0 + \sum_{k=1}^{n} (M_k - M_{k-1})$$

and the increments are orthogonal: $\forall s \leq r \leq t \leq z$ we have

(24) $$\mathbb{E}\left[(M_r - M_s)(M_z - M_t)\right] = 0.$$

and

(25) $$\mathbb{E}\left[M_s(M_z - M_t)\right] = 0.$$

So, we take squares of both sides in (23) to get (22).

Martingales bounded in L^2

Definition 3.7

$M = (M_n)$, $M_n \in L^2$ is bounded in L^2 if

(26) $$\sup_n \|M_n\|_2 < \infty \text{ that is } \sup_n \mathbb{E}\left[M_n^2\right] < \infty.$$

By the Pythagorean formula

(27) $$M \text{ is } L^2 \text{ bounded} \iff \sum_{k=1}^{\infty} \mathbb{E}\left[(M_k - M_{k-1})^2\right] < \infty.$$

Martingales bounded in L^2 cont.

Theorem 3.8

Assume that M is an L^2-bounded martingale. (That is (26) holds). Then

(28) $$M_n \to M_\infty \text{ both a.s. and in } L^2.$$

Proof.

Assume that M is L^2-bounded. Then M is also L^1-bounded. So, we can apply Doob's Convergence Theorem (Theorem 3.5). □

Martingales bounded in L^2 cont.

Proof cont.

This implies that the following limit exists and finite

$$\lim_{n\to\infty} M_n = M_\infty, \qquad \text{a.s.}$$

By the Pythagorean thm.:

(29) $$\mathbb{E}\left[(M_{n+r} - M_n)^2\right] = \sum_{k=n+1}^{n+r} \mathbb{E}\left[(M_k - M_{k-1})^2\right]$$

Let $r \to \infty$ on both sides to obtain:

Martingales bounded in L^2 cont.

Proof cont.

(30) $$\mathbb{E}\left[(M_\infty - M_n)^2\right] \leq \sum_{k=n+1}^{\infty} \mathbb{E}\left[(M_k - M_{k-1})^2\right]$$

Hence,

(31) $$\lim_{n\to\infty} \mathbb{E}\left[(M_\infty - M_n)^2\right] = 0.$$

That is $M_n \to M_\infty$ also in L^2. ■

Martingales bounded in L^2 cont.

We remark that it follows from putting together (3) and (31) that there is equality in (30). That is

(32) $$\mathbb{E}\left[(M_\infty - M_n)^2\right] = \sum_{k=n+1}^{\infty} \mathbb{E}\left[(M_k - M_{k-1})^2\right].$$

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Sums of zero-mean independent r.v.

Theorem 4.1

Assume that $X_1, X_2, \dots$ *are independent and* $\sigma_k^2 := \mathrm{Var}(X_k) < \infty$.

(a) *If* $\sum\limits_{k=1}^{\infty} \sigma_k^2 < \infty$ *then* $\sum\limits_{k=1}^{\infty} X_k$ *converges a.s.*.

(b) *If* $X = (X_n)$ *is bounded (that is* $\exists K$ *s.t.* $\forall n, \forall \omega$, *we have* $|X_n(\omega)| < K$*) and* $\sum\limits_{k=1}^{\infty} X_k$ *converges a.s. then* $\sum\limits_{k=1}^{\infty} \sigma_k^2 < \infty$.

Note that it follows from Kolomogov's $0-1$ law that $\mathbb{P}(X_k \text{ converges }) = 0$ or 1.

Sums of zero-mean independent r.v. cont.

In the proof we use the following

Definition 4.2

$$\mathcal{F}_n := \sigma\{X_1, \dots, X_n\}, \quad \mathcal{F}_0 := \{\emptyset, \Omega\}$$

$$M_n := X_1 + \cdots + X_n, \quad M_0 := 0$$

Further, $A_0 := 0$, $N_0 := 0$ and let

(33) $$A_n := \sum_{k=1}^{n} \sigma_k^2, \quad N_n := M_n^2 - A_n.$$

Sums of zero-mean independent r.v. cont.

Proof of part (a)

We know that M is a martingale with

(34) $$\mathbb{E}\left[(M_k - M_{k-1})^2\right] = \mathbb{E}\left[X_k^2\right] = \sigma_k^2.$$

So, by (22) we get

(35) $$\mathbb{E}\left[M_n^2\right] = \sum_{k=1}^{n} \sigma_k^2 = A_n.$$

If $\sum\limits_{k=1}^{\infty} \sigma_k^2 < \infty$ then $M = (M_n)$ is bounded in L^2, so $\lim\limits_{n\to\infty} M_n$ exists almost surely.

Sums of zero-mean independent r.v. cont.

Proof of part (b)

Similarly to (34) we have

(36)
$$\mathbb{E}\left[(M_k - M_{k-1})^2 | \mathcal{F}_{k-1}\right] = \mathbb{E}\left[X_k^2 | \mathcal{F}_{k-1}\right] = \mathbb{E}\left[X_k^2\right] = \sigma_k^2.$$

since X_k is independent of $\mathcal{F}_{k-1}$. Using this and the fact that $M_{k-1} \in \mathcal{F}_{k-1}$, we obtain that

$$\sigma_k^2 = \mathbb{E}\left[M_k^2 | \mathcal{F}_{k-1}\right] - 2M_{k-1}\mathbb{E}\left[M_k | \mathcal{F}_{k-1}\right] + M_{k-1}^2$$
$$= \mathbb{E}\left[M_k^2 | \mathcal{F}_{k-1}\right] - M_{k-1}^2.$$

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

That is $\mathbb{E}\left[M_k^2 | \mathcal{F}_{k-1}\right] - \sigma_k^2 = M_{k-1}^2$. This implies that

(37) $$\mathbb{E}\left[\underbrace{M_n^2 - \sum_{k=1}^{n} \sigma_k^2}_{N_n} | \mathcal{F}_{n-1}\right] = \underbrace{M_{n-1}^2 - \sum_{k=1}^{n-1} \sigma_k^2}_{N_{n-1}}.$$

So, we have just proved that N_n (defined in (33)) is a martingale.

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

For a fixed $c > 0$, we define the stopping time

$$T := \inf\{r : |M_r| > c\}.$$

We defined the stopped process on slide # 35 of File A as

$$N_n^T(\omega) := N_{T\wedge n}(\omega).$$

It follows from Theorem 6.1 of File A that N^T is also a martingale since N is a martingale as we pointed out above.

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

So, by Theorem 6.1, we have

(38) $$0 = \mathbb{E}[N_0] = \mathbb{E}\left[N_n^T\right] = \mathbb{E}\left[(M_n^T)^2\right] - A_{T\wedge n}.$$

Now we prove that

(39) $$|M_n^T| = |M_{T\wedge n}| \le K + c, \quad \forall n.$$

Namely, if $n < T(\omega) \le \infty$ then $|M_n^T(\omega)| = |M_n(\omega)| \le c$ by the definition of T. So, in this case (39) holds.

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

On the other hand, if $T(\omega) \le n$ then

(40) $$M_{T(\omega)\wedge n}(\omega) = M_{T(\omega)}(\omega).$$

It follows from our assumption that whenever T is finite we have

$$|M_{T(\omega)} - M_{T(\omega)-1}(\omega)| = |X_T(\omega)| < K.$$

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

However, by the definition of T, whenever $T < \infty$:

$$|M_{T-1}| \leq c.$$

Using the last two inequalities and the triangular inequality we have that whenever $T < \infty$ we have

$$|M_T| \leq K + c.$$

Putting together this and (54) we obtain that (39) holds also for those ω for which $T(\omega) \leq n$. So we have verified (39)

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

Using (39) and (38) we get

$$\mathbb{E}\left[A_{T\wedge n}\right] \leq (K+c)^2, \quad \forall n.$$

Remember that we assumed that $\sum\limits_{k=1}^{n} X_k = M_n$ a.s. convergent. This means that the partial sums $\{M_n(\omega)\}_{n=1}^{\infty}$ is bounded for a.a. ω.

Sums of zero-mean independent r.v. cont.

Proof of part (b) cont.

So, there is an L such that on a set H of positive measure, $\mathbb{P}(H) > 0$, the partial sums are smaller than L in modulus. So for any $c > L$ we have

$$T(\omega) = \infty, \quad \forall \omega \in H.$$

Hence for all $\omega \in H$ and for all n,

$$\sum_{k=1}^{n} \sigma_k^2 = A_n = A_{T\wedge n}(\omega) \overset{by (38)}{=} \mathbb{E}\left[M_{n\wedge T}^2\right] < (K+c)^2.$$

In the last proof we used only that X is bounded and the partial sums are bounded on a set of positive measure.

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Doob decomposition

Theorem 5.1 (Doob decomposition)

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$. Let $X = (X_n)$ be an adapted process with $X_n \in L^1$ for all n. Then X has a Doob decomposition:

$$X = X_0 + M + A, \tag{41}$$

where

- $M = (M_n)$ *is a martingale with* $M_0 = 0$
- $A = (A_n)$ *is previsible (that is* $A_n \in \mathcal{F}_{n-1}$*) with* $A_0 = 0$. *(*A_n *is called compensator of* X_n*).*

The decomposition is unique mod zero.

Doob decomposition cont.

The last comment means that: If

$$X = X_0 + \widetilde{M} + \tilde{A}$$

is another decomposition then

$$\mathbb{P}\left(M_n = \widetilde{M}_n, A_n = \tilde{A}_n, \forall n\right) = 1.$$

Corollary 5.2

X is a submartingale iff A in its the Doob decomposition (41), is an increasing process. That is

$$\mathbb{P}\left(A_n \leq A_{n+1}\right) = 1. \tag{42}$$

Doob decomposition cont.

Proof Thm 5.1

Strategy: Find our what the compensator A should be. If X has decomposition given by (41)

$$\begin{aligned} \mathbb{E}\left[X_n - X_{n-1} | \mathcal{F}_{n-1}\right] &= \mathbb{E}\left[M_n - M_{n-1} | \mathcal{F}_{n-1}\right] + \mathbb{E}\left[A_n - A_{n-1} | \mathcal{F}_{n-1}\right] \\ &= 0 + A_n - A_{n-1}. \end{aligned} \tag{43}$$

So, the decomposition in (41) comes from:

$$A_n := \sum_{k=1}^{n} \mathbb{E}\left[X_k - X_{k-1} | \mathcal{F}_{k-1}\right]. \tag{44}$$

Doob decomposition cont.

Proof Thm 5.1 cont.

Namely, let $M_n := X_n - A_n$. Then $M = (M_n)$ is a martingale:

$$\begin{aligned} \mathbb{E}\left[M_n | \mathcal{F}_{n-1}\right] &= \mathbb{E}\left[X_n - A_n | \mathcal{F}_{n-1}\right] \\ & \mathbb{E}\left[X_n | \mathcal{F}_{n-1} - A_n\right] \\ & \mathbb{E}\left[X_n | \mathcal{F}_{n-1}\right] - A_{n-1} - \mathbb{E}\left[X_n | \mathcal{F}_{n-1}\right] + X_{n-1} \\ & M_{n-1}. \end{aligned}$$

Doob decomposition cont.

Proof Corollary 5.2

$X = (X_n)$ is a submartingale if

$$\mathbb{E}[X_k - X_{k-1} | \mathcal{F}_{k-1}] \geq 0.$$

On the other hand, by (43) this happens when

$$A_n \geq A_{n-1}.$$

Doob decomposition cont.

Remark 5.3

Assume taht $X = (X_n)$ is an L^2 martingale, with $X_0 = 0$. Then (by Theorem **??** of File E of the Course Stochastic Processes), $\varphi(X_n)$ is a submartingale for any convex function φ. In particular,

(45) X_n martingale $\Longrightarrow$ X_n^2 submartingale.

Let $A = (A_n)$ be the compensator of X. That is

$$X_n^2 - A_n \text{ is a martingale.}$$

We say that (A_n) is the quadratic variation of (X_n).

Doob decomposition cont.

Remark 5.4

Let $X = (X_n)$ be an L^2 martingale. Then the quadratic variation is

$$A_n = \sum_{k=1}^{n} \mathbb{E}\left[(X_k - X_{k-1})^2 | \mathcal{F}_{k-1}\right].$$

Recall:

$$X_n = Z_0 + Z_1 + \cdots + Z_n$$

is a random walk if Z_n are iid and L^1. Let $\mu := \mathbb{E}[X_i]$. Then $X_n - n\mu$ is a martingale. This was Example **??** in File E in the course "stochastic Processes".

Doob decomposition cont.

Remark 5.5

Let $X = (X_n)$ be an L^2 random walk that is a martingale. Then the quadratic variation is $A_n = n\sigma^2$. The proof is a home work # 22.

Remark 5.6

Let X_n be an L^2 martingale with quadratic variation $A = (A_n)$. Let C_n be previsible and also L^2. Then the quadratic variation $B = (B_n)$ of their martingale transform $Y = C \bullet A$ is $B = C^2 \bullet X$. The proof is assigned as home work # 23.

The angle bracket process

Let M be an L^2 martingale with $M_0 = 0$. We write

(46) $$M^2 = N + A,$$

where N is a martingale and A is previsible and increasing with

(47) $$N_0 = 0 \text{ and } A_0 = 0.$$

We write

(48) $$< M > := A.$$

So it makes sense to define

$$A_\infty := \lim_{n\to\infty} A_n, \quad \text{a.s.}.$$

The angle bracket process cont.

Since $\mathbb{E}[N_n] = \mathbb{E}[N_0] = 0$, we have

$$\mathbb{E}\left[M_n^2\right] = \mathbb{E}[A_n].$$

So, by the monotone convergence theorem:

(49) M is bounded in $L^2 \iff \mathbb{E}[A_\infty] < \infty$.

Note that

(50) $$A_n - A_{n-1} = \mathbb{E}\left[M_n^2 - M_{n-1}^2 | \mathcal{F}_{n-1}\right] = \mathbb{E}\left[(M_n - M_{n-1})^2 | \mathcal{F}_{n-1}\right].$$

Convegence of M_n vs. $A_\infty < \infty$

Theorem 5.7

Let M be an L^2 martingale with $M_0 = 0$ and let A be a version of $< M >$. Then

(a) *For almost all ω for which $A_\infty(\omega) < \infty$ we have $\lim_{n\to\infty} M_n(\omega)$ exists.*

(b) *Assume that M has uniformly bounded increments. Then for almost all ω for which $\lim_{n\to\infty} M_n(\omega)$ exists, we have $A_\infty(\omega) < \infty$.*

Convegence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a)

Recall: A is previsible, that is

(51) $$A_{z+1} \in \mathcal{F}_z, \quad \forall z \geq 0.$$

Hence

$$S(k) := \inf\{n : A_{n+1} > k\}$$

is a stopping time. $S(k) = \infty$ means that $A_r \leq k$ for all $r \in \mathbb{N}$.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

The relevance of $S(k)$:

(52) $$\{A_\infty < \infty\} = \bigcup_k \{S(k) = \infty\}.$$

So, the right hand side above is a partition of $\{A_\infty < \infty\}$.

Fact 5.8

The stopped process $A^{S(k)}$ is previsible.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

To prove the Fact we need to verify that

(53) $$A_{n\wedge S(k)}(B) \in \mathcal{F}_{n-1}, \quad \forall B \in \mathcal{R}, \ \forall n \geq 1.$$

To see this, we fix an n and a $B \in \mathcal{R}$ (Borel set on the line) and we represent $\left\{A_{n\wedge S(k)} \in B\right\}$ as:

$$\left\{A_{n\wedge S(k)} \in B\right\} = F_1 \cup F_2,$$

where F_1, F_2 correspond to $S(k) = r < n$, $S(k) \geq n$ respectively. That is:

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

$$F_1 := \bigcup_{r=0}^{n-1} \{S(k) = r; A_r \in B\}$$
$$= \bigcup_{r=0}^{n-1} \left(\{A_r \in B\} \bigcap \bigcap_{\ell=1}^{r} \{A_\ell \leq k\} \bigcap \{A_{r+1} > k\}\right).$$

Using that $A_{r+1} \in \mathcal{F}_r$ for every r, (A is previsible), and $r+1 \leq n$ above, we get that

(54) $$F_1 \in \mathcal{F}_{n-1}.$$

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

$$F_2 = \{A_n \in B\} \bigcap \{S(k) \geq n\}$$
$$= \{A_n \in B\} \bigcap \bigcap_{r=1}^{n} \{A_r \leq k\}.$$

Clearly, all of the events in the previous line are in $\mathcal{F}_{n-1}$. This completes the proof of (53). So, we have verified Fact 5.8 above.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

So, we know that

- $A^{S(k)}$ is previsible (Fact 5.8),
- $N^{S(k)}$ is a martingale. This is so because
  - (a) $N = M^2 - A$ (defined in (46)) is a martingale (by definition),
  - (b) $S(k)$ is a stopping time

  so, $N^{S(k)}$ is a martingale by Theorem 6.1 of File A (this theorem says that in general, a martingale raised to the power of a stopping time is a martingale).

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

By the uniqueness of Doob decomposition of

$$(M^{S(k)})^2 = \text{martingale} + \text{previsible}$$

and since

$$(M^{S(k)})^2 = N^{S(k)} + A^{S(k)},$$

we obtain that

(55) $$< M^{S(k)} >= A^{S(k)}.$$

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

By the definition of $S(k)$ we have that

$$A_n^{S(k)} \leq k, \quad \forall n.$$

So,

$$\mathbb{E}\left[A_\infty^{S(k)}\right] < \infty.$$

Using this, we can apply (49) for $M^{S(k)}$ instead of M to obtain that $M^{S(k)}$ is an L^2 bounded martingale.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

Finally, we can apply Theorem 3.8 (L^2 bounded martingale convergence thm.) to get

(56) $$\lim_{n\to\infty} M_{n\wedge S(k)} \text{ exists almost surely.}$$

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (a) cont.

However,

$$(57) \qquad \{A_\infty < \infty\} = \bigcup_k \{S(k) = \infty\}.$$

So part (a) follows from the combination of (56) and (57). Namely, if $A_\infty(\omega) < \infty$ then by (57) there exists a $k = k(\omega)$ such that $S(k)(\omega) = \infty$. Then

$$M_{n \wedge S(k)(\omega)}(\omega) = M_n(\omega) \quad \forall n.$$

Hence by (56) we obtain that part (a) holds.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (b)

We argue by contradiction. Assume that part (b) of Theorem 5.7 is not true. Then we have

$$(58) \qquad \mathbb{P}\left(A_\infty = \infty, \sup_n |M_n| < \infty\right) > 0.$$

Then $\exists c$ s.t. $\mathbb{P}\left(A_\infty = \infty, \sup_n |M_n| < c\right) > 0$. That is

$$(59) \qquad \mathbb{P}\left(A_\infty = \infty, T(c) = \infty\right) > 0,$$

where $T(c) = \inf\{r : |M_r| > c\}$

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (b) cont.

By Theorem 6.1 of File A we have

$$(60) \qquad 0 = \mathbb{E}[N_0] = \mathbb{E}\left[N^n_{T(c)}\right] = \mathbb{E}\left[N_{T(c)\wedge n}\right] = \mathbb{E}\left[M^2_{T(c)\wedge n}\right] - \mathbb{E}\left[A_{T(c)\wedge n}\right].$$

Now we prove that there exists a K s.t.

$$(61) \qquad |M_{T(c)\wedge n}| \le K + c.$$

Namely, by assumption the increments of M are uniformly bounded.

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (b) cont.

That is, $\exists K$ s.t.

$$(62) \qquad |M_n(\omega) - M_{n-1}(\omega)| < K, \quad \forall \omega \text{ and } \forall n.$$

Observe that if $n < T(c)$ then

$$|M_{T(c)\wedge n}| = |M_n| < c.$$

On the other hand, if $n \ge T(c)$ then by the definition of $T(c)$ and by (62)

Convergence of M_n vs. $A_\infty < \infty$ cont.

Proof of Theorem 5.7 part (b) cont.

$$|M_{T(c)\wedge n}| = |M_{T(c)}| \le K + c$$

This and (60) imply that

$$(63) \qquad \mathbb{E}\left[A_{T(c)\wedge n}\right] \le (c + K)^2, \quad \forall n.$$

However, by monotone convergence Theorem we get that (59) and (63) contradict to each other.■

Cesáro's Lemma

Given two sequences of real numbers $\{b_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ satisfying the following conditions:

$$0 < b_1, \quad \text{for all } n: \ b_n < b_{n+1}, \quad \lim_{n\to\infty} b_n = \infty.$$

and

$$\lim_{n\to\infty} v_n = v_\infty \in \mathbb{R}.$$

Then

$$(64) \qquad \frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1}) \cdot v_k \to v_\infty.$$

Kronecker's Lemma

Let

- $\{b_n\}_{n=1}^\infty$ be defined as in Cesáro's Lemma,
- $\{x_n\}_{n=1}^\infty$ be an arbitrary sequence of real numbers
- $s_n := x_1 + \cdots + x_n$.

Then

$$(65) \qquad \left(\sum_{n=1}^{\infty} \frac{x_n}{b_n} \text{ converges}\right) \Longrightarrow \left(\frac{s_n}{b_n} \to 0\right).$$

Strong law under variance cond.

Theorem 5.9

Let $X_1, X_2, \dots$ be independent r.v. s.t.

$$(66) \qquad \mathbb{E}[X_n] = 0, \text{ and } \sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n)}{n^2} < \infty.$$

Then

$$(67) \qquad \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = 0 \quad \text{a.s.}.$$

Strong law under variance cond. cont.

Proof of Theorem 5.9

By Kronecker's Lemma we only need to prove that

$$\sum_{n=1}^{\infty} \frac{X_n}{n}, \quad \text{converges a.s..}$$

We get this if we apply part (a) of Theorem 4.1 for the r.v. X_n/n since their mean is zero and $\mathrm{Var}(X_n/n) = \mathrm{Var}(X_n)/n^2$.

Strong law for L^2 martingales

Theorem 5.10

Let M be an L^2 martingale with $M_0 = 0$ and let $A =< M >$ (defined in (48)). Then

$$(68) \qquad A_\infty(\omega) = \infty \Longrightarrow \lim_{n\to\infty} \frac{M_n(\omega)}{A_n(\omega)} = 0.$$

Proof

First we claim that

$$(69) \qquad W_n := \sum_{k=1}^{n} \frac{M_k - M_{k-1}}{1 + A_k} \quad \text{is a martingale.}$$

Strong law for L^2 martingales cont.

Proof cont.

This is so because for $C_n := \left\{(1 + A_n)^{-1}\right\}$ the process $C = (C_n)$ is a bounded and previsible. Hence

$$(70) \qquad W_n = C \bullet M$$

is a martingale too.

Fact 5.11

$$(71) \qquad < W >_\infty \leq 1.$$

Strong law for L^2 martingales cont.

Proof cont.

Since $< W >$ is a non-negative increasing process, in order to prove Fact 5.11, it is enough to show that

$$(72) \qquad < W >_n = \sum_{k=1}^{n} (< W >_k - < W >_{k-1}) \overset{\text{by (50)}}{=} \sum_{k=1}^{n} \mathbb{E}\left[(W_n - W_{n-1})^2 | \mathcal{F}_{n-1}\right] \leq 1.$$

In the one but last step we used (50).

Strong law for L^2 martingales cont.

Proof cont.

To verify (72) we prove another claim about the yellow part of the previous formula:
Now we claim that

$$(73) \qquad \mathbb{E}\left[(W_n - W_{n-1})^2 | \mathcal{F}_{n-1}\right] \leq \frac{1}{1+A_{n-1}} - \frac{1}{1+A_n}.$$

To verify this, note that:

Strong law for L^2 martingales cont.

Proof cont.

$$\mathbb{E}\left[(W_n - W_{n-1})^2 | \mathcal{F}_{n-1}\right] = \mathbb{E}\left[\frac{(M_n - M_{n-1})^2}{(1 + A_n)^2} | \mathcal{F}_{n-1}\right]$$

$$= (1 + A_n)^{-2} \cdot \left(\underbrace{\mathbb{E}\left[\underbrace{M_n^2}_{N_n + A_n} | \mathcal{F}_{n-1}\right]}_{N_{n-1} + A_n} - \underbrace{M_{n-1}^2}_{N_{n-1} + A_{n-1}} \right)$$

$$= (1 + A_n)^{-2} \cdot (N_{n-1} + A_n - (N_{n-1} + A_{n-1}))$$

$$= (1 + A_n)^{-2} \cdot (A_n - A_{n-1})$$

Strong law for L^2 martingales cont.

Proof cont.

However, using that A_n is increasing we get

$$(74) \qquad \frac{A_n - A_{n-1}}{(1 + A_n)^2} \leq \frac{1}{1 + A_{n-1}} - \frac{1}{1 + A_n}.$$

This and the argument on the previous slide verifies that (73) holds.

Strong law for L^2 martingales cont.

Proof cont.

Observe that in (74) we have a telescopic sum on the right hand side. This immediately implies that (72) holds which, in turn, implies that Fact 5.11 holds.From (71) and from Theorem 5.7 part (a) we get that

$$\lim_{n\to\infty} W_n \text{ exists.}$$

We can use Kronecker's Lemma if $b_k = 1 + A_k \to \infty$ that is $A_\infty = \infty$. In this case by Kronecker's Lemma we get $\lim_{n\to\infty} M_n/A_n = 0$.

Borel Cantelli Lemma

Let E_n be events on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We write

$$E_\infty := \limsup_{n\to\infty} E_n := \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_n.$$

Clearly, E_∞ is the set of ω for which infinitely many of E_n occur.

Lemma 5.12 (Borel-Cantelli Lemma)

(a) *If* $\sum_{k=1}^{\infty} \mathbb{P}(E_n) < \infty$ *then* $\mathbb{P}(E_\infty) = 0$.

(b) *If* $\{E_i\}_i$ *are independent and* $\sum_{k=1}^{\infty} \mathbb{P}(E_n) = \infty$ *then* $\mathbb{P}(E_\infty) = 1$.

Borel Cantelli Lemma cont.

Corollary 5.13 (Infinite Monkey theorem)

Monkey typing random on a typewriter for infinitely time will type the complete works of Shakespeare eventually.

Figure: Picture is from Wikipedia. Proof is a homework

Levy's extension of BC Lemma

Theorem 5.14

Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ *and a sequence of events* $E_n \in \mathcal{F}_n$. *Let*

$$Z_n := \sum_{k=1}^{n} \mathbb{1}_{E_k}, \quad \xi_k := \mathbb{P}(E_k|\mathcal{F}_{k-1}), \quad Y_n := \sum_{k=1}^{n} \xi_k.$$

Then for almost all ω

(a) *If* $Y_\infty(\omega) < \infty$ *then* $Z_\infty(\omega) < \infty$.

(b) *If* $Y_\infty(\omega) = \infty$ *then* $\lim_{n\to\infty} \frac{Z_n(\omega)}{Y_n(\omega)} = 1$.

Proof of BC Lemma as a corollary of Thm 5.14

Proof of BC Lemma as a corollary of Thm 5.14.

Using that $\mathbb{E}[\xi_k] = \mathbb{P}(E_k)$ we can see that part (a) of Theorem 5.14 immediately implies the first part of BC Lemma.

On the other hand if $\{E_n\}$ are independent and $\mathcal{F}_n := \sigma(E_1, \dots E_n)$ then $\xi_k = \mathbb{P}(E_k)$. Hence the second part of BC follows from part (b) of Theorem 5.14. □

Levy's extension of BC Lemma cont.

Proof of Thm 5.14

Observe that $Z = (Z_n)$ is submartingale with $Z_0 := 0$. Namely, $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[\mathbb{1}_{E_{n+1}}|\mathcal{F}_n] + Z_n \geq Z_n$. Now observe that

- Y is previsible since $\xi_k \in \mathcal{F}_{k-1}$.
- $M := Z - Y$ is a martingale since by $E_k \in \mathcal{F}_k$:

$$\mathbb{E}[Z_{n+1} - Y_{n+1}|\mathcal{F}_n] = Z_n + \mathbb{E}[\mathbb{1}_{E_{n+1}}|\mathcal{F}_n] - (Y_n + \mathbb{E}[\mathbb{1}_{E_{n+1}}|\mathcal{F}_n]) = Z_n - Y_n.$$

Levy's extension of BC Lemma cont.

Proof of Thm 5.14 cont.

That is $Z = M + Y$ be the Doob decomposition of Z. Now we observe that

$$(75) \qquad A_n := < M >_n = \sum_{k=1}^{n} \xi_k(1 - \xi_k) \leq Y_n.$$

To see this we substitute X_k for M_k^2 into (44) and then an immediate calculation yields (75).

Now we prove part (a): If $Y_\infty(\omega) < \infty$ then $A_\infty(\omega) < \infty$. So, by Theorem 5.7 the limit $\lim_{n\to\infty} M_n(\omega)$ exists. In this way $Z_\infty < \infty$ since $Y_\infty < \infty$ by assumption and $\lim_{n\to\infty} M_n(\omega)$ exists.

Levy's extension of BC Lemma cont.

Proof of Thm 5.14 cont.

Now we prove part (b):

- If $Y_\infty = \infty$ and $A_\infty < \infty$ then $\lim_{n\to\infty} M_n$ exists and it is clear that $Z_n/Y_n \to 1$.
- If $Y_\infty = \infty$ and $A_\infty = \infty$. It follows from Theorem 5.10 that $M_n/A_n \to 0$. So, by the last part of (75) we have $M_n/Y_n \to 0$. From here and from the definition of M we get $Z_n/Y_n \to 1$.

$\dim_{\mathrm{H}}$

Closure

This Section is built on Chapter 10.8.5 of Resnik's book [20]. Here we discuss the following problem:

Let $X = (X_n)$ be a positive martingale w.r.t. the filtration $\mathcal{F}_n$. Then we know by Corollary 3.6 that $X_n \to X_\infty$ a.s.. However, in general this does NOT mean that $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$.

In this short Section first we give a counter example which verifies the last assertion. Then we state a useful theorem (Theorem 6.3) related to this problem and an important corollary of this theorem. The proof of Theorem 6.3 is available in [20, Section 10.8.5]. We omit this proof here.

Closure cont.

Example 6.1

Let Z_n be a branching process and assume that the mean of the offspring distribution $\mu < 1$. Then the population dies out so, $Z_\infty = 0$. On the other hand $W_n := Z_n/\mu^n$ is a non-negative martingale so it converges to $W_\infty = 0$. However, $Z_n/\mu^n \neq 0$ for all n.

Definition 6.2

A martingale $\{X_n, \mathcal{F}_n\}$ is closed (or right-closed) if there exists an L^1 r.v. $X_\infty \in \mathcal{F}_\infty = \sigma(\mathcal{F}_n, n \in \mathbb{N})$ s.t.

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n] \text{ for all } n.$$

Closure cont.

Theorem 6.3

Given a $p \geq 1$ and a rv. $X \in L^p$. Also given a filtration $\mathcal{F}_n$ and we write $\mathcal{F}_\infty := \sigma(\mathcal{F}_n, n \in \mathbb{N})$. Then for

$$X_n := \mathbb{E}[X | \mathcal{F}_n] \text{ and } X_\infty := \mathbb{E}[X | \mathcal{F}_\infty] \tag{76}$$

we have X_n is a closed martingale which converges a.s. and in L^p:

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_\infty] \tag{77}$$

$$X_n \to X_\infty \text{ a.s. and in } L^p. \tag{78}$$

Closure cont.

Corollary 6.4

For $p \geq 1$ the class of L^p convergent non-negative martingales is equal to the class:

$$\mathcal{C} := \{(\mathbb{E}[X|\mathcal{F}_n])_{n=0}^\infty, X \in L^p, X \geq 0\}.$$

Proof of the Corollary

If $(X_n) \in \mathcal{C}$ then by Theorem 6.3 X_n is L^p convergent. Conversely, let $X_n \geq 0$ be an L^p convergent martingale. If $r \geq n$ then $\mathbb{E}[X_r | \mathcal{F}_n] = X_n$.

Closure cont.

Proof of the Corollary cont.

By assumption that $X_r \xrightarrow{L^p} X_\infty$ and $\mathbb{E}[\bullet | \mathcal{F}_n]$ is continuous in L^p metric (this follows from the L^p-non expansive property (property (e)) of the conditional expectation, (see #133 slide if File "Some basic Facts from probability theory"). So, as $r \to \infty$ we have

$$X_n = \mathbb{E}[X_r | \mathcal{F}_n] \xrightarrow{L^p} \mathbb{E}[X_\infty | \mathcal{F}_n].$$

So $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$. ∎

$\dim_H$

Recall Fatou Lemma

From now we follow again Williams book [23, Chapter 14] about the uniformly integrable martingales. First we recall some very well known lemmas:

Lemma 7.1 (Fatou Lemma)

Let $X_1, X_2, \dots$ be non-negative r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

May be both sides are infinite.

Reverse Fatou Lemma

Lemma 7.2 (reverse Fatou Lemma)

Let $\{X_n\}$ be sequence of r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\exists$ a non-negative r.v. $Y \in \mathcal{F}$ such that $X_n \leq Y$ and $\mathbb{E}[Y] < \infty$. Then

$$\mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \geq \limsup_{n\to\infty} \mathbb{E}[X_n].$$

Proof.

Apply Fatou Lemma for $Y - X_n$. □

Theorem BDD

We will need the following version of Bounded Convergence Theorem (abbreviated as BDD):

Theorem 7.3 (Theorem BDD)

Let X_n, X be rv. on $(\Omega, \mathcal{F}, \mathbf{P})$. We assume that

(a) $X_n \xrightarrow{P} X$ *(X_n tends to X is Probability)* ***AND***

(b) $\exists K$ *s.t.* $\forall \omega \in \Omega$, $|X_n(\omega)| < K$. *(That is the process $X = (X_n)$ is bounded.)*

Then

$$X_n \xrightarrow{L^1} X, \text{ that is } \mathbb{E}[|X_n - X|] \to 0.$$

Theorem BDD cont

Proof of Thm BDD

We claim that

(79) $$|X| \le K \text{ a.s.}.$$

Namely, for any $k \in \mathbb{N}$

$$\mathbb{P}\left(|X| > K + k^{-1}\right) \le \mathbb{P}\left(|X - X_n| > k^{-1}\right), \quad \forall n,$$

Using assumption (a) the right hand side tends to zero for every k. This yields that (79) holds.

Theorem BDD cont

Proof of Thm BDD cont.

For a given ε choose n_0 s.t. for all $n > n_0$ we have

$$\mathbb{P}\left(|X_n - X| > \frac{1}{3}\varepsilon\right) < \frac{\varepsilon}{3K}, \quad \text{for } n \ge n_0.$$

So, whenever $n \ge n_0$ we have

Theorem BDD cont

Proof of Thm BDD cont.

$$\begin{aligned}\mathbb{E}[|X_n - X|] =& \mathbb{E}\left[|X_n - X|; |X_n - X| > \frac{\varepsilon}{3}\right] \\ &+ \mathbb{E}\left[|X_n - X|; |X_n - X| \le \frac{\varepsilon}{3}\right] \\ =& 2K\mathbb{P}\left(|X_n - X| > \frac{\varepsilon}{3}\right) + \frac{\varepsilon}{3} \\ \le& \ \varepsilon. \blacksquare\end{aligned}$$

a.s. convergence vs conv. in prob.

The proof of the following theorem is available in [14, p.137].

Theorem 7.4

Given X, X_n be rv on $(\Omega, \mathcal{F}, \mathbb{P})$. Let

$$Y_n := \sup_{k \ge n} |X_k - X|.$$

Then

(80) $$X_n \xrightarrow{a.s.} X \iff Y_n \xrightarrow{P} 0.$$

Absolute continuity

Theorem 7.5

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

(81) *If $F \in \mathcal{F}$ and $\mathbb{P}(F) < \delta \Longrightarrow \mathbb{E}[|X|; F] < \varepsilon$.*

Proof

We argue by contradiction. Assume that $\exists \varepsilon_0 > 0$ and $\{F_n\}$ s.t. $F_n \in \mathcal{F}$ s.t.

$$\mathbb{P}(F_n) < 2^{-n} \text{ and } \mathbb{E}[|X|; F_n] > \varepsilon_0.$$

Absolute continuity cont.

Proof cont.

Let

$$H := \limsup_{n \to \infty} F_n.$$

That is $\mathbb{1}_H = \limsup_{n \to \infty} \mathbb{1}_{F_n}$. Then by Borel-Cantelli Lemma,

$$\mathbb{P}(H) = 0.$$

On the other hand by the reversed Fatou Lemma applied for the indicator functions of F_n we have:

$$\mathbb{E}[|X|; H] > \varepsilon_0.$$

The last two displayed formulas contradict. $\blacksquare$

Absolute continuity cont.

Corollary 7.6

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\varepsilon > 0$. Then $\exists K \in [0, \infty)$ such that

(82) $$\mathbb{E}[|X|; |X| > K] < \varepsilon.$$

Proof.

Choose δ for ε as in Theorem 7.5. Choose a K s.t. $\mathbb{P}(|X| > K) \le \frac{\mathbb{E}[|X|]}{K} < \delta$. Then by the previous theorem $\mathbb{E}[|X|; |X| > K] < \varepsilon$. $\square$

Homework # 31 is an extension of this Corollary.

$\dim_H$

UI martingales, definition

Definition 8.1

Let $\mathcal{C}$ be a class of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\mathcal{C}$ is uniformly integrable (UI) if

$$(83) \quad \forall \varepsilon > 0, \exists K \text{ s.t. } \forall X \in \mathcal{C}, \quad \mathbb{E}[|X|; |X| > K] < \varepsilon.$$

Example 8.2 (Example of non-UI martingale)

Let the probability space be $([0,1], \mathcal{R}[0,1], \mathcal{L}_1)$ and $X_n = n \cdot \mathbb{1}_{[0,n^{-1}]}$. $\forall K$ if $n > K$ then $\mathbb{E}[X_n; |X_n| > K] = 1$ still $X_n \to 0$ a.s. (but $X_n \not\to 0$ in L^1).

UI families cont.

Remark 8.3

We have seen above that an L^1-bounded family is NOT necessarily UI but it is obvious that every UI family is L^1-bounded.

How to check if a family is UI? Two simple ways are as follows:

UI families cont.

Lemma 8.4

Let $\mathcal{C}$ is a class of random variables of $(\Omega, \mathcal{F}, \mathbb{P})$. Then either of the following two conditions imply that $\mathcal{C}$ is UI.

(a) *If $\exists p > 1$ and $A \in \mathbb{R}$ such that $\mathbb{E}[|X|^p] < A$ for all $X \in \mathcal{C}$. (L^p bounded for some $p > 1$.)*

(b) *$\exists Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, s.t. $\forall X \in \mathcal{C}$ we have $|X(\omega)| \leq Y(\omega)$. ($\mathcal{C}$ is dominated by an integrable (non-negative) r.v..)*

The proofs are easy and left as homeworks.

UI families cont.

Theorem 8.5

Let $X, X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for every n. The the following two assertions are equivalent:

1. *$X_n \xrightarrow{L^1} X$ (that is $X_n \to X$ in L^1 that is $\mathbb{E}[|X_n - X|] \to 0$).*
2. *Both of the following conditions hold:*
   (a) *$X_n \xrightarrow{P} X$ (that is $\forall \varepsilon > 0$: $\lim\limits_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$) AND*
   (b) *The sequence $\{X_n\}_{n=1}^{\infty}$ is UI.*

The complete proof is available: [23, p. 131]. Here we prove only the "if" part.

UI families cont.

Proof of the if part (a)

Assume that (a) and (b) hold. For a $K \in [0, \infty)$ we define

$$\varphi_K(x) := \begin{cases} K, & \text{if } x > K; \\ x, & \text{if } -K \leq x \leq K; \\ -K, & \text{if } x < -K. \end{cases}$$

Fix an $\varepsilon > 0$. Now we apply the fact that X_n is UI and Corollary 7.6 in this order to conclude that $\exists K$ s.t.

$$(84) \quad \mathbb{E}[|\varphi_K(X_n) - X_n|] < \frac{\varepsilon}{3}; \ \mathbb{E}[|\varphi_K(X) - X|] < \frac{\varepsilon}{3}.$$

UI families cont.

Proof of the if part (a) cont.

We claim that

$$(85) \quad \varphi_K(X_n) \xrightarrow{L^1} \varphi_K(X).$$

Using that $|\varphi_K(x) - \varphi_K(y)| \leq |x - y|$ and using that $X_n \xrightarrow{P} X$, we obtain that $\varphi_K(X_n) \xrightarrow{P} \varphi_K(X)$. Observe that the process $\{\varphi_K(X_n)\}$ is bounded (by K), so we can apply Theorem BDD which yields that (85) holds.

UI families cont.

Proof of the if part (a) cont.

Hence for a fixed $\varepsilon > 0$ we can find n_0 s.t. for $n \geq n_0$ we have

$$\mathbb{E}[|\varphi_K(X_n) - \varphi_K(X)|] < \frac{\varepsilon}{3}.$$

Putting together this and the two inequalities in (84) we obtain that for $n \geq n_0$:

$$\mathbb{E}[|X_n - X|] < \varepsilon. \blacksquare$$

UI martingales definition

Definition 8.6 (UI martingale)

$M = (M_n)$ is a UI martingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n, \mathbb{P}\})$ if

- M is a martingale,
- $\{M_n\}_{n=0}^{\infty}$ is a UI family.

Conditional expectation vs. UI

Theorem 8.7

Let $X \in L^1$. Then the following family is UI:

(86) $$\mathcal{C} := \{\mathbb{E}[X|\mathcal{G}] : \mathcal{G} \text{ is a sub-algebra of } \mathcal{F}\}.$$

More precisely, $\mathcal{C}$ consists of the versions of $\mathbb{E}[X|\mathcal{G}]$, for some $\mathcal{G} \subset \mathcal{F}$ sub-σ-algebra.

Conditional expectation vs. UI cont.

Proof

Fix an $\varepsilon > 0$. Using Theorem 7.5 there exist $\delta > 0$ s.t.

(87) $$\forall F \in \mathcal{F},\ \mathbb{P}(F) < \delta \Longrightarrow \mathbb{E}[|X|; F] < \varepsilon.$$

Choose a K s.t.

(88) $$K^{-1}\mathbb{E}[|X|] < \delta.$$

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let Y be a version of $\mathbb{E}[X|\mathcal{G}]$. Then by Jensen inequality:

(89) $$|Y| \leq \mathbb{E}[|X||\mathcal{G}], \quad \text{a.s.}$$

Conditional expectation vs. UI cont.

Proof cont.

So, $\mathbb{E}[|Y|] \leq \mathbb{E}[|X|]$ and

$$K\mathbb{P}(|Y| > K) \leq \mathbb{E}[|Y|] \leq \mathbb{E}[|X|].$$

From here and (88) we get

$$\mathbb{P}(|Y| > K) < \delta.$$

Clearly, $\{|Y| > K\} \in \mathcal{G}$, so by (89) we get

$$\mathbb{E}[|Y|; |Y| \geq K] = \mathbb{E}[|\mathbb{E}[X|\mathcal{G}]|; |Y| \geq K] \leq \mathbb{E}[|X|; |Y| \geq K] < \varepsilon. \blacksquare$$

UI martingales cont.

Theorem 8.8

Let M be a UI martingale. Then

(90) $$M_\infty := \lim_{n\to\infty} M_n \text{ exists a.s. and in } L^1.$$

Further,

(91) $$M_n = \mathbb{E}[M_\infty|\mathcal{F}_n].$$

Proof

It follows from Theorem 3.5 that $M_\infty := \lim_{n\to\infty} M_n$ a.s. exists since M is an L^1-bounded martingale.

UI martingales cont.

Proof cont.

We know that almost sure convergence implies convergence in probability. So, both of the conditions of the second point of Theorem 8.5 are satisfied. Hence $M_n \xrightarrow{L^1} M_\infty$. This completes the proof of (90).
To prove (91), we only need to verify:

(92) $$\mathbb{E}[M_\infty; F] = \mathbb{E}[M_n; F], \quad \forall F \in \mathcal{F}_n.$$

UI martingales cont.

Proof cont.

Let $r > n$. Then $\mathbb{E}[M_r|\mathcal{F}_n] = M_n$. Hence, for $\forall F \in \mathcal{F}_n$ we have

(93) $$\mathbb{E}[M_n; F] = \mathbb{E}[M_r; F] \to \mathbb{E}[M_\infty; F]$$

since $M_r \to M_\infty$ is L^1. This implies that (92) holds, which completes the proof of (91).

Lévy's Upward Theorem

Theorem 8.9 (Lévy's Upward Theorem)

Let $X \in L^1$ and on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ and let

$$M_n := \mathbb{E}[X|\mathcal{F}_n] \text{ and } Y := \mathbb{E}[X|\mathcal{F}_\infty],$$

where $\mathcal{F}_\infty := \sigma(\mathcal{F}_n, n \in \mathbb{N})$. Then

(a) *$M = (M_n)$ is a UI martingale and*

(b) *$M_n \to Y$ a.s. and in L^1.*

Lévy's Upward Theorem cont.

Note that Theorem 8.9 is weaker than Theorem 6.3 (which we did not prove), but we prove Theorem 8.9.

Proof of Thm 8.9 part (a)

M is a martingale since by the tower property:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[\underbrace{\mathbb{E}[X|\mathcal{F}_{n+1}]}_{M_{n+1}}|\mathcal{F}_n\right] = M_n.$$

M is UI by Theorem 8.7.

Lévy's Upward Theorem cont.

So, M is UI. Then by Theorem 8.8 $\exists M_\infty$ s.t.

(94) $$M_n \longrightarrow M_\infty \text{ a.s. and in } L^1.$$

So, we only need to prove that

(95) $$Y = M_\infty.$$

Proof of Thm 8.9 part (b)

WLOG (acronym for without loss of generality) we can (and do) assume that $X \geq 0$. Let ν_1, ν_2 be measures on the measurable space $(\Omega, \mathcal{F}_\infty)$ defined by:

$$\nu_1(F) := \mathbb{E}[Y; F] \text{ and } \nu_2(F) := \mathbb{E}[M_\infty; F], \ F \in \mathcal{F}_\infty.$$

Lévy's Upward Theorem cont.

Proof of Thm 8.9 part (b) cont.

First observe that by tower property:

$$\mathbb{E}[Y|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_n] = \mathbb{E}[M_\infty; F], \quad \forall F \in \mathcal{F}_n.$$

Namely, the first equality follows from the tower property, the second one was checked in (92).
So, ν_1, ν_2 coincide on the π-system $\cup \mathcal{F}_n$ (which is actually an algebra). So, ν_1 is equal to ν_2 also on $\mathcal{F}_\infty$.

Lévy's Upward Theorem cont.

Proof of Thm 8.9 part (b) cont.

Both Y and M_∞ are $\mathcal{F}_\infty$ measurable. So

$$F := \{\omega : Y(\omega) > M_\infty(\omega)\} \in \mathcal{F}_\infty.$$

Hence $\mathbb{E}[Y; F] = \nu_1(F) = \nu_2(F) = \mathbb{E}[M_\infty; F]$. That is

$$\mathbb{E}[Y - M_\infty; Y - M_\infty] = 0.$$

That is $\mathbb{P}(Y > M_\infty) = 0$. Similarly we can see that $\mathbb{P}(M_\infty > Y) = 0$ and this completes the proof of (95).
■

Kolomorov's $0-1$ law

Theorem 8.10 (Kolmogorov's $0-1$ law)

Let $X_1, X_2, \dots$ be a sequence of independent rv.

$$\mathcal{T}_n := \sigma(X_{n+1}, X_{n+2}, \dots), \quad \mathcal{T} := \bigcap_{n=1}^{\infty} \mathcal{T}_n.$$

We say that $\mathcal{T}$ is the tail σ-algebra. Then

(96) $$\mathbb{P}(F) = 0 \text{ or } 1, \quad \forall F \in \mathcal{T}.$$

Martingale proof for Kolomorov $0-1$ law

Proof

Let

$$\mathcal{F}_n := \sigma(X_1, \dots, X_n),$$

Fix an $F \in \mathcal{T}$, and put $Y := \mathbb{1}_F$. Using the fact that $Y \in \mathcal{F}_\infty$ in the first equality and Levy's upward Thm. in the second one we get:

(97) $$Y = \mathbb{E}[Y|\mathcal{F}_\infty] = \lim_{n\to\infty} \mathbb{E}[Y|\mathcal{F}_n], \quad \text{a.s.}$$

On the other hand, $\forall n$:

(98) $$Y \in \mathcal{T}_n \Longrightarrow Y \text{ is independent of } \mathcal{F}_n.$$

Martingale proof for Kolomorov $0-1$ law cont.

Proof cont.

(99) $$\mathbb{E}[Y|\mathcal{F}_n] = \mathbb{E}[Y] = \mathbb{P}(F), \quad \text{a.s.}$$

Putting together (97) and (99) we obtain that

$$Y = \mathbb{P}(F).$$

Since by definition Y is either zero or one we obtain that $\mathbb{P}(F)$ is also either zero or one holds for all $F \in \mathcal{T}_\infty$. ■

Levy's Downward Theorem

Theorem 8.11 (Levy's Downward Theorem)

Let $\{\mathcal{G}_{-n} : n \in \mathbb{N}\}$ be a sequence of sub-σ-algebras of $\mathcal{F}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying:

$$\mathcal{G}_{-\infty} := \bigcap_{k=1}^{\infty} \mathcal{G}_{-k} \cdots \subset \mathcal{G}_{-(m+1)} \subset \mathcal{G}_{-m} \cdots \subset \mathcal{G}_{-1}.$$

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $M_{-n} := \mathbb{E}[X|\mathcal{G}_{-n}]$. Then

(a) $M_{-\infty} := \lim_{n\to\infty} M_{-n}$ *exists a.s. and in L^1,*
(b) $M_{-\infty} = \mathbb{E}[X|\mathcal{G}_{-\infty}]$ *a.s.*

Levy's Downward Theorem cont.

Proof

Observe that

(100) $(M_k, \mathcal{G}_k : -N \le k \le -1)$.

is a FINITE martingale sequence. So, we can apply Doob's Upcrossing Lemma (slide # 25) for it. Clearly, $\mathbb{E}[|M_{-k}|] \le \mathbb{E}[|X|]$ so M_{-n} is L^1 bounded. Therefore we can apply the steps of both Corollary 3.3 and Doob's Forward Convergence Theorem (slide # 29) to obtain that $\exists M_{-\infty}$ s.t.

(101) $M_{-\infty} = \lim_{n\to\infty} M_{-n}$, a.s.

Levy's Downward Theorem cont.

Proof cont.

Recall from the Figure on slide # 103 of the File "Some basic Facts in Probability Theory" that

(102)

a.s. convergence $\Longrightarrow$ convergence in probability.

We apply this for M_{-n} and Theorem 8.7 (in this order) to get

(a) $M_{-n} \xrightarrow{P} M_{-\infty}$

(b) (M_{-n}) is UI.

Using Theorem 8.5 we conclude that $M_{-n} \xrightarrow{L^1} M_{-\infty}$. This completes the proof of part (a).

Levy's Downward Theorem cont.

Proof part (b)

In particular this implies that

(103) $\lim_{r\to\infty} \mathbb{E}[M_{-r}; G] = \mathbb{E}[M_{-\infty}; G]$, $\forall G \in \mathcal{G}_\infty$.

In order to verify that

(104) $M_{-\infty} = \mathbb{E}[X|\mathcal{G}_{-\infty}]$

it is enough to show that $\forall G \in \mathcal{G}_{-\infty}$:

(105) $\mathbb{E}[M_{-\infty}; G] = \mathbb{E}[X; G] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_{-\infty}]; G]$.

Levy's Downward Theorem cont.

Proof of part (b) cont.

However, this follows from (103) if we can check that

(106) $\mathbb{E}[X; G] = \mathbb{E}[M_{-r}; G]$, $\forall G \in \mathcal{G}_{-\infty} \subset \mathcal{G}_{-r}$.

This is immediate from the definition of conditional expectation since by definition $M_{-r} = \mathbb{E}[X|\mathcal{G}_r]$. This completes the proof of (104). ■

Now we give a martingale proof for SLLN (Strong Law of Large Numbers). At the end of the course Probability 1 you have seen a proof for the special case when the 4-th moments existed.

Martingale proof for SSLN

Theorem 8.12 (SLLN)

Let $X_1, X_2, \dots$ be iid. rv. with $\mathbb{E}[|X_1|] < \infty$. Let

$$\mu := \mathbb{E}[X_k], \text{ and } S_n := X_1 + \cdots + X_n.$$

Then

(107) $\lim_{n\to\infty} \frac{S_n}{n} = \mu$, *a.s. and in* L^1.

Also in the course Probability I, only the almost sure convergence was stated.

Martingale proof for SSLN cont.

Below we mention an important Corollary. It follows easily from SLLN. This proof is assigned as homework # 35.

Corollary 8.13

Let $X_1, X_2, \dots$ be iid. rv. with $\mathbb{E}[X^+] = \infty$ and $\mathbb{E}[X^-] < \infty$. (Recall $X = X^+ - X^-$ and $X^+, X^- \ge 0$.) We can use SLLN to prove that

$$S_n/n \to \infty, \quad a.s.$$

where $S_n := X_1 + \cdots + X_n$.

Martingale proof for SLLN cont.

Proof of Thm 8.12

Let

$$\mathcal{G}_{-n} := \sigma(S_n, S_{n+1}, \dots), \quad \mathcal{G}_{-\infty} := \bigcap_{n=1}^{\infty} \mathcal{G}_{-n}.$$

Clearly,

$$\mathbb{E}[X_1|\mathcal{G}_{-n}] = \mathbb{E}[X_1|S_n].$$

However, it is easy to see (assigned as homework # 36) that

(108) $\mathbb{E}[X_1|S_n] = S_n/n$.

So we obtain that

Martingale proof for SLLN cont.

Proof of Thm 8.12 cont.

(109) $M_{-n} := \mathbb{E}[X_1|\mathcal{G}_{-n}] = \frac{S_n}{n}$, a.s.

We apply Levy's Downward Theorem for (M_{-n}) to obtain that the following limits exists:

(110) $L := \lim_{n\to\infty} \frac{S_n}{n}$.

Observe that for each k

(111) $L = \lim_{n\to\infty} \frac{X_{k+1} + \cdots + X_n}{n}$.

Martingale proof for SLLN cont.

Proof of Thm 8.12 cont.

That is $L \in \mathcal{T}_K := \sigma(X_{K+1}, X_{K+2}, \dots)$. So, $L \in \mathcal{T}_\infty$. Using Kolmogov's $0-1$ law this means that the rv L is almost sure constant. Then it cannot be anything but

$$L = \mathbb{E}[L] = \lim_{n\to\infty} \mathbb{E}\left[\frac{S_n}{n}\right] = \mu. \blacksquare$$

Doob's submartingale inequality

Theorem 8.14 (Doob's submartingale inequality)

Let Z be a non-negative submartingale and $L > 0$. Then

(112)
$$L \cdot \mathbb{P}\left(\sup_{k\le n} Z_k \ge L\right) \le \mathbb{E}\left[Z_n; \sup_{k\le n} Z_k\right] \le \mathbb{E}[Z_n].$$

Proof

$F_0 := \{Z_0 \ge L\}$,

$F_k := \{Z_0 < L\} \cap \{Z_1 < L\} \cap \dots \cap \{Z_{k-1} < L\} \cap \{Z_k \ge L\}$.

Doob's submartingale inequality

Proof cont.

That is for $T := \min\{k : Z_k \ge k\}$ we have $F_k = \{T = k\}$.

Clearly, $F_k \in \mathcal{F}_k$ and $Z_k \ge L$ on F_k. That is for $k \le n$:

(113)
$$\mathbb{E}[Z_n; F_k] = \mathbb{E}[\mathbb{E}[Z_n|\mathcal{F}_k]; F_k] \ge \mathbb{E}[Z_k; F_k] \ge L\mathbb{P}(F_k).$$

To complete the proof it is enough to sum up for $k \le n$ since $Z_n \ge 0$ and

$$\left\{\sup_{k\le n} Z_k \ge L\right\} = F_0 \bigsqcup F_1 \bigsqcup \dots \bigsqcup F_n. \blacksquare$$

Recall a fact learned earlier

Remark 8.15

Recall that for a

- convex function φ and a
- martingale $M = (M_n)$

it follows from conditional Jensen's inequality (see slide # 133 of File "Some basic facts from probability theory") that

$$\mathbb{E}[|\varphi(M_n)|] < \infty \Longrightarrow \varphi(M_n) \text{ is a submartingale.}$$

Kolmogorov's inequality

Lemma 8.16 (Kolmogorov's inequality)

Given a sequence of rv. $(X_n, n \ge 1)$. We assume that

- $(X_n, n \ge 1)$ *are independent,*
- $\mathbb{E}[X_i] = 0$,
- $X_i \in L^2$.

We define

$S_n := X_1 + \dots + X_n$, $\quad V_n := \mathrm{Var}(S_n) = \sum_{k=1}^{n} \sigma_k^2$. *Then for any $L > 0$ we have*

(114)
$$L^2 \cdot \mathbb{P}\left(\sup_{k\le n} |S_k| \ge L\right) \le V_n.$$

Kolmogorov's inequality cont.

Proof.

$\mathcal{F}_n := \sigma\{X_1, \dots, X_n\}$. Then $S = (S_n)$ is a martingale. So, by Remark 8.15, S^2 is a non-negative submartingale. Hence, we can apply the Submartingale inequality for S^2. □

$\dim_H$

An estimate on normal distribution

Fact 9.1

Let $X \sim \mathcal{N}(0,1)$ and Φ and φ are the CDF and the density of X respectively. That is

$$\varphi(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}},$$

$$\mathbb{P}(X > x) = 1 - \Phi(x) = \int_x^\infty \varphi(y)dy.$$

Then for $x > 0$ we have

(a) $\mathbb{P}(X > x) \le x^{-1}\varphi(x)$
(b) $\mathbb{P}(X > x) \ge (x + x^{-1})^{-1}\varphi(x)$

An estimate on normal distribution cont.

Proof

It is easy to check that

(115)
$$\varphi'(x) = -x\varphi(x) \text{ and } \left(\frac{\varphi(x)}{x}\right)' = -(1+x^{-2})\varphi(x).$$

Using the first equality we get that for $x > 0$:

$$\varphi(x) = \int_x^\infty y\varphi(y)dy \geq x\int_x^\infty \varphi(y)dy.$$

Which yields (a).

An estimate on normal distribution cont.

Proof cont

To verify (b), we use the second part of (115):

$$x^{-1}\varphi(x) = \int_x^\infty (1+y^{-2})\varphi(y)dy$$
$$\leq (1+x^{-2})\int_x^\infty \varphi(y)dy.$$

which yields (b).

Kolmogov's Law of Iterated Logarithm

Theorem 9.2 (Kolmogov's Law of Iterated logarithm (LIL))

Given a sequence of rv $X = (X_n, n \geq 1)$ satisfying:

- $X_1, X_2, \dots$ *are* iid,
- $\mathbb{E}[X_i] = 0$,
- $\mathrm{Var}(X_i) = 1$.

As usual, we write $S_n := X_1 + \cdots + X_n$. Then almost surely,

(116)
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1.$$

L

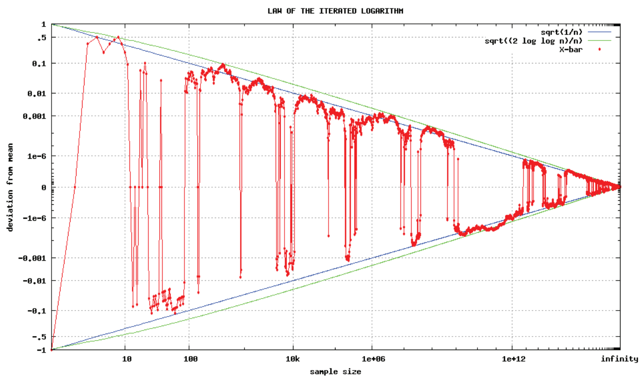

Figure: A plot of the average of n Bernoulli trials (each taking a value of +/- 1). Plot of (red), its variance given by CLT (blue) and its bound given by LIL (green). Figure is from Wikipedia.

A crude heuristics

By CLT:

$$\frac{S_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0,1).$$

If $\frac{S_n}{\sqrt{n}}$ was a $N \sim \mathcal{N}(0,1)$ then

$$\sum_n \mathbb{P}\left(N > \sqrt{(2+\varepsilon)\log\log n}\right) < \infty$$

and $\sum_n \mathbb{P}\left(N > \sqrt{(2-\varepsilon)\log\log n}\right) = \infty$ so the first one happens finitely many times and the second one happens infinitely many times by BC Lemmas. (This is not a proof just a heuristics. The second part of BC Lemma (Lemma 5.12) holds only if the vents are independent.)

Proof of LIL in a special case

We give the proof in the special case when

(117)
$$X_i \stackrel{D}{=} \mathcal{N}(0,1).$$

Proof of LIL assuming (117)

Let

$$h(n) := \sqrt{2n\log\log n}, \quad \text{for } n \geq 3.$$

First we verify that for every $c > 0$ and $n \geq 3$ we have:

(118)
$$\mathbb{P}\left(\sup_{k\leq n} S_k \geq c\right) \leq e^{-\frac{1}{2}c^2/n}.$$

Proof of LIL in a special case cont.

Proof of LIL assuming (117) cont.

Namely, we have learned that the moment generating function of the standard normal distribution X_i is $\mathbb{E}\left[e^{\theta X_i}\right] = e^{\frac{1}{2}\theta^2}$. Hence, $\mathbb{E}\left[e^{\theta S_n}\right] = e^{\frac{1}{2}\theta^2 n} < \infty$. For every fixed $\theta \in \mathbb{R}$ the function $x \to e^{\theta x}$ is convex. So, $e^{\theta S_n}$ is a submartingale. So, we can use Submartingale inequality for $\theta > 0$:

$$\mathbb{P}\left(\sup_{k\leq n} S_k \geq c\right) = \mathbb{P}\left(\sup_{k\leq n} e^{\theta S_k} \geq e^{\theta c}\right) \leq e^{-\theta c}\mathbb{E}\left[e^{\theta S_n}\right]$$
$$= e^{-\theta c}e^{\frac{1}{2}\theta^2 n}.$$

For $c > 0$ choosing $\theta = c/n$ we obtain that (118) holds.

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (upper bound)

We choose a $K > 1$ (actually K will be close to 1). Let

$$c_n := Kh(K^{n-1}).$$

Then

$$\mathbb{P}\left(\sup_{k\leq K^n} S_k \geq c_n\right) \leq \exp\left(-c_n^2/2K^n\right)$$
$$= (n-1)^{-K}(\log K)^{-K}.$$

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (upper bound) cont.

From this and from the first BC Lemma: almost surely for all $n \geq n_0(\omega)$ and for $k \in [K^{n-1}, K^n]$ we have

(119) $S_k \leq \sup_{k \leq K^n} S_k \leq c_n = Kh(K^{n-1}) \leq Kh(k)$, a.s.

So, for $K > 1$: $\limsup_{k \to \infty} h(k)^{-1} S_k \leq K$ a.s. By letting $K \downarrow 1$ we get that

$$\limsup h(k)^{-1} S_k \leq 1 \quad \text{a.s.}$$

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (lower bound)

Fix $N > 1$, $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Define the events:

$$F_n := \left\{ S_{N^{n+1}} - S_{N^n} > (1-\varepsilon) h\left(N^{n+1} - N^n\right) \right\}.$$

Then by Fact 9.1, for

$$y := (1-\varepsilon)\sqrt{2 \log\log\left(N^{n+1} - N^n\right)}$$

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (lower bound) cont.

we have

$$\mathbb{P}(F_n) = 1 - \Phi(y) \geq \frac{1}{\sqrt{2\pi} \cdot (y + y^{-1})} \exp(-y^2/2).$$

By ignoring the logarithmic terms we see that

$$\mathbb{P}(F_n) \sim (n \log N)^{-(1-\varepsilon)^2}$$

This yields that

(120) $\sum \mathbb{P}(F_n) = \infty$.

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (lower bound) cont.

The events F_n are independent so, we can apply BC Lemma (Lemma 5.12). This yields that almost surely infinitely many of F_n occurs. That is for infinitely many n:

(121) $S\left(N^{n+1}\right) > (1-\varepsilon) h\left(N^{n+1} - N^n\right) + S(N^n)$.

Observe that (119) is also valid if we substitute S with $-S$. For $K = 2$ this yields

$$S(N^n) > -2h(N^n).$$

Proof of LIL in a special case cont.

Proof of LIL assuming (117) (lower bound) cont.

So, by the last twi displayed formulas:

$$S\left(N^{n+1}\right) > (1-\varepsilon) h\left(N^{n+1} - N^n\right) - 2h(N^n)$$

From here, simple algebraic manipulations yield:

$$\limsup_{k \to \infty} h(k)^{-1} S_k \geq \limsup_{n \to \infty} h(N^{n+1})^{-1} S(N^{n+1})$$
$$\geq (1-\varepsilon)\sqrt{1 - N^{-1}} - \frac{2}{\sqrt{N}}.$$

This completes the proof since this holds for all N. ■

$\dim_{\mathrm{H}}$

An Auxiliary Lemma

Lemma 10.1

Given $X, Y \geq 0$ rv. staisfying:

(122) $c\mathbb{P}(X \geq c) \leq \mathbb{E}[Y; X \geq c]$, $\forall c > 0$.

For every $p > 1$ we define the conjugate q of p by

(123) $\frac{1}{p} + \frac{1}{q} = 1.$

Then we have

(124) $\|X\|_p \leq q \cdot \|Y\|_p$.

Proof of Lemma 10.1

Proof

$$L := \int_{c=0}^{\infty} pc^{p-1} \mathbb{P}(X \geq c)\, dc$$
$$\overset{(122)}{\leq} \int_{c=0}^{\infty} pc^{p-2} \mathbb{E}[Y; X \geq c]\, dc =: R$$

By Fubini Thm. we get

Proof of Lemma 10.1

Proof cont.

(125)
$$L = \int_{c=0}^{\infty} \left(\int_{\Omega} \mathbb{1}_{\{X \geq c\}}(\omega) d\mathbb{P}(\omega) \right) pc^{p-1} dc$$
$$= \int_{\Omega} \left(\int_{c=0}^{X(\omega)} pc^{p-1} dc \right) d\mathbb{P}(\omega) = \mathbb{E}\left[X^p\right].$$

Similarly we get

(126)
$$R = \mathbb{E}\left[q \cdot X^{p-1} \cdot Y\right].$$

By Hölder inequality:

Proof of Lemma 10.1

Proof cont.

(127)
$$\mathbb{E}\left[X^p\right] = L \leq R = \mathbb{E}\left[qX^{p-1}Y\right] \leq q\|Y\|_p\|X^{p-1}\|_q.$$

WLOG we may assume that $\|Y\|_p < \infty$.
If additionally, $\|X\|_p < \infty$ then by $(p-1)q = p$ we get

$$\|X^{p-1}\|_q = \mathbb{E}\left[X^p\right]^{1/q}.$$

Using this and (127) we get

$$\left(\mathbb{E}\left[X^p\right]\right)^{1-1/q} = \|X\|_p \leq q\|Y\|_p.$$

Proof of Lemma 10.1

Proof cont.

For general X we do the same for $X \wedge n$ to conclude that

$$\|X \wedge n\|_p \leq q\|Y\|_p, \quad \forall n.$$

Then we apply Monoton Conv. Thm. which concludes the proof of the Lemma.■

Doob's L^p inequality

Theorem (Doob's L^p inequality)

Fix a $p > 1$ and let q its conjugate, defined by $\frac{1}{p} + \frac{1}{q} = 1$. Given the process $Z = (Z_n)$ satisfying:

- non-negative,
- submartingale,
- $\sup_n \|Z_n\|_p < \infty$. ($Z$ is L^p-bounded.)

We use the standard notation:

$$Z^* := \sup_k Z_k.$$

Then $Z^* \in L^p$.

Doob's L^p inequality cont.

Theorem (Doob's L^p inequality cont.)

More precisely,

(128)
$$\|Z^*\|_p \leq q \cdot \sup_r \|Z_r\|_p.$$

So, the non-negative submartingale Z is L^p-dominated (by $Z^* \in L^p$). Further,

(129)
$$Z_\infty := \lim_{n\to\infty} Z_n \text{ exists a.s. and in } L^p \text{ and}$$

(130)
$$\|Z_\infty\|_p = \sup_r \|Z_r\|_p =\uparrow \lim_{r\to\infty} \|Z_r\|_p.$$

Doob's L^p inequality cont.

Theorem (Doob's L^p inequality cont.)

In the special case when $Z_n = |M_n|$ for a martingale $M = (M_n)$ which is bounded in L^p then

(131)
$$M_\infty := \lim_{n\to\infty} M_n \text{ exists a.s. and in } L^p.$$

and

(132)
$$Z_\infty = |M_\infty| \text{ a.s.}$$

Doob's L^p inequality cont.

Proof of Doob's L^p inequality

First we prove that (128) holds. Let

$$Z_n^* := \sup_{k \leq n} Z_k.$$

Doob's submartingale inequality yields:

(133)
$$c\mathbb{P}\left(Z_n^* \geq c\right) \leq \mathbb{E}\left[Z_n; \{Z_n^* \geq c\}\right]$$

Now we apply Lemma 10.1 with

$$X = Z_n^* \text{ and } Y = Z_n.$$

Doob's L^p inequality cont.

Proof of Doob's L^p inequality cont.

to obtain that

$$\|Z_n^*\|_p \leq q\|Z_n\|_p \leq q \sup_r \|Z_r\|_p.$$

That is Z^* is L^p-bounded. Observe that $-Z$ is an L^p-bounded supermartingale. Hence, $-Z$ is also an L^1-bounded supermartingale. Doob's Forward Convergence Thm. (Theorem 3.5) yields that

(134)
$$Z_\infty := \lim_{n\to\infty} Z_n \text{ exists a.s.}$$

Doob's L^p inequality cont.

Proof of Doob's L^p inequality cont.

On the other hand,

$$|Z_n - Z_\infty|^p \le (2Z^*)^p \in L^p.$$

Using Dominated Conv. Thm. we get

$$Z_n \xrightarrow{L^p} Z_\infty.$$

Now we verify that

(135) $\{\|Z_r\|_p\}_r$ is monotone increasing.

Doob's L^p inequality cont.

Proof of Doob's L^p inequality cont.

Namely, consider the convex function

$$\varphi(x) := |x|^p.$$

By the Conditional Jensen's inequality (File Some Basic Facts from Probability Theory, ...133) we have:

$$\|Z_{r+1}\|_p^p = \mathbb{E}\left[\varphi(Z_{r+1})\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi(Z_{r+1})|\mathcal{F}_r\right]\right] \overset{cond.Jensen}{\ge} \mathbb{E}\left[\varphi\left(\mathbb{E}\left[Z_{r+1}|\mathcal{F}_r\right]\right)\right] \ge \mathbb{E}\left[\varphi\left(Z_r\right)\right] = \|Z_r\|_p^p.$$

Doob's L^p inequality cont.

Proof of Doob's L^p inequality cont.

To verify (131) and (132) is a homework. ■

$\dim_{\mathrm{H}}$

Product Martingales

Kakutani's Theorem

Given $X = (X_n)$ satisfying:

- $X_1, X_2, \dots$ are independent
- X_k are non-negative for all k.
- $\mathbb{E}\left[X_k\right] = 1$ for all k.

Then for $M_n := \prod_{i=1}^{n} X_i$ is a non-negative martingale (see File E of the course Stochastic processes). Hence $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. We introduce:

(136) $$a_n := \mathbb{E}\left[\sqrt{X_n}\right].$$

Product Martingales cont.

Kakutani's Theorem cont.

Then the following conditions are equivalent

(a) $\mathbb{E}\left[M_\infty\right] = 1$,

(b) $M_n \xrightarrow{L^1} M_\infty$,

(c) M is UI.

(d) $\prod_{n=1}^{\infty} a_n > 0$,

(e) $\sum_{n=1}^{\infty} (1 - a_n) < \infty$.

If any of these conditions does not hold then

(137) $$\mathbb{P}\left(M_\infty = 0\right) = 1.$$

Product Martingales cont.

Proof of Kakutani's Theorem cont.

First observe that by Jensen's inequality:

(138) $$0 \le = a_n = \mathbb{E}\left[\sqrt{X_n}\right] \le \sqrt{\mathbb{E}\left[X_n\right]} = 1.$$

Assume that (d) holds. Let

(139) $$N_n := \prod_{k=1}^{n} \frac{\sqrt{X_k}}{a_k}.$$

Clearly, N_n is a martingale (product of independent nonnegative rv having mean 1).

Product Martingales cont.

Proof of Kakutani's Theorem

By independence and (138):

(140) $$\mathbb{E}\left[N_n^2\right] \le \frac{1}{\prod_{k=1}^{n} a_k^2} \le \frac{1}{\prod_{k=1}^{\infty} a_k^2} < \infty.$$

That is N_n is L^2 bounded. So, we can apply Doob's L^2 inequality (Theorem 172 with $p = 2$) for N_n^2 in the second step below:

(141)
$$\mathbb{E}\left[\sup_n |M_n|\right] \le \mathbb{E}\left[\sup_n N_n^2\right] \le 4 \sup_n \mathbb{E}\left[N_n^2\right] < \infty.$$

Product Martingales cont.

Proof of Kakutani's Theorem

That is

$$M^* := \sup_n |M_n| \in L^1.$$

That is M is a UI martingale which implies that properties (a), (b), (c) hold.

On the other hand,
Assume that $\prod_{n=1}^{\infty} a_n = 0$. Then N defined in (139) is a nonnegative martingale, so its limit N_∞ exists a.s. This implies that $M_\infty = 0$ a.s. since $\prod a_n = 0$. ■

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